MI Preprint Series

Mathematics for Industry
Kyushu University

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Yoshiyuki Kagei, Takaaki Nishida & Yuka Teramoto

MI 2017-2

(Received February 14, 2017)

Institute of Mathematics for Industry
Graduate School of Mathematics
Kyushu University
Fukuoka, JAPAN

On the spectrum for the artificial compressible system

Yoshiyuki Kagei[1], Takaaki Nishida[2] and Yuka Teramoto[3]

Abstract

Stability of stationary solutions of the incompressible Navier-Stokes system and the corresponding artificial compressible system is considered. Both systems have the same sets of stationary solutions and the incompressible system is obtained from the artificial compressible one in the zero limit of the artificial Mach number ϵ which is a singular limit. It is proved that if a stationary solution of the incompressible system is asymptotically stable and the velocity field of the stationary solution satisfies an energy-type stability criterion by variational method with admissible functions being only potential flow parts of velocity fields, then it is also stable as a solution of the artificial compressible one for sufficiently small ϵ. The result is applied to the Taylor problem.

Mathematics Subject Classification (2000). 35Q30, 76N15.

Keywords. Incompressible Navier-Stokes system, artificial compressible system, singular perturbation, stability, Taylor vortex.

1 Introduction

This paper is concerned with the incompressible Navier-Stokes system

$$\operatorname{div} \boldsymbol{v} = 0, \tag{1.1}$$

$$\partial_t \boldsymbol{v} - \nu \Delta \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla p = \boldsymbol{g}, \tag{1.2}$$

[1]Faculty of Mathematics, Kyushu University, Fukuoka 819-0395, Japan
[2]Department of Applied Complex System, Kyoto University, Kyoto, 606-8317, Japan
[3]Graduate School of Mathematics, Kyushu University, Fukuoka 819-0395, Japan

and the artificial compressible system for (1.1)–(1.2):

$$\epsilon^2 \partial_t p + \operatorname{div} \boldsymbol{v} = 0, \quad (1.3)$$

$$\partial_t \boldsymbol{v} - \nu \Delta \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla p = \boldsymbol{g}. \quad (1.4)$$

Here $\boldsymbol{v} = {}^\top(v^1(x,t), v^2(x,t), v^3(x,t))$ and $p = p(x,t)$ denote the unknown velocity field and pressure, respectively, at time $t > 0$ and position $x \in \Omega$, where Ω is a bounded domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$; $\boldsymbol{g} = \boldsymbol{g}(x)$ is a given external force and $\epsilon > 0$ is a small parameter, called the artificial Mach number.

We consider (1.1)–(1.2) and (1.3)–(1.4) under the boundary condition

$$\boldsymbol{v}|_{\partial\Omega} = \boldsymbol{v}_*. \quad (1.5)$$

Here $\boldsymbol{v}_*$ is a given velocity field satisfying $\int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n} \, dS = 0$, where $\boldsymbol{n}$ denotes the unit outward normal to $\partial\Omega$.

In this paper we are interested in the relation of stability properties between stationary solutions of (1.1)–(1.2) and (1.3)–(1.4) under the boundary condition (1.5).

The artificial compressible system (1.3)–(1.4) was proposed by A. Chorin ([1, 2, 3]) and R. Temam ([18, 19]). In [1, 2, 3], the system (1.3)–(1.4) was introduced to find numerically stationary solutions of the incompressible Navier-Stokes equation (1.1)–(1.2). The idea is as follows. Obviously, the set of stationary solutions of (1.1)–(1.2) is the same as that of (1.3)–(1.4). If solutions of the artificial compressible system (1.3)–(1.4) converge to a function $u_s = {}^\top(p_s, \boldsymbol{v}_s)$ as $t \to \infty$, then the limit u_s is a stationary solution of (1.3)–(1.4), and thus, u_s is a stationary solution of (1.1)–(1.2). By using this method, Chorin numerically obtained stationary cellular convection patterns of the Bénard convection problem described by the Oberbeck-Boussinesq equation

$$\operatorname{div} \boldsymbol{v} = 0, \quad (1.6)$$

$$\mathrm{Pr}^{-1} (\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}) - \Delta \boldsymbol{v} + \nabla p - \mathrm{Ra}\theta \boldsymbol{e}_3 = \boldsymbol{0}, \quad (1.7)$$

$$\partial_t \theta + \boldsymbol{v} \cdot \nabla \theta - \Delta \theta - \mathrm{Ra} \boldsymbol{v} \cdot \boldsymbol{e}_3 = 0 \quad (1.8)$$

in the infinite layer $\{x = (x', x_3); x' = (x_1, x_2) \in \mathbb{R}^2, 0 < x_3 < 1\}$. Here $\theta(x,t)$ is the temperature deviation from the heat conductive state; $\boldsymbol{e}_3 = {}^\top(0,0,1) \in \mathbb{R}^3$; $\mathrm{Pr} > 0$ and $\mathrm{Ra} > 0$ are non-dimensional parameters, called the Prandtl and Rayleigh numbers, respectively

Since the limit function u_s in Chorin's method is a large time limit of solutions of (1.3)–(1.4), u_s is stable as a solution of (1.3)–(1.4). The following questions were then addressed in [15]:

(a) whether u_s is stable as a solution of (1.1)–(1.2) with (1.5), in other words, whether u_s represents an observable stationary flow in the real world,

(b) conversely, what kind of stationary flows can be computed by Chorin's method.

These questions were formulated in [15] as stability problems of stationary solutions of the incompressible system and the corresponding artificial compressible one. Since the incompressible system (1.1)–(1.2) is obtained from the artificial compressible one (1.3)–(1.4) as the limit $\epsilon \to 0$, one could expect that solutions of (1.1)–(1.2) would be approximated by solutions of (1.3)–(1.4) with $\epsilon \ll 1$. However, the limiting procedure is a singular limit, so it is not straightforward to conclude that stability properties of u_s as a solution of (1.1)–(1.2) are the same as those as a solution of (1.3)–(1.4) even if $0 < \epsilon \ll 1$.

The convergence of solutions as $\epsilon \to 0$ was discussed in [18, 19, 20] for the system (1.3)–(1.4) with the additional stabilizing nonlinear term $+\frac{1}{2}(\operatorname{div} \boldsymbol{v})\boldsymbol{v}$ on the left of (1.4); and it was shown that there exists a weak solution ${}^\top(p_\epsilon, \boldsymbol{v}_\epsilon)$ for each $\epsilon > 0$ such that $\boldsymbol{v}_{\epsilon'} \to \boldsymbol{v}$ in $L^2(0,T; L^2(\Omega)^3)$ and $\nabla p_{\epsilon'} \to \nabla p$ weakly in $H^{-1}(\Omega \times (0,T))$ for all $T > 0$ along a sequence $\epsilon' \to 0$, where ${}^\top(p, \boldsymbol{v})$ is a weak solution of (1.1)–(1.2). We also mention interesting works by Donatelli [5, 6] and Donatelli and Marcati [7, 8] where similar convergence results were obtained in the case of unbounded domains by using the wave equation structure of the pressure and the dispersive estimates. For the stability questions, we need to investigate the spectrum of the linearized operators around a stationary solution. The purpose of this paper is to investigate whether (1.3)–(1.4) gives a good approximation of (1.1)–(1.2), when $0 < \epsilon \ll 1$, from the view point of the stability of stationary solutions.

In [15], the above questions were considered for the Oberbeck-Boussinesq equation (1.6)–(1.8) in the infinite layer under the boundary condition $\boldsymbol{v} = \boldsymbol{0}$, $\theta = 0$ on $\{x_3 = 0, 1\}$ and a periodic boundary condition in $x' = (x_1, x_2)$. The results can be restated for the systems (1.1)–(1.2) and (1.3)–(1.4) in the following way.

We introduce the linearized operators around a stationary solution $u_s = {}^\top(p_s, \boldsymbol{v}_s)$ associated with the systems (1.1)–(1.2) and (1.3)–(1.4) under (1.5).

Here and in what follows ${}^\top\cdot$ stands for the transposition. Let $L : L^2_\sigma(\Omega) \to L^2_\sigma(\Omega)$ be the operator defined by

$$L = -\nu\mathbb{P}\Delta + \mathbb{P}(\boldsymbol{v}_s \cdot \nabla + {}^\top(\nabla \boldsymbol{v}_s))$$

with domain $D(L) = [H^2(\Omega) \cap H^1_0(\Omega)]^3 \cap L^2_\sigma(\Omega)$. Here $H^k(\Omega)$ denotes the k th order L^2-Sobolev space on Ω, $\mathbb{P}$ is the orthogonal projection, called the Helmholtz projection, from $L^2(\Omega)^3$ to $L^2_\sigma(\Omega)$, and $L^2_\sigma(\Omega)$ denotes the set of all L^2-vector fields $\boldsymbol{w}$ on Ω satisfying $\operatorname{div} \boldsymbol{w} = 0$ and $\boldsymbol{w} \cdot \boldsymbol{n}|_{\partial\Omega} = 0$, where $\boldsymbol{n}$ denotes the unit outward normal to $\partial\Omega$. We define the operator $L_\epsilon : H^1_*(\Omega) \times L^2(\Omega)^3 \to H^1_*(\Omega) \times L^2(\Omega)^3$, acting on $u = {}^\top(p, \boldsymbol{w})$, by

$$L_\epsilon = \begin{pmatrix} 0 & \frac{1}{\epsilon^2}\operatorname{div} \\ \nabla & -\nu\Delta + \boldsymbol{v}_s \cdot \nabla + {}^\top(\nabla \boldsymbol{v}_s) \end{pmatrix}$$

with domain $D(L_\epsilon) = H^1_*(\Omega) \times [H^2(\Omega) \cap H^1_0(\Omega)]^3$. Here $H^1_*(\Omega)$ denotes the set of H^1 functions on Ω that have zero mean value over Ω.

As for the question (a), it was proved in [15] that if there exists a positive number b_0 such that $\rho(-L_{\epsilon_n}) \supset \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \geq -b_0\}$ for some sequence $\epsilon_n \to 0$ as $n \to \infty$, then there exists a positive constant b_1 such that $\rho(-L) \supset \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \geq -b_1\}$. Therefore, a stationary solution obtained by Chorin's method with $0 < \epsilon \ll 1$ is stable as a solution of the incompressible system (1.1)–(1.2). Furthermore, the instability result was proved: if $\sigma(-L) \cap \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda > 0\} \neq \emptyset$, then $\sigma(-L_\epsilon) \cap \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda > 0\} \neq \emptyset$ for $0 < \epsilon \ll 1$. This shows that unstable stationary solutions of (1.1)–(1.2) cannot be obtained by Chorin's method with $0 < \epsilon \ll 1$.

As for the question (b), it was shown in [15] that if $\rho(-L) \supset \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \geq -b_0\}$ for some positive constant b_0, then there exist positive constants δ_0 and b_1 such that $\rho(-L_\epsilon) \supset \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \geq -b_1\}$ for $0 < \epsilon \ll 1$, provided that

$$\inf_{\boldsymbol{w} \in H^1_0(\Omega)^3, \boldsymbol{w} \neq \boldsymbol{0}} \frac{\operatorname{Re}(\boldsymbol{w} \cdot \nabla \boldsymbol{v}_s, \boldsymbol{w})_{L^2}}{\|\nabla \boldsymbol{w}\|^2_{L^2}} \geq -\delta_0. \tag{1.9}$$

This gives a sufficient condition for u_s to be computed by Chorin's method with $0 < \epsilon \ll 1$. The corresponding result for the Oberbeck-Boussinesq system (1.6)–(1.8) is stated exactly in the same form; and the result is applicable to stable bifurcating cellular convective patterns of the system (1.6)–(1.8), such as roll pattern, hexagonal pattern and etc., since they bifurcate from

$\boldsymbol{v} = \boldsymbol{0}$, $\theta = 0$, and hence, the condition (1.9) is satisfied near the bifurcation point. (Observe that the condition (1.9) is independent of θ_s and θ.) However, the condition (1.9) is stringent since most of its applications might be limited to stationary flows whose velocity fields are sufficiently small. We note that the condition (1.9) seems to be the standard energy stability criterion; but this is not the case; the standard energy stability criterion for the Oberbeck-Boussinesq system (1.6)–(1.8) should be formulated in order for the functional

$$\|\nabla \boldsymbol{w}\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 - \mathrm{Re}\,\{2\mathrm{Ra}(\boldsymbol{w}\cdot\boldsymbol{e}_3, \theta)_{L^2} - \mathrm{Pr}^{-1}(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s, \boldsymbol{w})_{L^2} - (\boldsymbol{w}\cdot\nabla\theta_s, \theta)_{L^2}\}$$

to be positive definite. Here $u_s = {}^\top(p_s, \boldsymbol{v}_s, \theta_s)$ is a stationary solution of (1.6)–(1.8). In fact, the condition (1.9) arises from a *compressible* aspect of the system (1.3)–(1.4), more precisely, eigenvalues related to diffusion waves whose imaginary part of order $O(\epsilon^{-1})$. This suggests us one direction to investigate the nature of the condition (1.9).

In this paper we give an improvement of the condition (1.9) in such a way that we could treat stationary flows whose velocity fields are not necessarily small and also could see some of compressible aspects of the system (1.3)–(1.4) with (1.5). We show that the condition (1.9) can be replaced by

$$\inf_{\boldsymbol{w}\in H_0^1(\Omega)^3, \boldsymbol{w}\neq\boldsymbol{0}} \frac{\mathrm{Re}\,((\mathbb{Q}\boldsymbol{w})\cdot\nabla\boldsymbol{v}_s, \mathbb{Q}\boldsymbol{w})_{L^2}}{\|\nabla\mathbb{Q}\boldsymbol{w}\|_{L^2}^2} \geq -\delta_0. \tag{1.10}$$

Here $\mathbb{Q} = I - \mathbb{P}$ is the orthogonal projection from $L^2(\Omega)^3$ to the space $G^2(\Omega) = \{\nabla p; p \in H_*^1(\Omega)\}$ which is the orthogonal complement of $L^2_\sigma(\Omega)$. The same result also holds for the case of the Oberbeck-Boussinesq system (1.6)–(1.8). For simplicity, in this paper, we consider the above improvement only in the context of the Navier-Stokes system (1.1)–(1.2).

As an application, we consider the Taylor problem, namely, a flow between two concentric infinite cylinders, whose inner cylinder rotates with a uniform speed and outer one is at rest. If the rotation speed is sufficiently small, then a laminar flow, called the Couette flow, is stable. When the rotation speed increases, beyond a certain value of the rotation speed, the Couette flow is getting unstable, and a vortex pattern is observed. The vortex pattern is periodic in the direction of the axis of the cylinders and it is called the Taylor vortex. This phenomenon has been studied mathematically as a bifurcation problem for the incompressible system (1.1)–(1.2) (see [4, 12, 13, 16, 21]). The velocity field near the bifurcation point of the Taylor vortex is not necessarily small, but one can show that the condition (1.10) is satisfied with $\boldsymbol{v}_s$ being the Taylor vortex under *axi-symmetric perturbations* (i.e., $\boldsymbol{w}$ in (1.10) are *axi-symmetric*). This means that the Taylor vortex can be computed by Chorin's method. In section 5, we discuss not only the stability of the Taylor vortex but also the stability and instability of the Couette flow by applying the main result of this paper and the instability result of [15].

We briefly explain the idea of the proof of the main result. As was shown in [15], the spectrum of $-L_\epsilon$ near the imaginary axis is divided into two parts. One part locates in a region with imaginary part of $O(1)$ as $\epsilon \to 0$, and this part is obtained by small perturbations of eigenvalues of $-L$ when $0 < \epsilon \ll 1$. The other part locates in a region with imaginary part of $O(\epsilon^{-1})$, and this part consists of eigenvalues arising from a *compressible* aspect of $-L_\epsilon$. In [15] the condition (1.9) was used to show that the eigenvalues in this region have negative real parts. The idea to obtain the condition (1.10) instead of (1.9) is as follows. By using the Helmholtz decomposition, $\boldsymbol{w}$ in (1.9) is written as a sum of the incompressible part $\mathbb{P}\boldsymbol{w}$ and the potential flow part $\nabla\phi = \mathbb{Q}\boldsymbol{w}$. Since $-L$ is sectorial, we see that $\|(\lambda + L)^{-1}\boldsymbol{F}\|_{L^2} = O(|\mathrm{Im}\,\lambda|^{-1})\|\boldsymbol{F}\|_{L^2}$ as $|\mathrm{Im}\,\lambda| \to \infty$. So, if $|\mathrm{Im}\,\lambda| = O(\epsilon^{-1})$, then we expect that the incompressible part $\mathbb{P}\boldsymbol{w}$ becomes small in L^2 as $\epsilon \to 0$ since $|\mathrm{Im}\,\lambda|^{-1} = O(\epsilon) \to 0$, which implies that the parts including $\mathbb{P}\boldsymbol{w}$ of the numerator in (1.9) is getting smaller when $\epsilon \to 0$. In fact, we can show that if $\boldsymbol{w}$ is the velocity component of $(\lambda + L_\epsilon)^{-1}F$ with λ near the imaginary axis and $|\mathrm{Im}\,\lambda| = O(\epsilon^{-1})$, then

$$\|\mathbb{P}\boldsymbol{w}\|_{L^2(\Omega)} \leq C\{\epsilon\|F\|_{H^1(\Omega)\times L^2(\Omega)} + \epsilon^{\frac{1}{4}}\|\mathbb{Q}\boldsymbol{w}\|_{L^2(\Omega)}\}. \tag{1.11}$$

Using this estimate, one can show that the condition (1.9) can be replaced by the condition (1.10). To obtain (1.11), we establish the estimate

$$\begin{aligned} &|\lambda|\|\boldsymbol{v}\|_{L^2(\Omega)} + \|\boldsymbol{v}\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} \\ &\quad \leq C\{\|\boldsymbol{g}\|_{L^2(\Omega)} + |\lambda|^{\frac{3}{4}}\|\boldsymbol{\psi}\|_{L^2(\partial\Omega)} + \|\boldsymbol{\psi}\|_{H^{\frac{3}{2}}(\partial\Omega)}\} \end{aligned} \tag{1.12}$$

for a solution ${}^\top(p, \boldsymbol{v})$ of the Stokes system with nonhomogeneous boundary data:

$$\left\{ \begin{array}{rcl} \mathrm{div}\,\boldsymbol{v} & = & 0, \\ \lambda\boldsymbol{v} - \Delta\boldsymbol{v} + \nabla p & = & \boldsymbol{g}, \\ \boldsymbol{v}|_{\partial\Omega} & = & \boldsymbol{\psi}, \end{array} \right.$$

where $\lambda \in \{\lambda \in \mathbb{C};\ |\mathrm{arg}\lambda| \leq \pi - a\}$ for some $0 < a < \frac{\pi}{2}$, $\boldsymbol{g} \in L^2(\Omega)$ and $\boldsymbol{\psi} \in H^{\frac{3}{2}}(\partial\Omega)$ with $\boldsymbol{\psi}\cdot\boldsymbol{n}|_{\partial\Omega} = 0$. See Lemma 3.7 below.

This paper is organized as follows. In section 2 we introduce notations used in this paper and state the main result of this paper. In section 3 we prove the main result, namely, we show that the condition (1.9) can be replaced by the condition (1.10). Section 4 is devoted to the proof of the estimate (1.12). In section 5 we discuss an application of the main result to the Taylor problem.

2 Main Result

We first introduce notation used in this paper. For $1 \le p \le \infty$ we denote by $L^p(D)$ the usual Lebesgue space over D and its norm is denoted by $\|\cdot\|_{L^p(D)}$. The mth order L^2 Sobolev space over D is denoted by $H^m(D)$, and its norm is denoted by $\|\cdot\|_{H^m(D)}$. When $D = \Omega$, we simply denote these norms by $\|\cdot\|_p$, $\|\cdot\|_{H^m}$. The inner product of $L^2(D)$ is denoted by $(\cdot,\cdot)_{L^2(D)}$, i.e.,

$$(f,g)_{L^2(D)} = \int_D f(x)\overline{g(x)}dx.$$

Here $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. When $D = \Omega$ we simply denote $(\cdot,\cdot)_{L^2(D)}$ by $(\cdot,\cdot)$.

We set

$$\begin{aligned} H_0^1(D) &= \text{the}\, H^1(D)\text{-closure of } C_0^\infty(D), \\ H^{-1}(D) &= \text{the dual space of } H_0^1(D), \\ \dot{H}^1(D) &= \{f \in L^2_{loc}(D) : \|\nabla f\|_{L^2(D)} < \infty\}, \\ \dot{H}^{-1}(D) &= \text{the dual space of } \dot{H}^1(D). \end{aligned}$$

We define $L^2_*(\Omega)$ and $H^k_*(\Omega)$ by

$$L^2_*(\Omega) = \{f \in L^2(\Omega);\ \int_\Omega f(x)dx = 0\},$$

$$H^k_*(\Omega) = H^k(\Omega) \cap L^2_*(\Omega) \ \ (k \ge 1).$$

We set

$$L^2_\sigma(\Omega) = \{\boldsymbol{v} \in L^2(\Omega)^3;\ \operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega,\ \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega} = 0\}.$$

Here and in what follows, $\boldsymbol{n}$ denotes the unit outward normal to $\partial\Omega$. It is known that $(L^2(\Omega))^3 = L^2_\sigma(\Omega) \oplus G^2(\Omega)$, where $G^2(\Omega) = \{\nabla p;\ p \in H^1_*(\Omega)\}$ is orthogonal complement of $L^2_\sigma(\Omega)$.

The orthogonal projection $\mathbb{P}$ from $L^2(\Omega)^3$ onto $L^2_\sigma(\Omega)$ is called the Helmholtz projection. We set $\mathbb{Q} = I - \mathbb{P}$.

We denote the resolvent set of an operator A by $\rho(A)$ and the spectrum of A by $\sigma(A)$.

We state the main result of this paper. We introduce the linearized operators for the Navier-Stokes and the corresponding artificial compressible systems. Let $u_s = {}^\top(p_s, \boldsymbol{v}_s)$ be a smooth stationary solution of (1.1)–(1.2), (1.5). The equations for the perturbation is then written as

$$\operatorname{div} \boldsymbol{w} = 0, \tag{2.1}$$

$$\partial_t \boldsymbol{w} - \nu\Delta\boldsymbol{w} + \boldsymbol{v}_s \cdot \nabla\boldsymbol{w} + \boldsymbol{w} \cdot \nabla\boldsymbol{v}_s + \boldsymbol{w} \cdot \nabla\boldsymbol{w} + \nabla p = \mathbf{0}. \tag{2.2}$$

The boundary condition is the non-slip one:

$$\boldsymbol{w}|_\Omega = \mathbf{0}. \tag{2.3}$$

Applying the Helmholtz projection $\mathbb{P}$ we have

$$\frac{d\boldsymbol{w}}{dt} + L\boldsymbol{w} + \mathbb{P}(\boldsymbol{w} \cdot \nabla\boldsymbol{w}) = \mathbf{0}. \tag{2.4}$$

Here L is the linearized operator around $\boldsymbol{v}_s$ on $L^2_\sigma(\Omega)$ defined by

$$D(L) = (H^2(\Omega) \cap H^1_0(\Omega))^3 \cap L^2_\sigma(\Omega),$$
$$L\boldsymbol{w} = -\nu\mathbb{P}\Delta\boldsymbol{w} + \mathbb{P}(\boldsymbol{v}_s \cdot \nabla\boldsymbol{w} + \boldsymbol{w} \cdot \nabla\boldsymbol{v}_s), \quad (\boldsymbol{w} \in D(L)).$$

The corresponding artificial system is written as

$$\frac{du}{dt} + L_\epsilon u + N(u, u) = 0. \tag{2.5}$$

Here $u = {}^\top(p, \boldsymbol{w})$; L_ϵ is the linearized operator around u_s on $H^1_*(\Omega) \times L^2(\Omega)^3$ defined by

$$D(L_\epsilon) = H^1_*(\Omega) \times (H^2(\Omega) \cap H^1_0(\Omega))^3,$$
$$L_\epsilon = \begin{pmatrix} 0 & \frac{1}{\epsilon^2}\operatorname{div} \\ \nabla & -\nu\Delta + \boldsymbol{v}_s \cdot \nabla + {}^\top(\nabla\boldsymbol{v}_s) \end{pmatrix};$$

and $N(u, u)$ is the nonlinear operator given by $N(u, u) = {}^\top(0, \boldsymbol{w} \cdot \nabla\boldsymbol{w})$ $(u = {}^\top(p, \boldsymbol{w}))$.

The main result of this paper is stated as follows.

Theorem 2.1. *Suppose that $\rho(-L) \supset \{\lambda \in \mathbb{C};\, \mathrm{Re}\,\lambda \geq -b_0\}$ for some positive constant b_0. Then there exist positive constants ϵ_0, δ_0 and b_1 such that if*

$$\inf_{\boldsymbol{w}\in H_0^1(\Omega)^3,\, \boldsymbol{w}\neq \boldsymbol{0}} \frac{\mathrm{Re}\,((\mathbb{Q}\boldsymbol{w})\cdot\nabla \boldsymbol{v}_s,\, \mathbb{Q}\boldsymbol{w})}{\|\nabla \mathbb{Q}\boldsymbol{w}\|_2^2} \geq -\delta_0, \tag{2.6}$$

then $\rho(-L_\epsilon) \supset \{\lambda \in \mathbb{C};\, \mathrm{Re}\,\lambda \geq -b_1\}$ for all $0 < \epsilon \leq \epsilon_0$.

Remark 2.2. *If we consider flows in an unbounded domain Ω which is translation invariance in the unbounded directions of Ω, such as an infinite layer and a cylindrical domain, then we can consider a stationary flow $\boldsymbol{v}_s$ which is periodic in the unbounded directions of Ω. For definiteness, let us consider a cylindrical domain whose axis is the x_1-axis. The problem is then formulated in the basic period domain Ω_{per} under the periodic boundary condition in x_1. In this case, if the stationary solution $\boldsymbol{v}_s$ which is periodic in x_1 satisfies $\partial_{x_1}\boldsymbol{v}_s \neq 0$, then 0 is always an eigenvalue of $-L$ due to the translation invariance in x_1. As in [15], one can prove the following result instead of Theorem 2.1.*

Suppose that $\rho(-L) \supset \{\lambda \in \mathbb{C};\, \mathrm{Re}\,\lambda \geq -b_0\} \setminus \{0\}$ for some positive constant b_0 and 0 is a simple eigenvalue with $\mathrm{Ker}\,(-L) = \mathrm{span}\,\{\partial_{x_1}\boldsymbol{v}_s\}$. Then there exist positive constants ϵ_0, δ_0 and b_1 such that if

$$\inf_{\boldsymbol{w}\in H_{0,per}^1(\Omega_{per})^3,\, \boldsymbol{w}\neq \boldsymbol{0}} \frac{\mathrm{Re}\,((\mathbb{Q}\boldsymbol{w})\cdot\nabla \boldsymbol{v}_s,\, \mathbb{Q}\boldsymbol{w})}{\|\nabla \mathbb{Q}\boldsymbol{w}\|_2^2} \geq -\delta_0,$$

then $\rho(-L_\epsilon) \supset \{\lambda \in \mathbb{C};\, \mathrm{Re}\,\lambda \geq -b_1\} \setminus \{0\}$ for all $0 < \epsilon \leq \epsilon_0$ and 0 is a simple eigenvalue with $\mathrm{Ker}\,(-L_\epsilon) = \mathrm{span}\,\{\partial_{x_1} u_s\}$. Here $H_{0,per}^1(\Omega_{per})$ denotes the set of all H^1 functions on Ω_{per} which vanish on $\partial\Omega$ and satisfy the periodic boundary condition in x_1; and $u_s = {}^\top(p_s, \boldsymbol{v}_s)$ with p_s being the corresponding pressure.

Remark 2.3. *On can easily see from the proofs of Theorem 2.1 and [15, Theorem 3.3] that the same result also holds for the case of the Oberbeck-Boussinesq system* (1.6)–(1.8).

3 Proof of Theorem 2.1

In this section we give a proof of Theorem 2.1. We consider the resolvent problem for $-L_\epsilon$:

$$\lambda u + L_\epsilon u = F, \quad u = {}^\top(p, \boldsymbol{w}) \in D(L_\epsilon), \tag{3.1}$$

where $F = {}^\top(f, \boldsymbol{g}) \in H^1_*(\Omega) \times L^2(\Omega)^3$ is given.

For simplicity we set $\nu = 1$. The problem (3.1) is then written as

$$\epsilon^2 \lambda p + \operatorname{div} \boldsymbol{w} = \epsilon^2 f, \tag{3.2}$$

$$\lambda \boldsymbol{w} - \Delta \boldsymbol{w} + \boldsymbol{v}_s \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v}_s + \nabla p = \boldsymbol{g}, \tag{3.3}$$

$$\boldsymbol{w}|_{\partial\Omega} = \mathbf{0}. \tag{3.4}$$

Throughout this section we assume that

$$\rho(-L) \supset \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -b_0\} \tag{3.5}$$

for some positive constant b_0.

The following two propositions imply that the part of the spectrum $\sigma(-L_\epsilon)$ near the imaginary axis can lie only in a region $\operatorname{Im} \lambda = O(\epsilon^{-1})$ under the assumption (3.5).

Proposition 3.1. *There exist positive constants a and b such that $\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -a\epsilon^2 |\operatorname{Im} \lambda|^2 + b\} \subset \rho(-L_\epsilon)$ for all $0 < \epsilon \leq 1$.*

Proposition 3.1 can be proved by the standard Matsumura-Nishida energy method as in the proof of [15, Proposition 6.1]. We omit the proof.

Proposition 3.2. *There exist positive numbers ϵ_1 and a_1 such that*

$$\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -b_0, |\lambda| \leq a_1 \epsilon^{-1}\} \subset \rho(-L_\epsilon)$$

for all $0 < \epsilon \leq \epsilon_1$.

Proof. Similarly to the proof of [15, Proposition 4.3 (ii)], by using the solvability result for the nonhomogeneous Stokes problem (see, e.g., [9, 11]), one can show that $0 \in \rho(-L_\epsilon)$, i.e., $R(L_\epsilon)$(= the range of L_ϵ)= $H^1_*(\Omega) \times L^2(\Omega)^3$. Proposition 3.2 can then be proved by the same perturbation argument as in the proof of [15, Proposition 6.3]. This completes the proof. □

Remark 3.3. *Under the situation of Remark 2.2, Proposition 3.2 is stated as*

$$\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -b_0, |\lambda| \leq a_1 \epsilon^{-1}\} \setminus \{0\} \subset \rho(-L_\epsilon) \tag{3.6}$$

and 0 is a simple eigenvalue of L_ϵ. In fact, by [15, Proposition 4.3], we see that 0 is a simple eigenvalue of L_ϵ. Furthermore, by [15, Proposition 6.2], one can see that if $\lambda \in \Sigma \setminus \{0\}$, then the argument of the proof of [15, *Proposition 6.3] works well without restricting to the subspace* $Y_{1,\epsilon} = R(L_\epsilon)$ *to conclude* (3.6)*, with the replacement of the estimate in the end of the proof of [15, Proposition 6.3] by*

$$\|(\lambda + L_\epsilon)^{-1}F\|_{H^1\times L^2\times L^2} \leq \frac{C}{|\lambda|}\{\epsilon\|f\|_{H^1} + \|\boldsymbol{F}\|_2\}.$$

The argument in the proof of [15, Proposition 6.3] to obtain the Neumann series on $Y_{1,\epsilon} = R(L_\epsilon)$ *does not work since* $Y_{1,\epsilon} = R(L_\epsilon)$ *is not invariant under* $\mathscr{L}_{\epsilon,\lambda}^{-1}J$*; but the Neumann series can be obtained just by removing the restriction to* $Y_{1,\epsilon} = R(L_\epsilon)$*; and, with this modification, the proof of [15, Proposition 6.3] can be completed.*

Theorem 2.1 follows from Propositions 3.1 and 3.2 without the condition (2.6) if $\sqrt{b/a} < a_1$. In the case $\sqrt{b/a} \geq a_1$, there exists some range of λ near the imaginary axis with $\mathrm{Im}\,\lambda = O(\epsilon^{-1})$ which should be proved to be in $\rho(-L_\epsilon)$ for $0 < \epsilon \ll 1$.

To complete the proof of Theorem 2.1, it suffices to deduce the a priori estimate for solutions of (3.1) uniformly for $\lambda = \mu + i\frac{\eta}{\epsilon}$ with $-\mu_0 \leq \mu \leq \mu_1$ and $a_1/2 \leq |\eta| \leq 2\sqrt{b/a}$, where μ_0 and μ_1 are some positive constants. In fact, if we obtain such a uniform a priori estimate, then it follows that $\{\lambda = \mu + i\frac{\eta}{\epsilon}; -\mu_0 \leq \mu \leq \mu_1,\, a_1/2 \leq |\eta| \leq 2\sqrt{b/a}\} \subset \rho(-L_\epsilon)$ by a standard continuation argument since $\lambda = \pm i\frac{a_1}{\epsilon} \in \rho(-L_\epsilon)$ for $0 < \epsilon \leq \epsilon_1$ by Proposition 3.2. We will establish an appropriate a priori estimate under the condition (2.6).

We begin with estimating the H^1-norm of the velocity component $\boldsymbol{w}$ of a solution $u = {}^\top(p, \boldsymbol{w})$ of (3.1) for λ near the imaginary axis with $\mathrm{Im}\,\lambda = O(\epsilon^{-1})$.

Proposition 3.4. *Let* $\lambda = \mu + i\frac{\eta}{\epsilon}$ *with* $\mu, \eta \in \mathbb{R}$*. Suppose that* $u = {}^\top(p, \boldsymbol{w}) \in D(L_\epsilon)$ *is a solution of* (3.1)*. For given positive numbers* μ_1 *and* η_* *there exist positive constants* δ_1 *and* $C' = C'(\|\boldsymbol{v}_s\|_{C^1}, \beta, \Omega)$ *such that if*

$$\inf\left\{\tfrac{\mathrm{Re}\,(\nabla\varphi\cdot\nabla\boldsymbol{v}_s, \nabla\varphi)}{\|\Delta\varphi\|_2^2}; \varphi \in H^2_*(\Omega),\, \varphi \neq 0,\, \tfrac{\partial\varphi}{\partial\boldsymbol{n}}|_{\partial\Omega} = 0\right\} \geq -\delta_1$$

and

$$-\frac{\beta^2}{128} \leq \mu \leq \mu_1,\ \ \eta_* \leq \eta \leq C'\epsilon^{-1},$$

then

$$(\eta^3 + \beta^2\eta)\|\boldsymbol{w}\|_2^2 + \eta\|\nabla\boldsymbol{w}\|_2^2 \leq C\left\{(\eta + \frac{\epsilon^2}{\eta})\|\boldsymbol{g}\|_2^2 + \frac{\epsilon^2}{\eta}\|f\|_{H^1}^2\right\}$$

for all $0 < \epsilon \le C' \min\{1, \eta_*, \sqrt{\frac{\eta_*}{\mu_*}}, \frac{\eta_*}{\mu_*}, \eta_* \mu_*^{-\frac{2}{3}}, \sqrt{\frac{1}{\mu_*}}\}$ *with* $\mu_* = \max\{\frac{\beta^2}{128}, \mu_1\}$.

To prove Proposition 3.4, we prepare several estimates for a solution of (3.1).

The following estimate can be proved in a similar manner to the proof of [15, Proposition 6.5]. For the convenience of the readers we give its proof.

Proposition 3.5. *Let* μ_1 *and* η_* *be given positive numbers. Let* $u = {}^\top(p, \boldsymbol{w}) \in D(L_\epsilon)$ *be a solution of* (3.1) *with* $\lambda = \mu + i\frac{\eta}{\epsilon}$, $\mu, \eta \in \mathbb{R}$. *There exists a positive constant* $C' = C'(\|\boldsymbol{v}_s\|_{C^1}, \beta, \Omega)$ *such that if*

$$\epsilon \le C' \min\{1, \eta_*, \frac{\eta_*}{\mu_*}, \frac{1}{\sqrt{\mu_1}}\}, \ -\frac{\beta^2}{128} \le \mu \le \mu_1, \ \eta_* \le \eta \le \frac{1}{4\epsilon}$$

with $\mu_* = \max\{\frac{\beta^2}{128}, \mu_1\}$, *then*

$$(\eta^3 + 2\beta^2\eta)\|\boldsymbol{w}\|_2^2 + \eta\|\nabla \boldsymbol{w}\|_2^2 \le -64\eta \mathrm{Re}\,(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s, \boldsymbol{w}) + C(\epsilon^2\eta^2 + \epsilon)\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2.$$

Here $\boldsymbol{G}_\lambda = \lambda\boldsymbol{g} - \nabla f$; *and* C *is a positive constant depending only on* $\|\boldsymbol{v}_s\|_{C^1}$ *and* Ω.

Proof. Let $u = {}^\top(p, \boldsymbol{v}) \in D(L_\epsilon)$ be a solution of (3.1). Then, by (3.2), we have

$$p = -\frac{1}{\epsilon^2\lambda}\mathrm{div}\,\boldsymbol{w} + \frac{1}{\lambda}f.$$

Substituting this into (3.3), we obtain

$$\epsilon^2\lambda^2\boldsymbol{w} - \epsilon^2\lambda\Delta\boldsymbol{w} - \nabla\mathrm{div}\,\boldsymbol{w} + \epsilon^2\lambda(\boldsymbol{v}_s\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{v}_s) = \epsilon^2\boldsymbol{G}_\lambda. \tag{3.7}$$

We take the inner product of (3.7) with $\boldsymbol{w}$. It follows that

$$\epsilon^2\lambda^2\|\boldsymbol{w}\|_2^2 + \epsilon^2\lambda\|\nabla\boldsymbol{w}\|_2^2 + \|\mathrm{div}\,\boldsymbol{w}\|_2^2 = -\epsilon^2\lambda(\boldsymbol{v}_s\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{v}_s, \boldsymbol{w}) + \epsilon^2(\boldsymbol{G}_\lambda, \boldsymbol{w}). \tag{3.8}$$

Since $\lambda^2 = \left(\mu^2 - \frac{\eta^2}{\epsilon^2}\right) + 2i\frac{\mu\eta}{\epsilon}$, the real part of (3.8) yields

$$\begin{aligned}(\epsilon^2\mu^2 - \eta^2)\|\boldsymbol{w}\|_2^2 + \epsilon^2\mu\|\nabla\boldsymbol{w}\|_2^2 + \|\mathrm{div}\,\boldsymbol{w}\|_2^2 \\ = -\epsilon^2\mu\mathrm{Re}\,(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s, \boldsymbol{w}) + \epsilon\eta\mathrm{Im}\,(\boldsymbol{v}_s\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{v}_s, \boldsymbol{w}) + \epsilon^2\mathrm{Re}\,(\boldsymbol{G}_\lambda, \boldsymbol{w}).\end{aligned} \tag{3.9}$$

Therefore,

$$\begin{aligned}(\eta^2-\epsilon^2\mu^2)\|\boldsymbol{w}\|_2^2 \;\le\;& (\epsilon^2\mu+3+\epsilon\eta)\|\nabla\boldsymbol{w}\|_2^2\\ &+(\epsilon^2\mu\|\nabla\boldsymbol{v}_s\|_\infty+\epsilon\eta(\|\boldsymbol{v}_s\|_\infty^2+\|\nabla\boldsymbol{v}_s\|_\infty))\|\boldsymbol{w}\|_2^2\\ &+\epsilon^2\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2.\end{aligned} \tag{3.10}$$

The imaginary part of (3.8) yields

$$\begin{aligned}&2\mu\eta\|\boldsymbol{w}\|_2^2+\eta\|\nabla\boldsymbol{w}\|_2^2\\ &= -\epsilon\mu\mathrm{Im}\,(\boldsymbol{v}_s\cdot\nabla\boldsymbol{w}+\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s,\boldsymbol{w})-\eta\mathrm{Re}\,(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s,\boldsymbol{w})+\epsilon\mathrm{Im}\,(\boldsymbol{G}_\lambda,\boldsymbol{w})\\ &\le -\eta\mathrm{Re}\,(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s,\boldsymbol{w})+\left(\epsilon^2\frac{\mu^2}{2\eta}\|\boldsymbol{v}_s\|_\infty^2+\epsilon|\mu|\|\nabla\boldsymbol{v}_s\|_\infty\right)\|\boldsymbol{w}\|_2^2\\ &\quad+\frac{1}{2}\eta\|\nabla\boldsymbol{w}\|_2^2+\epsilon\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2,\end{aligned}$$

and hence,

$$\begin{aligned}&2\mu\eta\|\boldsymbol{w}\|_2^2+\frac{1}{2}\eta\|\nabla\boldsymbol{w}\|_2^2\\ &\le -\eta\mathrm{Re}\,(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s,\boldsymbol{w})+\left(\epsilon^2\frac{\mu^2}{\eta}\|\boldsymbol{v}_s\|_\infty^2+\epsilon|\mu|\|\nabla\boldsymbol{v}_s\|_\infty\right)\|\boldsymbol{w}\|_2^2+\epsilon\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2.\end{aligned} \tag{3.11}$$

By (3.10) and (3.11), we have

$$\begin{aligned}&\left(\frac{\eta^3}{12}-\epsilon^2\frac{\eta\mu^2}{12}+2\mu\eta\right)\|\boldsymbol{w}\|_2^2+\frac{\eta}{4}(1-\epsilon^2\mu-\epsilon\eta)\|\nabla\boldsymbol{w}\|_2^2\\ &\le -\eta\mathrm{Re}\,(\boldsymbol{w}\cdot\boldsymbol{v}_s,\boldsymbol{w})\\ &\quad+C\left\{(\epsilon^2\mu\eta+\epsilon\eta^2+\epsilon|\mu|)\|\nabla\boldsymbol{v}_s\|_\infty+(\epsilon\eta^2+\epsilon^2\frac{\mu^2}{\eta})\|\boldsymbol{v}_s\|_\infty^2\right\}\|\boldsymbol{w}\|_2^2\\ &\quad+C(\epsilon^2\eta+\epsilon)\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2,\end{aligned}$$

and consequently, if $\epsilon\le\frac{1}{2\sqrt{\mu_*}}$ and $\eta\le\frac{1}{4\epsilon}$, then

$$\begin{aligned}&\left(\frac{\eta^3}{12}-\epsilon^2\frac{\eta\mu^2}{12}+2\mu\eta+\frac{1}{16}\beta^2\eta\right)\|\boldsymbol{w}\|_2^2+\frac{1}{16}\eta\|\nabla\boldsymbol{w}\|_2^2\\ &\le -\eta\mathrm{Re}\,(\boldsymbol{w}\cdot\boldsymbol{v}_s,\boldsymbol{w})\\ &\quad+C\left\{(\epsilon^2\mu\eta+\epsilon\eta^2+\epsilon|\mu|)\|\nabla\boldsymbol{v}_s\|_\infty+(\epsilon\eta^2+\epsilon^2\frac{\mu^2}{\eta})\|\boldsymbol{v}_s\|_\infty^2\right\}\|\boldsymbol{w}\|_2^2\\ &\quad+C(\epsilon^2\eta+\epsilon)\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2.\end{aligned}$$

Therefore, there exists a positive constant $C' = C'(\|\boldsymbol{v}_s\|_{C^1}, \beta)$ such that if

$$\epsilon \le C' \min\{1, \eta_*, \frac{\eta_*}{\mu_1}, \frac{1}{\sqrt{\mu_1}}\}, \; -\frac{\beta^2}{128} \le \mu \le \mu_*, \; \eta_* \le \eta \le \frac{1}{4\epsilon},$$

then

$$\frac{1}{32}(\eta^3+\beta^2\eta)\|\boldsymbol{w}\|_2^2+\frac{1}{16}\eta\|\nabla\boldsymbol{w}\|_2^2 \le -\eta\mathrm{Re}\,(\boldsymbol{w}\cdot\nabla\boldsymbol{w}_s, \boldsymbol{w})+C(\epsilon^2\eta^2+\epsilon)\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2.$$

This completes the proof. □

We next estimate incompressible part of $\boldsymbol{w}$.

Proposition 3.6. *Let μ_0, μ_1 and η_* be given positive numbers. Let $u = {}^\top(p, \boldsymbol{w})$ be a solution of (3.1) with $\lambda = \mu + i\frac{\eta}{\epsilon}$, $-\mu_0 \le \mu \le \mu_1$, $\eta \ge \eta_*$. If $\boldsymbol{w} = \boldsymbol{v} + \nabla\varphi$ is the Helmholtz decomposition of $\boldsymbol{w}$, then*

$$\begin{aligned}\|\boldsymbol{v}\|_2^2 \;\; &\le \;\; C\left\{\frac{\epsilon^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}\|\nabla\varphi\|_{H^1}^2 + \frac{\epsilon^2}{\eta^2}\|\nabla\varphi\|_{H^2}^2 + \frac{\epsilon^2}{\eta^2}\|\boldsymbol{g}\|_2^2\right. \\ &\qquad \left. + \frac{\epsilon^2}{\eta^2}\|\boldsymbol{v}_s\|_\infty^2\|\nabla\boldsymbol{w}\|_2^2 + \frac{\epsilon^2}{\eta^2}\|\nabla\boldsymbol{v}_s\|_\infty^2\|\boldsymbol{w}\|_2^2\right\},\end{aligned}$$

$$\|\boldsymbol{v}\|_{H^2}^2 \le C\left\{\frac{\eta^{\frac{3}{2}}}{\epsilon^{\frac{3}{2}}}\|\nabla\varphi\|_{H^1}^2 + \|\nabla\varphi\|_{H^2}^2 + \|\boldsymbol{g}\|_2^2 + \|\boldsymbol{v}_s\|_\infty^2\|\nabla\boldsymbol{w}\|_2^2 + \|\nabla\boldsymbol{v}_s\|_\infty^2\|\boldsymbol{w}\|_2^2\right\}.$$

To prove Proposition 3.6 we make use of the following estimate for the Stokes system with nonhomogeneous boundary data.

Lemma 3.7. *Suppose that ${}^\top(p, \boldsymbol{v}) \in H_*^1(\Omega) \times H^2(\Omega)$ is a solution of*

$$\left\{\begin{array}{rcl}\mathrm{div}\,\boldsymbol{v} & = & 0, \\ \lambda\boldsymbol{v} - \Delta\boldsymbol{v} + \nabla p & = & \boldsymbol{g}, \\ \boldsymbol{v}|_{\partial\Omega} & = & \boldsymbol{\psi},\end{array}\right. \tag{3.12}$$

with $\lambda \in \{\lambda \in \mathbb{C}; |\arg\lambda| \le \pi - \omega\}$ for some $0 < \omega < \frac{\pi}{2}$, $\boldsymbol{g} \in L^2(\Omega)$ and $\boldsymbol{\psi} \in H^{\frac{3}{2}}(\partial\Omega)$ satisfying $\boldsymbol{\psi}\cdot\boldsymbol{n}|_{\partial\Omega} = 0$. Then there exists a positive constant $C = C(\omega, \Omega)$ such that

$$|\lambda|\|\boldsymbol{v}\|_2 + \|\boldsymbol{v}\|_{H^2} + \|p\|_{H^1} \le C\{\|\boldsymbol{g}\|_2 + |\lambda|^{\frac{3}{4}}\|\boldsymbol{\psi}\|_{L^2(\partial\Omega)} + \|\boldsymbol{\psi}\|_{H^{\frac{3}{2}}(\partial\Omega)}\}.$$

Lemma 3.7 will be proved in section 4.

Proof of Proposition 3.6. Let $u = {}^\top(p, \boldsymbol{w}) \in D(L_\epsilon)$ be a solution of (3.2)–(3.4) and let $\boldsymbol{w} = \boldsymbol{v} + \nabla\varphi$ be the Helmholtz decomposition of $\boldsymbol{w}$. Then, $\operatorname{div}\boldsymbol{v} = 0$, $\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega} = 0$, $\frac{\partial\varphi}{\partial\boldsymbol{n}}\big|_{\partial\Omega} = \boldsymbol{w}\cdot\boldsymbol{n}|_{\partial\Omega}$ and $\int_\Omega \varphi dx = 0$. Since $\boldsymbol{w}|_{\partial\Omega} = \boldsymbol{0}$, we see that

$$\frac{\partial\varphi}{\partial\boldsymbol{n}}\bigg|_{\partial\Omega} = 0$$

and

$$\begin{cases} \operatorname{div}\boldsymbol{v} &= 0, \\ \lambda\boldsymbol{v} - \Delta\boldsymbol{v} + \nabla q &= \boldsymbol{g} - (\boldsymbol{v}_s\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{v}_s), \\ \boldsymbol{v}|_{\partial\Omega} &= -\nabla\varphi|_{\partial\Omega}\,. \end{cases}$$

Here

$$q = \lambda\varphi - \Delta\varphi + p.$$

Note that

$$\int_\Omega q\,dx = \int_\Omega (\lambda\varphi - \Delta\varphi + p)\,dx = -\int_{\partial\Omega}\frac{\partial\varphi}{\partial\boldsymbol{n}}\,d\sigma = 0.$$

By Lemma 3.7 with $\boldsymbol{\psi}$ replaced by $-\nabla\varphi|_{\partial\Omega}$, we have

$$\begin{aligned} &|\lambda|\|\boldsymbol{v}\|_2 + \|\boldsymbol{v}\|_{H^2} + \|q\|_{H^1} \\ &\quad\le C\{|\lambda|^{\frac{3}{4}}\|\boldsymbol{\psi}\|_{L^2(\partial\Omega)} + \|\boldsymbol{\psi}\|_{H^{\frac{3}{2}}(\partial\Omega)} + \|\boldsymbol{g}\|_2 + \|\boldsymbol{v}_s\|_\infty\|\nabla\boldsymbol{w}\|_2 + \|\nabla\boldsymbol{v}_s\|_\infty\|\boldsymbol{w}\|_2\} \\ &\quad\le C\{|\lambda|^{\frac{3}{4}}\|\nabla\varphi\|_{H^1} + \|\nabla\varphi\|_{H^2} + \|\boldsymbol{g}\|_2 + \|\boldsymbol{v}_s\|_\infty\|\nabla\boldsymbol{w}\|_2 + \|\nabla\boldsymbol{v}_s\|_\infty\|\boldsymbol{w}\|_2\}. \end{aligned}$$

Therefore, if $\lambda = \mu + i\frac{\eta}{\epsilon}$, $-\mu_0 \le \mu \le \mu_1$ and $\eta \ge \eta_*$, then

$$\begin{aligned} \|\boldsymbol{v}\|_2^2 &\le C\{|\lambda|^{-\frac{1}{2}}\|\nabla\varphi\|_{H^1}^2 + |\lambda|^{-2}\|\nabla\varphi\|_{H^2}^2 + |\lambda|^{-2}\|\boldsymbol{g}\|_2^2 \\ &\qquad + |\lambda|^{-2}(\|\boldsymbol{v}_s\|_\infty^2\|\nabla\boldsymbol{w}\|_2^2 + \|\nabla\boldsymbol{v}_s\|_\infty^2\|\boldsymbol{w}\|_2^2)\} \\ &\le C\Big\{\frac{\epsilon^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}\|\nabla\varphi\|_{H^1}^2 + \frac{\epsilon^2}{\eta^2}\|\nabla\varphi\|_{H^2}^2 + \frac{\epsilon^2}{\eta^2}\|\boldsymbol{g}\|_2^2 \\ &\qquad + \frac{\epsilon^2}{\eta^2}(\|\boldsymbol{v}_s\|_\infty^2\|\nabla\boldsymbol{w}\|_2^2 + \|\nabla\boldsymbol{v}_s\|_\infty^2\|\boldsymbol{w}\|_2^2)\Big\} \end{aligned}$$

and

$$\begin{aligned} \|\boldsymbol{v}\|_{H^2}^2 + \|q\|_{H^1}^2 &\le C\Big\{\eta^{\frac{3}{2}}\epsilon^{-\frac{3}{2}}\|\nabla\varphi\|_{H^1}^2 + \|\nabla\varphi\|_{H^2}^2 + \|\boldsymbol{g}\|_2^2 \\ &\qquad + \|\boldsymbol{v}_s\|_\infty^2\|\nabla\boldsymbol{w}\|_2^2 + \|\nabla\boldsymbol{v}_s\|_\infty^2\|\boldsymbol{w}\|_2^2\Big\}. \end{aligned}$$

This completes the proof. □

The potential flow part $\nabla\varphi$ satisfies the following estimates.

Proposition 3.8. *Let $\boldsymbol{w} = \boldsymbol{v} + \nabla\varphi$ be as in Proposition 3.6. Then there exists a positive constant $C' = C'(\|\boldsymbol{v}_s\|_{C^1})$ such that if $0 < \epsilon \leq C' \min\{1, \frac{\eta_*}{\mu_*}, \eta_*\}$ with $\mu_* = \max\{\mu_0, \mu_1\}$, the following estimates*

$$\|\Delta\varphi\|_2^2 \leq C_1 \left\{ \eta^2 \|\boldsymbol{w}\|_2^2 + \epsilon\eta \|\nabla \boldsymbol{w}\|_2^2 + \frac{\epsilon^4}{\eta^2} \|\boldsymbol{G}_\lambda\|_2^2 \right\},$$

$$\frac{1}{\eta^2} \|\nabla\Delta\varphi\|_2^2 \leq C_1 \left\{ \eta^2 \|\boldsymbol{w}\|_2^2 + \epsilon\eta \|\nabla \boldsymbol{w}\|_2^2 + \epsilon^2 \|\Delta \boldsymbol{v}\|_2^2 + \frac{\epsilon^4}{\eta^2} \|\boldsymbol{G}_\lambda\|_2^2 \right\},$$

hold with $C_1 > 0$ independent of μ_0, μ_1, η_*, ϵ, and Ω.

Proof. Let $\boldsymbol{w} = \boldsymbol{v} + \nabla\varphi$ be the Helmholtz decomposition of $\boldsymbol{w}$. Since $\operatorname{div} \boldsymbol{w} = \Delta\varphi$, we see from (3.9) that

$$\begin{aligned}
\|\Delta\varphi\|_2^2 &= -(\epsilon^2\mu^2 - \eta^2)\|\boldsymbol{w}\|_2^2 - \epsilon^2\mu\|\nabla \boldsymbol{w}\|_2^2 \\
&\quad -\epsilon^2\mu \mathrm{Re}\,(\boldsymbol{w}\cdot\nabla \boldsymbol{v}_s, \boldsymbol{w}) + \epsilon\eta \mathrm{Im}\,(\boldsymbol{v}_s\cdot\nabla \boldsymbol{w} + \boldsymbol{w}\cdot\nabla \boldsymbol{v}_s, \boldsymbol{w}) \\
&\quad +\epsilon^2 \mathrm{Re}\,(\boldsymbol{G}_\lambda, \boldsymbol{w}) \\
&\leq \eta^2\|\boldsymbol{w}\|_2^2 + \epsilon^2\mu_0\|\nabla \boldsymbol{w}\|_2^2 + \epsilon^2\mu_1\|\nabla \boldsymbol{v}_s\|_\infty\|\boldsymbol{w}\|_2^2 \\
&\quad +\epsilon\eta\|\nabla \boldsymbol{w}\|_2^2 + \epsilon\eta(\|\boldsymbol{v}_s\|_\infty^2 + \|\nabla \boldsymbol{v}_s\|_\infty)\|\boldsymbol{w}\|_2^2 \\
&\quad +\epsilon^2 \mathrm{Re}\,(\boldsymbol{G}_\lambda, \boldsymbol{w}),
\end{aligned}$$

and hence,

$$\begin{aligned}
\|\Delta\varphi\|_2^2 &\leq \eta^2\|\boldsymbol{w}\|_2^2 + (\epsilon^2\mu_0 + \epsilon\eta)\|\nabla \boldsymbol{w}\|_2^2 \\
&\quad +\{\epsilon^2\mu_1\|\nabla \boldsymbol{v}_s\|_\infty + \epsilon\eta(\|\boldsymbol{v}_s\|_\infty^2 + \|\nabla \boldsymbol{v}_s\|_\infty)\}\|\boldsymbol{w}\|_2^2 \\
&\quad +\epsilon^2 \mathrm{Re}\,(\boldsymbol{G}_\lambda, \boldsymbol{w}).
\end{aligned}$$

It follows that if $\epsilon \leq \frac{\eta_*}{\mu_*}$, then

$$\begin{aligned}
\|\Delta\varphi\|_2^2 &\leq (2\eta^2 + 2\epsilon\eta(\|\boldsymbol{v}_s\|_\infty^2 + \|\nabla \boldsymbol{v}_s\|_\infty))\|\boldsymbol{w}\|_2^2 \\
&\quad +2\epsilon\eta\|\nabla \boldsymbol{w}\|_2^2 + \frac{\epsilon^4}{\eta^2}\|\boldsymbol{G}_\lambda\|_2^2.
\end{aligned} \tag{3.13}$$

Therefore there exists a positive constant $C' = C'(\|\boldsymbol{v}_s\|_{C^1})$ such that if $\epsilon \leq C' \min\left\{\frac{\eta_*}{\mu_*},\, \eta_*\right\}$, then

$$\|\Delta\varphi\|_2^2 \leq C_1 \left\{\eta^2\|\boldsymbol{w}\|_2^2 + \epsilon\eta\|\nabla\boldsymbol{w}\|_2^2 + \frac{\epsilon^4}{\eta^2}\|\boldsymbol{G}_\lambda\|_2^2\right\}.$$

We next estimate $\|\nabla\Delta\varphi\|_2^2$. We take the inner product of (3.7) with $-\nabla\Delta\varphi$ to obtain

$$\begin{aligned} &-\epsilon^2\lambda^2(\boldsymbol{w}, \nabla\Delta\varphi) + \epsilon^2\lambda(\Delta\boldsymbol{w}, \nabla\Delta\varphi) + \|\nabla\Delta\varphi\|_2^2 \\ &= \epsilon^2\lambda(\boldsymbol{v}_s \cdot \nabla\boldsymbol{w} + \boldsymbol{w} \cdot \nabla\boldsymbol{v}_s, \nabla\Delta\varphi) - \epsilon^2(\boldsymbol{G}_\lambda, \nabla\Delta\varphi). \end{aligned} \tag{3.14}$$

Since $\boldsymbol{w}|_{\partial\Omega} = 0$ and $\operatorname{div}\boldsymbol{w} = \Delta\varphi$, we have

$$-\epsilon^2\lambda^2(\boldsymbol{w}, \nabla\Delta\varphi) = \epsilon^2\lambda^2(\operatorname{div}\boldsymbol{w}, \Delta\varphi) = \epsilon^2\lambda^2\|\Delta\varphi\|_2^2,$$

$$\epsilon^2\lambda(\Delta\boldsymbol{w}, \nabla\Delta\varphi) = \epsilon^2\lambda(\Delta\boldsymbol{v}, \nabla\Delta\varphi) + \epsilon^2\lambda\|\nabla\Delta\varphi\|_2^2.$$

Taking the real part of (3.14), we thus have

$$\begin{aligned} &(\epsilon^2\mu^2 - \eta^2)\|\Delta\varphi\|_2^2 + \epsilon^2\mu\|\nabla\Delta\varphi\|_2^2 + \|\nabla\Delta\varphi\|_2^2 \\ &\leq \frac{1}{2}\|\nabla\Delta\varphi\|_2^2 + 3\epsilon^4|\lambda|^2\{\|\boldsymbol{v}_s\|_\infty^2\|\nabla\boldsymbol{w}\|_2^2 + \|\nabla\boldsymbol{v}_s\|_\infty^2\|\boldsymbol{w}\|_2^2\} \\ &\quad + \frac{3}{2}\epsilon^4\|\boldsymbol{G}_\lambda\|_2^2 + \frac{3}{2}\epsilon^4|\lambda|^2\|\Delta\boldsymbol{v}\|_2^2. \end{aligned}$$

This implies that, if $\lambda = \mu + i\frac{\eta}{\epsilon}$ with $-\mu_0 \leq \mu \leq \mu_1$, $\eta \geq \eta_*$, then

$$\begin{aligned} &\epsilon^2\mu^2\|\Delta\varphi\|_2^2 + (\frac{1}{2} - \epsilon^2\mu_*)\|\nabla\Delta\varphi\|_2^2 \\ &\quad \leq \eta^2\|\Delta\varphi\|_2^2 + \frac{3}{2}(\epsilon^4\mu^2 + \epsilon^2\eta^2)\|\Delta\boldsymbol{v}\|_2^2 \\ &\quad\quad + 3(\epsilon^4\mu^2 + \epsilon^2\eta^2)\{\|\boldsymbol{v}_s\|_\infty^2\|\nabla\boldsymbol{w}\|_2^2 + \|\nabla\boldsymbol{v}_s\|_\infty^2\|\boldsymbol{w}\|_2^2\} + \frac{3}{2}\epsilon^4\|\boldsymbol{G}_\lambda\|_2^2. \end{aligned} \tag{3.15}$$

By (3.13) and (3.15), if $\epsilon \leq \frac{\eta_*}{\mu_*}$ and $\epsilon^2 \leq \frac{1}{4\mu_*}$, then

$$\begin{aligned} &\epsilon^2\frac{\mu^2}{\eta^2}\|\Delta\varphi\|_2^2 + \frac{1}{4\eta^2}\|\nabla\Delta\varphi\|_2^2 \\ &\leq 2\eta^2\|\boldsymbol{w}\|_2^2 + 2\epsilon\eta\|\nabla\boldsymbol{w}\| + 2\epsilon\eta(\|\boldsymbol{v}_s\|_\infty^2 + \|\nabla\boldsymbol{v}_s\|_\infty)\|\boldsymbol{w}\|_2^2 \\ &\quad + \frac{3}{2}(\epsilon^4\frac{\mu^2}{\eta^2} + \epsilon^2)\|\Delta\boldsymbol{v}\|_2^2 + 3(\epsilon^4\frac{\mu^2}{\eta^2} + \epsilon^2)\|\boldsymbol{v}_s\|_\infty^2\|\nabla\boldsymbol{w}\|_2^2 \\ &\quad + 3(\epsilon^4\frac{\mu^2}{\eta^2} + \epsilon^2)\|\nabla\boldsymbol{v}_s\|_\infty^2\|\boldsymbol{w}\|_2^2 + \frac{5}{2}\frac{\epsilon^4}{\eta^2}\|\boldsymbol{G}_\lambda\|_2^2. \end{aligned}$$

Therefore, there exists a positive constant $C' = C'(\|\boldsymbol{v}_s\|_{C^1})$ such that if $\epsilon \le C' \min\{1, \eta_*, \sqrt{\frac{\eta_*}{\mu_*}}, \frac{\eta_*}{\mu_*}, \eta_* \mu_*^{-\frac{2}{3}}, \frac{1}{\sqrt{\mu_*}}\}$, then

$$
\begin{aligned}
&\frac{1}{\eta^2}\|\nabla\Delta\varphi\|_2^2 \\
&\le \left(8\eta^2 + 12(\epsilon^4\frac{\mu^2}{\eta^2} + \epsilon^2)\|\nabla \boldsymbol{v}_s\|_\infty^2 + 8\epsilon\eta(\|\boldsymbol{v}_s\|_\infty^2 + \|\nabla\boldsymbol{v}_s\|_\infty)\right)\|\boldsymbol{w}\|_2^2 \\
&\quad + \left(8\epsilon\eta + 12(\epsilon^4\frac{\mu^2}{\eta^2} + \epsilon^2)\|\boldsymbol{v}_s\|_\infty^2\right)\|\nabla\boldsymbol{w}\|_2^2 \\
&\quad + 6(\epsilon^4\frac{\mu^2}{\eta^2} + \epsilon^2)\|\Delta\boldsymbol{v}\|_2^2 + \frac{10\epsilon^4}{\eta^2}\|\boldsymbol{G}_\lambda\|_2^2 \\
&\le C\{\eta^2\|\boldsymbol{w}\|_2^2 + \epsilon\eta\|\nabla\boldsymbol{w}\|_2^2 + \epsilon^2\|\Delta\boldsymbol{v}\|_2^2 + \frac{\epsilon^4}{\eta^2}\|\boldsymbol{G}_\lambda\|_2^2\}.
\end{aligned}
$$

This completes the proof. □

We are now in a position to prove Proposition 3.4.

Proof of Proposition 3.4. Let $\boldsymbol{w} = \boldsymbol{v} + \nabla\varphi$ be the Helmholtz decomposition of $\boldsymbol{w}$. Then

$$
\begin{aligned}
-\eta \mathrm{Re}\,(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s, \boldsymbol{w}) &\le -\eta\mathrm{Re}\,(\nabla\varphi\cdot\nabla\boldsymbol{v}_s, \nabla\varphi) + \eta\{|\mathrm{Re}\,(\boldsymbol{v}\cdot\nabla\boldsymbol{v}_s, \nabla\varphi)| \\
&\quad + |\mathrm{Re}\,(\nabla\varphi\cdot\nabla\boldsymbol{v}_s, \boldsymbol{v})| + |\mathrm{Re}\,(\boldsymbol{v}\cdot\nabla\boldsymbol{v}_s, \boldsymbol{v})|\} \\
&\le -\eta\mathrm{Re}\,(\nabla\varphi\cdot\nabla\boldsymbol{v}_s, \nabla\varphi) \\
&\quad + \kappa\eta\|\nabla\boldsymbol{v}_s\|_\infty\|\nabla\varphi\|_2^2 + (1+\frac{1}{\kappa})\eta\|\nabla\boldsymbol{v}_s\|_\infty\|\boldsymbol{v}\|_2^2
\end{aligned}
$$

for any $\kappa > 0$. Choose $\kappa = \frac{\beta^2}{64\|\nabla\boldsymbol{v}_s\|_\infty}$. Then, since $\|\boldsymbol{w}\|_2^2 = \|\boldsymbol{v}\|_2^2 + \|\nabla\varphi\|_2^2$, we see from Proposition 3.5 that

$$
\begin{aligned}
&(\eta^3 + \beta^2\eta)\|\boldsymbol{w}\|_2^2 + \eta\|\nabla\boldsymbol{w}\|_2^2 \\
&\quad \le -c_0\eta\mathrm{Re}\,(\nabla\varphi\cdot\nabla\boldsymbol{v}_s, \nabla\varphi) + C\eta\|\nabla\boldsymbol{v}_s\|_\infty\|\boldsymbol{v}\|_2^2 + C(\epsilon^2\eta^2 + \epsilon)\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2,
\end{aligned}
\tag{3.16}
$$

where $c_0 = 64$.

We estimate the second term on the right-hand side of (3.16). By Propositions 3.6 and 3.8, there exists a positive constant $C = C(\Omega)$ such that

$$\begin{aligned}\frac{1}{\eta^2}\|\nabla\Delta\varphi\|_2^2 \;\le\; & C\Big\{\eta^2\|\boldsymbol{w}\|_2^2 + \epsilon\eta\|\nabla\boldsymbol{w}\|_2^2 + \epsilon^{\frac{1}{2}}\eta^{\frac{3}{2}}\|\nabla\varphi\|_{H^1}^2 + \epsilon^2\|\nabla\Delta\varphi\|_2^2 \\ & +\epsilon^2\|\boldsymbol{g}\|_2^2 + \epsilon^2\|\boldsymbol{v}_s\|_\infty^2\|\nabla\boldsymbol{w}\|_2^2 + \epsilon^2\|\nabla\boldsymbol{v}_s\|_\infty^2\|\boldsymbol{w}\|_2^2 \\ & +\frac{\epsilon^4}{\eta^2}\|\boldsymbol{G}_\lambda\|_2^2\Big\}.\end{aligned}$$

This implies that if $\eta^2 \le \frac{1}{2C\epsilon^2}$, then

$$\begin{aligned}\tfrac{1}{\eta^2}\|\nabla\Delta\varphi\|_2^2 \;\le\; & C\{(\eta^2 + \epsilon^2\|\nabla\boldsymbol{v}_s\|_\infty^2)\|\boldsymbol{w}\|_2^2 + (\epsilon\eta + \epsilon^2\|\boldsymbol{v}_s\|_\infty^2)\|\nabla\boldsymbol{w}\|_2^2 \\ & +\epsilon^{\frac{1}{2}}\eta^{\frac{3}{2}}\|\nabla\varphi\|_{H^1}^2 + \epsilon^2\|\boldsymbol{g}\|_2^2 + \tfrac{\epsilon^4}{\eta^2}\|\boldsymbol{G}_\lambda\|_2^2\Big\}.\end{aligned} \tag{3.17}$$

By (3.17), Proposition 3.6 and the elliptic estimates: $\|\nabla\varphi\|_{H^k} \le C\|\nabla^{k-1}\Delta\varphi\|_2$ $(k = 1, 2)$, we have

$$\begin{aligned}\|\boldsymbol{v}\|_2^2 \;\le\; & C\Big\{\Big(\tfrac{\epsilon^{\frac{1}{2}}}{\eta^{\frac{1}{2}}} + \epsilon^{\frac{5}{2}}\eta^{\frac{3}{2}}\Big)\|\Delta\varphi\|_2^2 + \epsilon^2(\eta^2 + \epsilon^2\|\nabla\boldsymbol{v}_s\|_\infty^2)\|\boldsymbol{w}\|_2^2 \\ & +\epsilon^2(\epsilon\eta + \epsilon^2\|\boldsymbol{v}_s\|_\infty^2)\|\nabla\boldsymbol{w}\|_2^2 + \epsilon^4\|\boldsymbol{g}\|_2^2 + \tfrac{\epsilon^6}{\eta^2}\|\boldsymbol{G}_\lambda\|_2^2 \\ & +\tfrac{\epsilon^2}{\eta^2}\|\boldsymbol{g}\|_2^2 + \tfrac{\epsilon^2}{\eta^2}\|\boldsymbol{v}_s\|_\infty^2\|\nabla\boldsymbol{w}\|_2^2 + \tfrac{\epsilon^2}{\eta^2}\|\nabla\boldsymbol{v}_s\|_\infty^2\|\boldsymbol{w}\|_2^2\Big\}.\end{aligned} \tag{3.18}$$

Furthermore, Proposition 3.8 gives

$$\eta\|\Delta\varphi\|_2^2 \le C_1\left\{\eta^3\|\boldsymbol{w}\|_2^2 + \epsilon\eta^2\|\nabla\boldsymbol{w}\|_2^2 + \frac{\epsilon^4}{\eta}\|\boldsymbol{G}_\lambda\|_2^2\right\}. \tag{3.19}$$

It then follows from (3.16), (3.18) and (3.19) that

$$\begin{aligned}& (\eta^3 + \beta^2\eta)\|\boldsymbol{w}\|_2^2 + \eta\|\nabla\boldsymbol{w}\|_2^2 + \frac{\eta}{2C_1}\|\Delta\varphi\|_2^2 \\ \le\; & -c_0\eta\mathrm{Re}\,(\nabla\varphi\cdot\nabla\boldsymbol{v}_s, \nabla\varphi) \\ & +\frac{1}{2}\eta^3\|\boldsymbol{w}\|_2^2 + \frac{\epsilon}{2}\eta^2\|\nabla\boldsymbol{w}\|_2^2 + \frac{\epsilon^4}{2\eta}\|\boldsymbol{G}_\lambda\|_2^2 \\ & +C\eta\|\nabla\boldsymbol{v}_s\|_\infty\Big[\Big(\epsilon^{\frac{1}{2}}\eta^{-\frac{1}{2}} + \epsilon^{\frac{5}{2}}\eta^{\frac{3}{2}}\Big)\|\Delta\varphi\|_2^2 \\ & +\epsilon^2(\eta^2 + \epsilon^2\|\nabla\boldsymbol{v}_s\|_\infty^2)\|\boldsymbol{w}\|_2^2 \\ & +\epsilon^2(\epsilon\eta + \epsilon^2\|\boldsymbol{v}_s\|_\infty^2)\|\nabla\boldsymbol{w}\|_2^2 + \frac{\epsilon^2}{\eta^2}\|\boldsymbol{v}_s\|_\infty\|\nabla\boldsymbol{w}\|_2^2 \\ & +\frac{\epsilon^2}{\eta^2}\|\nabla\boldsymbol{v}_s\|_\infty\|\boldsymbol{w}\|_2^2 + \epsilon^4\|\boldsymbol{g}\|_2^2 + \frac{\epsilon^2}{\eta^2}\|\boldsymbol{g}\|_2^2 + \frac{\epsilon^6}{\eta^2}\|\boldsymbol{G}_\lambda\|_2^2\Big] \\ & +C(\epsilon^2\eta^2 + \epsilon)\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2.\end{aligned}$$

From this we see that there exists a positive constant $C' = C'(\|\boldsymbol{v}_s\|_{C^1}, \beta, \Omega)$ such that if $\epsilon \le C' \min\{1,\, \eta_*\}$, $\eta \le \frac{C}{\epsilon}$, then

$$\begin{aligned}
&\frac{1}{4}(\eta^3 + \beta^2\eta)\|\boldsymbol{w}\|_2^2 + \frac{1}{2}\eta\|\nabla\boldsymbol{w}\|_2^2 + \frac{1}{4C_1}\eta\|\Delta\varphi\|_2^2 \\
&\quad \le -c_0\eta \mathrm{Re}\,(\nabla\varphi\cdot\nabla\boldsymbol{v}_s, \nabla\varphi) + C\epsilon^4\eta\|\boldsymbol{g}\|_2^2 + C(\epsilon^2\eta^2+\epsilon)\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2 \\
&\qquad + C\frac{\epsilon^4}{\eta}\|\boldsymbol{G}_\lambda\|_2^2 + C\frac{\epsilon^2}{\eta}\|\boldsymbol{g}\|_2^2. \\
&\quad \le \frac{1}{8}(\eta^3+\beta^2\eta)\|\boldsymbol{w}\|_2^2 - c_0\eta\mathrm{Re}\,(\nabla\varphi\cdot\nabla\boldsymbol{v}_s, \nabla\varphi) \\
&\qquad + C\Big(\frac{\epsilon^4}{\eta} + \epsilon^4\eta + \frac{\epsilon^2}{\beta^2\eta}\Big)\|\boldsymbol{G}_\lambda\|_2^2 + C\Big(\epsilon^4\eta + \frac{\epsilon^2}{\eta}\Big)\|\boldsymbol{g}\|_2^2.
\end{aligned}$$

Therefore, we conclude that if

$$\inf\left\{ \frac{\mathrm{Re}\,(\nabla\varphi\cdot\nabla\boldsymbol{v}_s, \nabla\varphi)}{\|\Delta\varphi\|_2^2} ; \varphi \in H^2_*(\Omega),\ \varphi \ne 0,\ \frac{\partial\varphi}{\partial\boldsymbol{n}}\Big|_{\partial\Omega} = 0 \right\} \ge -\frac{1}{8c_0C_1},$$

then

$$(\eta^3+\beta^2\eta)\|\boldsymbol{w}\|_2^2 + \eta\|\nabla\boldsymbol{w}\|_2^2 + \eta\|\Delta\varphi\|_2^2 \le C\Big\{ \Big(\eta + \frac{\epsilon^2}{\eta}\Big)\|\boldsymbol{g}\|_2^2 + \frac{\epsilon^2}{\eta}\|\nabla f\|_2^2 \Big\}.$$

This completes the proof. □

Remark 3.9. *The estimate* (1.11) *can be obtained by using Propositions 3.6 and 3.8.*

It remains to derive the estimates for $\|p\|_{H^1}$ and $\|\partial_x^2\boldsymbol{w}\|_2$.

Proposition 3.10. *Under the assumption of Proposition 3.4 (with C' suitably replaced),*

$$\|\partial_x^2\boldsymbol{w}\|_2 + \|\nabla p\|_2 \le C\Big\{ \Big(1 + \frac{\eta}{\epsilon}\Big)\|\boldsymbol{g}\|_2 + \|f\|_{H^1} \Big\}$$

for all $0 < \epsilon \le C' \min\{1, \eta_*, \sqrt{\frac{\eta_*}{\mu_*}}, \frac{\eta_*}{\mu_*}, \eta_*\mu_*^{-\frac{2}{3}}, \frac{1}{\sqrt{\mu_*}}\}$.

Proof. We see from (3.2)–(3.4) that

$$\begin{aligned}
\mathrm{div}\,\boldsymbol{w} &= \epsilon^2 f - \epsilon^2\lambda p, \\
-\Delta\boldsymbol{w} + \nabla p &= \boldsymbol{g} - \lambda\boldsymbol{w} - \boldsymbol{v}_s\cdot\nabla\boldsymbol{w} - \boldsymbol{w}\cdot\nabla\boldsymbol{v}_s, \\
\boldsymbol{w}|_{\partial\Omega} &= \boldsymbol{0}.
\end{aligned}$$

The elliptic estimate for the Stokes system ([11, 17]) then gives

$$\begin{aligned}\|\partial_x^2 \boldsymbol{w}\|_2 + \|\nabla p\|_2 \;\; &\le \;\; C\{\epsilon^2\|f\|_{H^1} + \epsilon^2|\lambda|\|p\|_{H^1} \\ &\quad + \|\boldsymbol{g}\|_2 + |\lambda|\|\boldsymbol{w}\|_2 + \|\boldsymbol{v}_s\|_\infty\|\nabla \boldsymbol{w}\|_2 + \|\nabla \boldsymbol{v}_s\|_\infty\|\boldsymbol{w}\|_2\}.\end{aligned}$$

By the Poincaré inequality, we have

$$\epsilon^2|\lambda|\|p\|_{H^1} \le C(\epsilon^2\mu_* + \epsilon\eta)\|\nabla p\|_2$$

with $C = C(\Omega) > 0$. Therefore, if $\epsilon \le \frac{C}{\sqrt{\mu_*}}$ and $\eta \le \frac{C}{\epsilon}$, then

$$\begin{aligned}&\|\partial_x^2 \boldsymbol{w}\|_2 + \|\nabla p\|_2 \\ &\quad \le \;\; C\{\epsilon^2\|f\|_{H^1} + \|\boldsymbol{g}\|_2 + |\lambda|\|\boldsymbol{w}\|_2 + \|\boldsymbol{v}_s\|_\infty\|\nabla \boldsymbol{w}\|_2 + \|\nabla \boldsymbol{v}_s\|_\infty\|\boldsymbol{w}\|_2\}.\end{aligned}$$

This, together with Proposition 3.4, implies the desired conclusion. This completes the proof. □

4 Proof of Lemma 3.7

In this section, we give a proof of Lemma 3.7. We first estimate a solution ${}^\top(p, \boldsymbol{v})$ of the following problem on the half-space $\mathbb{R}^3_+ = \{x = (x', x_3);\ x' = (x_1, x_2) \in \mathbb{R}^2,\ x_3 > 0\}$:

$$\left\{\begin{aligned} \operatorname{div} \boldsymbol{v} \;&=\; f, \\ \lambda \boldsymbol{v} - \Delta \boldsymbol{v} + \nabla p \;&=\; \boldsymbol{g}, \\ \boldsymbol{v}|_{x_3=0} \;&=\; \boldsymbol{\psi}. \end{aligned}\right. \tag{4.1}$$

Here $\lambda \in \mathbb{C}$ is a parameter; f and $\boldsymbol{g}$ are given functions on $\mathbb{R}^3_+$; and $\boldsymbol{\psi} = {}^\top(\psi^1(x'), \psi^2(x'), \psi^3(x'))$ is a given function on $\mathbb{R}^2$ satisfying $\boldsymbol{\psi} \cdot \boldsymbol{n}|_{\partial\mathbb{R}^3_+} = -\psi^3(x') = 0$ ($x' \in \mathbb{R}^2$). We denote by Σ_ω the sector defined by

$$\Sigma_\omega = \{\lambda \in \mathbb{C}; |\arg \lambda| \le \pi - \omega\}.$$

We have the following estimate.

Lemma 4.1. *Let ω be a positive number satisfying $0 < \omega < \frac{\pi}{2}$. If $\lambda \in \Sigma_\omega \backslash \{0\}$, then*

$$\begin{aligned}&|\lambda|\|\boldsymbol{v}\|_{L^2(\mathbb{R}^3_+)} + \|\partial_x^2 \boldsymbol{v}\|_{L^2(\mathbb{R}^3_+)} + \|\partial_x p\|_{L^2(\mathbb{R}^3_+)} \\ &\qquad \le C_\omega\{|\lambda|\|f\|_{\dot{H}^{-1}(\mathbb{R}^3_+)} + \|\partial_x f\|_{L^2(\mathbb{R}^3_+)} + \|\boldsymbol{g}\|_{L^2(\mathbb{R}^3_+)} \\ &\qquad\qquad + |\lambda|^{\frac{3}{4}}\|\boldsymbol{\psi}\|_{L^2(\mathbb{R}^2)} + \|\boldsymbol{\psi}\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^2)}\}.\end{aligned}$$

Here C_ω is a positive constant depending only on ω.

To prove Lemma 4.1, we introduce some notation. For $\lambda \in \mathbb{C}\setminus(-\infty, -|\xi'|^2]$, $\lambda \neq 0$ and $\xi' \in \mathbb{R}^2$, $\xi' \neq 0$, we denote the principal branch of the square root of $\lambda + |\xi'|^2$ by $\mu_1 = \mu_1(\lambda, \xi')$, i.e.,

$$\mu_1 = \mu_1(\lambda, \xi') = \sqrt{\lambda + |\xi'|^2}$$

with $\operatorname{Re}\mu_1 > 0$. For a complex number μ, we define functions $g_\mu^{(\pm)}(x_3, y_3)$ and $h_\mu(x_3)$ by

$$g_\mu^{(\pm)}(x_3, y_3) = \frac{1}{2\mu}\left\{e^{-\mu|x_3-y_3|} \pm e^{-\mu(x_3+y_3)}\right\}$$

and

$$h_\mu(x_3) = \frac{1}{\mu}e^{-\mu x_3}.$$

We also introduce functions $\widetilde{g}^{(\pm)}(x_3, y_3)$ and $\widetilde{h}(x_3)$ which are defined by

$$\widetilde{g}^{(\pm)}(x_3, y_3) = g_{\mu_1}^{(\pm)}(x_3, y_3) - g_{|\xi'|}^{(\pm)}(x_3, y_3)$$

and

$$\widetilde{h}(x_3) = h_{\mu_1}(x_3) - h_{|\xi'|}(x_3).$$

We denote by $\delta(x_3)$ the Dirac delta function.

For a function $K(x_3, y_3)$ on $(0,\infty)\times(0,\infty)$ we denote by Kf the integral operator $\int_0^\infty K(x_3, y_3)f(y_3)\,dy_3$. For a function $f = f(x')$ $(x' \in \mathbb{R}^2)$, we denote by $\mathcal{F}f = \hat{f}$ the Fourier transform of f:

$$(\mathcal{F}f)(\xi') = \hat{f}(\xi') = \int_{\mathbb{R}^2} f(x')e^{-i\xi'\cdot x'}\,dx', \quad \xi' \in \mathbb{R}^2.$$

The inverse Fourier transform is denoted by $\mathcal{F}^{-1}$.

The solution ${}^\top(p, \boldsymbol{v})$ of (4.1) is then written as

$$\begin{aligned} \begin{pmatrix} p(x) \\ \boldsymbol{v}(x) \end{pmatrix} &= \mathcal{F}^{-1}\hat{\Gamma}\mathcal{F}F(x) + \mathcal{F}^{-1}\hat{\Gamma}_B\mathcal{F}\begin{pmatrix} 0 \\ \boldsymbol{\psi} \end{pmatrix}(x) \\ &= \mathcal{F}^{-1}\left[\int_0^\infty \hat{\Gamma}(\lambda, \xi', x_3, y_3)\hat{F}(\xi', y_3)dy_3\right](x') \\ &\quad +\mathcal{F}^{-1}\left[\hat{\Gamma}_B(\lambda, \xi', x_3)\begin{pmatrix} 0 \\ \hat{\boldsymbol{\psi}}(\xi') \end{pmatrix}\right](x') \end{aligned} \tag{4.2}$$

with $F = {}^\top(f, \boldsymbol{g}), \boldsymbol{g} = {}^\top(\boldsymbol{g}', g^3)$. Here

$$\hat{\Gamma}(\lambda, \xi', x_3, y_3) = \hat{G}(\lambda, \xi', x_3, y_3) + \hat{H}(\lambda, \xi', x_3, y_3)$$

is the integral kernel of the solution operator for the problem (4.1) with $\boldsymbol{\psi} = \mathbf{0}$, and

$$\hat{\Gamma}_B(\lambda, \xi', x_3) = \partial_{y_3} \hat{G}(\lambda, \xi', x_3, 0) + \partial_{y_3} \hat{H}(\lambda, \xi', x_3, 0),$$

where

$$
\begin{aligned}
&\hat{G}(\lambda, \xi', x_3, y_3) \\
&= \delta(x_3 - y_3) Q_0 \\
&+ \begin{pmatrix} \lambda g^{(+)}_{|\xi'|}(x_3, y_3) & -i^\top \xi' g^{(+)}_{|\xi'|}(x_3, y_3) & -\partial_{x_3} g^{(-)}_{|\xi'|}(x_3, y_3) \\ -i\xi' g^{(+)}_{|\xi'|}(x_3, y_3) & 0 & 0 \\ -\partial_{x_3} g^{(+)}_{|\xi'|}(x_3, y_3) & 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & g^{(-)}_{\mu_1}(x_3, y_3) I_2 & 0 \\ 0 & 0 & g^{(-)}_{\mu_1}(x_3, y_3) \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\xi'^\top \xi'}{\lambda} \widetilde{g}^{(+)}(x_3, y_3) & -\frac{i\xi'}{\lambda} \partial_{x_3} \widetilde{g}^{(-)}(x_3, y_3) \\ 0 & -\frac{i^\top \xi'}{\lambda} \partial_{x_3} \widetilde{g}^{(+)}(x_3, y_3) & -\frac{1}{\lambda} \partial^2_{x_3} \widetilde{g}^{(-)}(x_3, y_3) \end{pmatrix}
\end{aligned}
$$

with Q_0 being the 4×4 diagonal matrix $Q_0 = \operatorname{diag}(1, 0, 0, 0)$; and

$$
\begin{aligned}
&\hat{H}(\lambda, \xi', x_3, y_3) \\
&= \begin{pmatrix} 0 & i^{\top}\xi' h_{|\xi'|}(x_3) e^{-\mu_1 y_3} & 0 \\ 0 & -\frac{\xi'^{\top}\xi'}{\lambda}\widetilde{h}(x_3) e^{-\mu_1 y_3} & 0 \\ 0 & \frac{i^{\top}\xi'}{\lambda}\partial_{x_3}\widetilde{h}(x_3) e^{-\mu_1 y_3} & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 & 0 \\ h_{\mu_1}(x_3)\beta_0(y_3) & h_{\mu_1}(x_3) B(y_3) & h_{\mu_1}(x_3)\beta_3(y_3) \\ 0 & 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} -i^{\top}\xi' h_{|\xi'|}(x_3)\beta_0(y_3) & -i^{\top}\xi' h_{|\xi'|}(x_3) B(y_3) & -i^{\top}\xi' h_{|\xi'|}(x_3)\beta_3(y_3) \\ \frac{\xi'^{\top}\xi'}{\lambda}\widetilde{h}(x_3)\beta_0(y_3) & \frac{\xi'^{\top}\xi'}{\lambda}\widetilde{h}(x_3) B(y_3) & \frac{\xi'^{\top}\xi'}{\lambda}\widetilde{h}(x_3)\beta_3(y_3) \\ -\frac{i^{\top}\xi'}{\lambda}\partial_{x_3}\widetilde{h}(x_3)\beta_0(y_3) & -\frac{i'^{\top}\xi'}{\lambda}\partial_{x_3}\widetilde{h}(x_3) B(y_3) & -\frac{i^{\top}\xi'}{\lambda}\partial_{x_3}\widetilde{h}(x_3)\beta_3(y_3) \end{pmatrix}
\end{aligned}
$$

with $\mu_1 = \mu_1(\lambda, \xi')$ and

$$
\begin{aligned}
\beta_0(y_3) = \beta_0(\lambda, \xi', y_3) &= \frac{i\lambda\xi'}{|\xi'|(\mu_1 - |\xi'|)} e^{-|\xi'| y_3}, \\
B(y_3) = B(\lambda, \xi', y_3) &= -\frac{\xi'^{\top}\xi'}{|\xi'|(\mu_1 - |\xi'|)} (e^{-\mu_1 y_3} - e^{-|\xi'| y_3}), \\
\beta_3(y_3) = \beta_3(\lambda, \xi', y_3) &= \frac{i\xi'}{\mu_1 - |\xi'|} (e^{-\mu_1 y_3} - e^{-|\xi'| y_3}).
\end{aligned}
$$

The solution formula (4.2) will be derived in the end of this section.

To estimate ${}^{\top}(p, \boldsymbol{v})$, we prepare elementary inequalities.

Lemma 4.2. *Let $\omega \in (0, \frac{\pi}{2})$. Then there exists a positive constant C depending only on ω such that the following inequalities hold uniformly for $\lambda \in \Sigma_\omega \backslash \{0\}$:*

(i) $C^{-1}(|\lambda|^{\frac{1}{2}} + |\xi'|) \leq |\mu_1| \leq C(|\lambda|^{\frac{1}{2}} + |\xi'|)$,

(ii) $C^{-1}|\mu_1| \leq \operatorname{Re}\mu_1 \leq C|\mu_1|$,

(iii) $|e^{-\mu_1 x_3} - e^{-|\xi'| x_3}| \leq C|\mu_1 - |\xi'|| \left(\frac{1}{|\xi'|} e^{-\frac{1}{2}\operatorname{Re}\mu_1 x_3} + \frac{1}{\operatorname{Re}\mu_1} e^{-\frac{1}{2}|\xi| x_3} \right)$.

Inequalities (i) and (ii) can be proved by an elementary computation; and estimate (iii) can be proved by using the mean value theorem. (Cf., [14, Lemma 4.1].)

We will use the following generalized Young inequality.

Lemma 4.3. *Let $K(x_3, y_3)$ be a function on $(0,\infty)\times(0,\infty)$ satisfying*

$$\sup_{y_3>0}\int_0^\infty |K(x_3,y_3)|\,dx_3 \le M_1 \quad \text{and} \quad \sup_{x_3>0}\int_0^\infty |K(x_3,y_3)|\,dy_3 \le M_2$$

for some constants $M_1 > 0$ and $M_2 > 0$. Set

$$KF(x_3) = \int_0^\infty K(x_3,y_3)F(y_3)\,dy_3.$$

Then

$$\|KF\|_{L^2(0,\infty)} \le M_1^{\frac{1}{2}} M_2^{\frac{1}{2}} \|F\|_{L^2(0,\infty)}.$$

See, e.g., [10] for a proof of Lemma 4.3.

Proof of Lemma 4.1. The estimate for $\mathcal{F}^{-1}\hat{\Gamma}\mathcal{F}F$ is well-known. In fact, one can obtain the desired estimate for $\mathcal{F}^{-1}\hat{\Gamma}\mathcal{F}F$ by using Lemmas 4.2 and 4.3. As for $\mathcal{F}^{-1}\hat{\Gamma}_B\mathcal{F}\left[{}^\top(0,\boldsymbol{\psi})\right]$, we set ${}^\top(p_B, \boldsymbol{v}_B) = \mathcal{F}^{-1}\hat{\Gamma}_B\mathcal{F}\left[{}^\top(0,\boldsymbol{\psi})\right]$. Then, by a direct computation, we see from (4.2) that

$$\hat{p}_B = -i(\mu_1 + |\xi'|)h_{|\xi'|}(x_3)\xi'\cdot\hat{\boldsymbol{\psi}}',$$

$$\hat{\boldsymbol{v}}_B = \begin{pmatrix} \mu_1 h_{\mu_1}(x_3)\hat{\boldsymbol{\psi}}' + \frac{\xi'}{|\xi'|}h_{\mu_1}(x_3)\xi'\cdot\hat{\boldsymbol{\psi}}' + \frac{\xi'}{\lambda}(\mu_1+|\xi'|)\tilde{h}(x_3)\xi'\cdot\hat{\boldsymbol{\psi}}' \\ -\frac{i}{\lambda}(\mu_1+|\xi'|)\partial_{x_3}\tilde{h}(x_3)\xi'\cdot\hat{\boldsymbol{\psi}}' \end{pmatrix}.$$

The desired estimates can be obtained by applying Lemma 4.2. This completes the proof. □

We turn to the proof of Lemma 3.7. We fix $\omega \in (0, \frac{\pi}{2})$. Let C_ω be the positive constant in Lemma 4.1. Then for each $\bar{x} \in \partial\Omega$ there exist an open neighborhood $\mathcal{O}_{\bar{x}}$ of $\bar{x}$, a local coordinate system with coordinates $\tilde{x} = (\tilde{x}', \tilde{x}_3)$ in $\bar{x}$ and a smooth function $h(\tilde{x}')$ such that

$$\mathcal{O}_{\bar{x}} \cap \Omega = \{(\tilde{x}', \tilde{x}_3);\ h(\tilde{x}') < \tilde{x}_3 < h(\tilde{x}') + \gamma,\ |\tilde{x}'| < r\},$$

$$\mathcal{O}_{\bar{x}} \cap \partial\Omega = \{(\tilde{x}', \tilde{x}_3);\ \tilde{x}_3 = h(\tilde{x}'),\ |\tilde{x}'| < r\},$$

$$(0, h(0)) = \bar{x}, \;\; \nabla_{\tilde{x}'} h(0) = 0, \;\; \sup_{|\tilde{x}'|<r} |\nabla_{\tilde{x}'} h(\tilde{x}')| \leq \frac{1}{24 C_\omega}$$

for some positive constants γ and r. By a translation and a rotation of the coordinate system we can write $\tilde{x}$ as x. We set $\Phi(x) = {}^\top(x', x_3 - h(x'))$. Then $y = \Phi(x)$ is a diffeomorphism from $\mathcal{O}_{\bar{x}}$ to an open set $\tilde{\mathcal{O}}_{\bar{x}}$ of $\mathbb{R}^3_+$ and

$$\begin{cases} \Phi(\mathcal{O}_{\bar{x}} \cap \Omega) = \{y = (y', y_3);\, |y'| < r,\, 0 < y_3 < \gamma\}, \\ \Phi(\mathcal{O}_{\bar{x}} \cap \partial\Omega) = \{y = (y', y_3);\, |y'| < r,\, y_3 = 0\}, \\ |\nabla_{y'} h(y')| \leq \frac{C_\omega}{24} \text{ for } |y'| < r. \end{cases} \tag{4.3}$$

Since $\partial\Omega$ is compact, there exist $x^{(1)}, \dots, x^{(N)} \in \partial\Omega$ such that $\partial\Omega \subset \cup_{m=1}^N \mathcal{O}_m$, $\mathcal{O}_m = \mathcal{O}_{x^{(m)}}$ and diffeomorphisms $\Phi_m : \mathcal{O}_m \to \tilde{\mathcal{O}}_m$, $\Phi_m(x) = {}^\top(x', x_3 - h_m(x'))$, $m = 1, \dots, N$, which satisfy (4.3) with $\mathcal{O}_{\bar{x}}$ replaced by $\mathcal{O}_m$ for some $h = h_m(x')$, $r = r_m$, and $\gamma = \gamma_m$.

We take an open set $\mathcal{O}_0$ such that $\overline{\mathcal{O}}_0 \subset \Omega$ and $\overline{\Omega} \subset \cup_{m=0}^N \mathcal{O}_m$, and introduce a partition of unity $\{\chi_m\}_{m=0}^N$ subordinate to $\{\mathcal{O}_m\}_{m=0}^N$ satisfying $\mathrm{supp}\chi_m \subset \mathcal{O}_m$, $\Sigma_{m=0}^N \chi_m = 1$ on $\overline{\Omega}$.

We make the following transformation of $\chi_m p$ and $\chi_m \boldsymbol{v}$:

$$\begin{aligned} \tilde{p}_m(y) &= (\chi_m p)(\Phi_m^{-1}(y)), \\ \tilde{\boldsymbol{v}}_m(y) &= {}^\top(\tilde{\boldsymbol{v}}'_m(y), \tilde{v}^3_m(y))), \\ \tilde{\boldsymbol{v}}'_m(y) &= (\chi_m \boldsymbol{v}')(\Phi_m^{-1}(y)), \\ \tilde{v}^3_m(y) &= -\nabla_{y'} h(y') \cdot (\chi_m \boldsymbol{v}')(\Phi_m^{-1}(y)) + (\chi_m v^3)(\Phi^{-1}(y)). \end{aligned}$$

Furthermore, $\chi_m \boldsymbol{g}$ is transformed into $\tilde{\boldsymbol{g}}_m = {}^\top(\tilde{\boldsymbol{g}}'_m(y), \tilde{g}^n_m(y))$ in a similar manner.

Under these transformation the localized problem for (3.12) on $\Omega_m = \mathcal{O}_m \cap \Omega$ is transformed into the following one on $\mathbb{R}^n_+$:

$$\begin{cases} \mathrm{div}\, \tilde{\boldsymbol{v}}_m = \widetilde{\boldsymbol{v} \cdot \nabla \chi_m}, \\ \lambda \tilde{\boldsymbol{v}}_m - \Delta \tilde{\boldsymbol{v}}_m + \nabla \tilde{p}_m = \tilde{\boldsymbol{G}}_m, \\ \tilde{\boldsymbol{v}}'_m|_{y_3=0} = \tilde{\boldsymbol{\psi}}'_m, \qquad \tilde{v}^3_m|_{y_3=0} = 0. \end{cases} \tag{4.4}$$

Here $\widetilde{\boldsymbol{v} \cdot \nabla \chi_m}$ denotes $(w \cdot \nabla \chi_m)(\Phi_m^{-1}(y))$ and $\tilde{\boldsymbol{G}}_m = {}^\top(\tilde{\boldsymbol{G}}'_m, \tilde{G}^3_m)$, where

$$\begin{aligned} \tilde{\boldsymbol{G}}'_m = {} & (\widetilde{p \nabla_{x'} \chi_m}) - 2\widetilde{\nabla_x \chi_m \cdot \nabla_x} \boldsymbol{v}' - \widetilde{\boldsymbol{v}' \Delta_x \chi_m} + \tilde{\boldsymbol{g}}'_m + (\nabla_{y'} h) \partial_{y_3} \tilde{p}_m \\ & -2(\nabla_{y'} h) \cdot \nabla_{y'} \partial_{y_3} \tilde{\boldsymbol{v}}'_m - (\Delta_{y'} h) \partial_{y_3} \tilde{\boldsymbol{v}}'_m - |\nabla_{y'} h|^2 \partial^2_{y_3} \tilde{\boldsymbol{v}}'_m, \end{aligned}$$

$$\begin{aligned}\tilde{G}^3_m &= -\nabla_{y'}h\cdot\tilde{\boldsymbol{G}}'_m + 2(\nabla_{y'}(\nabla_{y'}h))\cdot\nabla_{y'}\tilde{\boldsymbol{v}}'_m + (\Delta_{y'}\nabla_{y'}h)\cdot\tilde{\boldsymbol{v}}'_m + \nabla_{y'}h\cdot\nabla_{y'}\tilde{p}_m\\ &\quad+\widetilde{(p\partial_{x_3}\chi_m)} - 2\widetilde{\nabla_x\chi_m\cdot\nabla_x v^3} - \widetilde{v^3\Delta_x\chi_m} + \tilde{g}^3_m - 2(\nabla_{y'}h)\cdot\nabla_{y'}\partial_{y_3}\tilde{w}^3_m\\ &\quad-(\Delta_{y'}h)\partial_{y_n}\tilde{w}^3_m - |\nabla_{y'}h|^2\partial^2_{y_3}\tilde{w}^3_m.\end{aligned}$$

Here

$$\tilde{w}^3_m = \tilde{v}^3_m + \nabla' h\cdot\tilde{\boldsymbol{v}}'_m.$$

Applying Lemma 4.1 to (4.4), we have the following estimate.

Proposition 4.4. *Let* $\lambda\in\Sigma_\omega\backslash\{0\}$*. Then*

$$\begin{aligned}&|\lambda|\|\tilde{\boldsymbol{v}}_m\|_{L^2(\mathbb{R}^3_+)} + \|\partial^2_y\tilde{\boldsymbol{v}}_m\|_{L^2(\mathbb{R}^3_+)} + \|\partial_y\tilde{p}_m\|_{L^2(\mathbb{R}^3_+)}\\ &\quad\le C'_\omega\{\|\lambda\widetilde{\boldsymbol{v}\cdot\nabla\chi_m}\|_{\dot{H}^{-1}(\mathbb{R}^3_+)} + \|\nabla(\widetilde{\boldsymbol{v}\cdot\nabla\chi_m})\|_{L^2(\mathbb{R}^3_+)}\\ &\qquad+\|\widetilde{p\nabla_x\chi_m}\|_{L^2(\mathbb{R}^3_+)} + \|\widetilde{\nabla_x\chi_m\cdot\nabla_x\boldsymbol{v}}\|_{L^2(\mathbb{R}^3_+)} + \|\widetilde{\boldsymbol{v}\Delta_x\chi_m}\|_{L^2(\mathbb{R}^3_+)}\\ &\qquad+\|\tilde{\boldsymbol{v}}_m\|_{L^2(\mathbb{R}^3_+)} + \|\partial_y\tilde{\boldsymbol{v}}_m\|_{L^2(\mathbb{R}^3_+)}\\ &\qquad+\|\tilde{\boldsymbol{g}}_m\|_{L^2(\mathbb{R}^3_+)} + |\lambda|^{\frac{3}{4}}\|\tilde{\boldsymbol{\psi}}_m\|_{L^2(\mathbb{R}^2)} + \|\tilde{\boldsymbol{\psi}}_m\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^2)}\},\end{aligned}$$

where C'_ω *depends only on* ω.

Proof. We have

$$\begin{aligned}\|\tilde{\boldsymbol{G}}_m\|_{L^2(\mathbb{R}^3_+)} &\le 12\|\nabla_{y'}h\|_\infty\{\|\partial_y\tilde{p}_m\|_{L^2(\mathbb{R}^3_+)} + \|\partial^2_y\tilde{\boldsymbol{v}}_m\|_{L^2(\mathbb{R}^3_+)}\}\\ &\quad+C\{\|\widetilde{p\nabla_x\chi_m}\|_{L^2(\mathbb{R}^3_+)} + \|\widetilde{\nabla_x\chi_m\cdot\nabla_x\boldsymbol{v}}\|_{L^2(\mathbb{R}^3_+)} + \|\widetilde{\boldsymbol{v}\Delta_x\chi_m}\|_{L^2(\mathbb{R}^3_+)}\\ &\quad+\|\tilde{\boldsymbol{v}}_m\|_{L^2(\mathbb{R}^3_+)} + \|\partial_y\tilde{\boldsymbol{v}}_m\|_{L^2(\mathbb{R}^3_+)} + \|\tilde{\boldsymbol{g}}_m\|_{L^2(\mathbb{R}^3_+)}\}.\end{aligned}$$

Noting that $\|\nabla_{y'}h\|_\infty\le\frac{1}{24C_\omega}$ and applying Lemma 4.1 to (4.4), we obtain

$$\begin{aligned}&|\lambda|\|\tilde{\boldsymbol{v}}_m\|_{L^2(\mathbb{R}^3_+)} + \|\partial^2_y\tilde{\boldsymbol{v}}_m\|_{L^2(\mathbb{R}^3_+)} + \|\partial_y\tilde{p}_m\|_{L^2(\mathbb{R}^3_+)}\\ &\quad\le \frac{1}{2}\{\|\partial_y\tilde{p}_m\|_{L^2(\mathbb{R}^3_+)} + \|\partial^2_y\tilde{\boldsymbol{v}}_m\|_{L^2(\mathbb{R}^3_+)}\}\\ &\qquad+C''_\omega\{\|\lambda\widetilde{\boldsymbol{v}\cdot\nabla\chi_m}\|_{\dot{H}^{-1}(\mathbb{R}^3_+)} + \|\nabla(\widetilde{\boldsymbol{v}\cdot\nabla\chi_m})\|_{L^2(\mathbb{R}^3_+)}\\ &\qquad+\|\widetilde{p\nabla_x\chi_m}\|_{L^2(\mathbb{R}^3_+)} + \|\widetilde{\nabla_x\chi_m\cdot\nabla_x\boldsymbol{v}}\|_{L^2(\mathbb{R}^3_+)} + \|\widetilde{\boldsymbol{v}\Delta_x\chi_m}\|_{L^2(\mathbb{R}^3_+)}\\ &\qquad+\|\tilde{\boldsymbol{v}}_m\|_{L^2(\mathbb{R}^3_+)} + \|\partial_y\tilde{\boldsymbol{v}}_m\|_{L^2(\mathbb{R}^3_+)}\\ &\qquad+\|\tilde{\boldsymbol{g}}_m\|_{L^2(\mathbb{R}^3_+)} + |\lambda|^{\frac{3}{4}}\|\tilde{\boldsymbol{\psi}}_m\|_{L^2(\mathbb{R}^2)} + \|\tilde{\boldsymbol{\psi}}_m\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^2)}\},\end{aligned}$$

and hence,

$$
\begin{aligned}
&|\lambda|\|\tilde{\boldsymbol{v}}_m\|_{L^2(\mathbb{R}^3_+)} + \|\partial_y^2\tilde{\boldsymbol{v}}_m\|_{L^2(\mathbb{R}^3_+)} + \|\partial_y\tilde{p}_m\|_{L^2(\mathbb{R}^3_+)} \\
\leq\ & 2C''_\omega\{\|\lambda\widetilde{\boldsymbol{v}\cdot\nabla\chi_m}\|_{\dot{H}^{-1}(\mathbb{R}^3_+)} + \|\nabla(\widetilde{\boldsymbol{v}\cdot\nabla\chi_m})\|_{L^2(\mathbb{R}^3_+)} \\
& +\|\widetilde{p\nabla_x\chi_m}\|_{L^2(\mathbb{R}^3_+)} + \|\widetilde{\nabla_x\chi_m\cdot\nabla_x\boldsymbol{v}}\|_{L^2(\mathbb{R}^3_+)} + \|\widetilde{\boldsymbol{v}\Delta_x\chi_m}\|_{L^2(\mathbb{R}^3_+)} \\
& +\|\tilde{\boldsymbol{v}}_m\|_{L^2(\mathbb{R}^3_+)} + \|\partial_y\tilde{\boldsymbol{v}}_m\|_{L^2(\mathbb{R}^3_+)} \\
& +\|\tilde{\boldsymbol{g}}_m\|_{L^2(\mathbb{R}^3_+)} + |\lambda|^{\frac{3}{4}}\|\tilde{\boldsymbol{\psi}}_m\|_{L^2(\mathbb{R}^2)} + \|\tilde{\boldsymbol{\psi}}_m\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^2)}\}.
\end{aligned}
$$

This completes the proof. □

We then have the following estimate.

Proposition 4.5. *Let $\lambda \in \Sigma_\omega\backslash\{0\}$ with $|\lambda| \geq \delta$ for some $\delta > 0$. Then*

$$
\begin{aligned}
&|\lambda|\|\boldsymbol{v}\|_2 + \|\boldsymbol{v}\|_{H^2} + \|\partial_x p\|_{H^1} \\
\leq\ & C_1\{|\lambda|\|\boldsymbol{v}\|_{H^{-1}} + \|\boldsymbol{v}\|_{H^1} + \|p\|_2 + \|\boldsymbol{g}\|_2 + |\lambda|^{\frac{3}{4}}\|\boldsymbol{\psi}\|_{L^2(\partial\Omega)} + \|\boldsymbol{\psi}\|_{H^{\frac{3}{2}}(\partial\Omega)}\}
\end{aligned}
$$

Proof. By Proposition 4.4 we have

$$
\begin{aligned}
&|\lambda|\|\boldsymbol{v}\|_2 + \|\partial_x^2\boldsymbol{v}\|_2 + \|\partial_x p\|_2 \\
\leq\ & C\{|\lambda|\|\boldsymbol{v}\|_{H^{-1}} + \|\boldsymbol{v}\|_{H^1} + \|p\|_2 + \|\boldsymbol{g}\|_2 + |\lambda|^{\frac{3}{4}}\|\boldsymbol{\psi}\|_{L^2(\partial\Omega)} + \|\boldsymbol{\psi}\|_{\dot{H}^{\frac{3}{2}}(\partial\Omega)}\}.
\end{aligned}
$$

Since $|\lambda| \geq \delta$, using the Poincaré inequality: $\|p\|_{L^2(\Omega)} \leq \|\partial_x p\|_{L^2(\Omega)}$ for $p \in H^1_*(\Omega)$, we see that

$$
\begin{aligned}
&|\lambda|\|\boldsymbol{v}\|_2 + \|\boldsymbol{v}\|_2 + \|\partial_x^2\boldsymbol{v}\|_2 + \|p\|_{H^1} \\
&\leq C\{|\lambda|\|\boldsymbol{v}\|_{H^{-1}} + \|\boldsymbol{v}\|_{H^1} + \|p\|_2 + \|\boldsymbol{g}\|_2 + |\lambda|^{\frac{3}{4}}\|\boldsymbol{\psi}\|_{L^2(\partial\Omega)} + \|\boldsymbol{\psi}\|_{H^{\frac{3}{2}}(\partial\Omega)}\}.
\end{aligned}
\tag{4.5}
$$

Furthermore, since

$$
\|\nabla\boldsymbol{v}\|_2^2 = \int_{\partial\Omega}\boldsymbol{v}\frac{\partial\overline{\boldsymbol{v}}}{\partial\boldsymbol{n}}\,d\sigma - \int_\Omega \boldsymbol{v}\Delta\overline{\boldsymbol{v}}\,dx,
$$

we have

$$
\|\nabla\boldsymbol{v}\|_2 \leq C\{\|\boldsymbol{v}\|_{L^2(\partial\Omega)} + \|\boldsymbol{v}\|_2 + \|\partial_x^2\boldsymbol{v}\|_2\}.
$$

This, together with (4.5), gives the desired estimate. □

Lemma 3.7 follows from Proposition 4.5 by a compactness argument as given in [9].

We finally give a proof of solution formula (4.2). To this end, we introduce the integral kernel of the generalized resolvent for the adjoint problem for (4.1) with $\boldsymbol{\psi} = \mathbf{0}$:

$$\begin{cases} -\operatorname{div} \boldsymbol{v}^* &= q, \\ \bar{\lambda}\boldsymbol{v}^* - \Delta \boldsymbol{v}^* - \nabla p^* &= \boldsymbol{f}, \\ \boldsymbol{v}^*|_{x_3=0} &= \mathbf{0}. \end{cases} \tag{4.6}$$

The solution of (4.6) has a representation similar to (4.2). In fact, it holds that

$$\begin{pmatrix} p^*(x) \\ \boldsymbol{v}^*(x) \end{pmatrix} = \Gamma^* \begin{pmatrix} q \\ \boldsymbol{f} \end{pmatrix} (x),$$

where $\Gamma^*(x,y) = (\mathcal{F}^{-1}\hat{\Gamma}^*\mathcal{F})(x,y)$ with $\hat{\Gamma}^*(\lambda, \xi', x_3, y_3) = {}^\top\overline{\hat{\Gamma}(\bar{\lambda}, \xi', y_3, x_3)}$. We denote the integral kernel of $\mathcal{F}^{-1}\hat{\Gamma}\mathcal{F}$ by $\Gamma(x,y) = \int_{\mathbb{R}^2} e^{i(x'-y')\cdot\xi'}\hat{\Gamma}(\lambda, \xi', x_3, y_3)\, d\xi'$ by

$$\Gamma(x,y) = \begin{pmatrix} \Gamma_{11}(x,y) & \Gamma_{12}(x,y) \\ \Gamma_{21}(x,y) & \Gamma_{22}(x,y) \end{pmatrix}$$

where $\Gamma_{12}(x,y)$ is an $1 \times n$ matrix; $\Gamma_{21}(x,y)$ is an $n \times 1$ matrix; and $\Gamma_{22}(x,y)$ is an $n \times n$ matrix. Similarly we write

$$\Gamma^*(x,y) = \begin{pmatrix} \Gamma^*_{11}(x,y) & \Gamma^*_{12}(x,y) \\ \\ \Gamma^*_{21}(x,y) & \Gamma^*_{22}(x,y) \end{pmatrix}.$$

It follows that

$$p^*(x) = \Gamma^*_{11} q(x) + \Gamma^*_{12}\boldsymbol{f}(x), \;\; \boldsymbol{v}^*(x) = \Gamma^*_{21} q(x) + \Gamma^*_{22}\boldsymbol{f}(x).$$

Let $q \in C_0^\infty(\mathbb{R}^3_+)$ and $\boldsymbol{f} \in C_0^\infty(\mathbb{R}^3_+)^3$. Since $\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\mathbb{R}^3_+} = \boldsymbol{\psi} \cdot \boldsymbol{n} = -\psi^3 = 0$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3_+}(p(x)\overline{q(x)} + \boldsymbol{v}(x)\overline{\boldsymbol{f}(x)})dx \\
&= -\int_{\mathbb{R}^3_+} p(x)\overline{\mathrm{div}\,(\Gamma^*_{21}q(x) + \Gamma^*_{22}\boldsymbol{f}(x))}dx + \int_{\mathbb{R}^3_+} \boldsymbol{v}(x)\overline{(\bar{\lambda} - \Delta)(\Gamma^*_{21}q(x) + \Gamma^*_{22}\boldsymbol{f}(x))}dx \\
&\quad - \int_{\mathbb{R}^3_+} \boldsymbol{v}(x)\cdot\overline{\nabla(\Gamma^*_{11}q(x) + \Gamma^*_{12}\boldsymbol{f}(x))}dx \\
&= \int_{\mathbb{R}^3_+} \nabla p(x)\overline{(\Gamma^*_{21}q(x) + \Gamma^*_{22}\boldsymbol{f}(x))}dx \\
&\quad + \int_{\mathbb{R}^3_+}(\lambda - \Delta)\boldsymbol{v}(x)\overline{(\Gamma^*_{21}q(x) + \Gamma^*_{22}\boldsymbol{f}(x))}dx - \int_{\partial\mathbb{R}^3_+}\boldsymbol{v}(x)\overline{\left(\frac{\partial\Gamma^*_{21}q}{\partial\boldsymbol{n}_x}(x) + \frac{\partial\Gamma^*_{22}\boldsymbol{f}}{\partial\boldsymbol{n}_x}(x)\right)}dS_x \\
&\quad + \int_{\mathbb{R}^3_+}\mathrm{div}\,\boldsymbol{v}(x)\overline{(\Gamma^*_{11}q(x) + \Gamma^*_{12}\boldsymbol{f}(x))}dx \\
&= \int_{\mathbb{R}^3_+}\boldsymbol{g}(x)\cdot\overline{(\Gamma^*_{21}q(x) + \Gamma^*_{22}\boldsymbol{f}(x))}dx - \int_{\partial\mathbb{R}^3_+}\psi(x)\overline{\left(\frac{\partial\Gamma^*_{21}q}{\partial\boldsymbol{n}_x}(x) + \frac{\partial\Gamma^*_{22}\boldsymbol{f}}{\partial\boldsymbol{n}_x}(x)\right)}dS_x \\
&\quad + \int_{\mathbb{R}^3_+} f(x)\overline{(\Gamma^*_{11}q(x) + \Gamma^*_{12}\boldsymbol{f}(x))}dx \\
&= \left(\int_{\mathbb{R}^3_+}\Gamma_{12}(y,x)\boldsymbol{g}(x)dx - \int_{\partial\mathbb{R}^3_+}\frac{\partial\Gamma_{12}}{\partial\boldsymbol{n}_x}(y,x)\psi(x)dS_x + \int_{\mathbb{R}^3_+}\Gamma_{11}(y,x)f(x)dx, q\right)_{L^2_y} \\
&\quad + \left(\int_{\mathbb{R}^3_+}\Gamma_{22}(y,x)\boldsymbol{g}(x)dx - \int_{\partial\mathbb{R}^3_+}\frac{\partial\Gamma_{22}}{\partial\boldsymbol{n}_x}(y,x)\psi(x)dS_x + \int_{\mathbb{R}^3_+}\Gamma_{21}(y,x)f(x)dx, \boldsymbol{f}\right)_{L^2_y}.
\end{aligned}
$$

We thus obtain

$$
p(x) = \Gamma_{11}f(x) + \Gamma_{12}\boldsymbol{g}(x) - \int_{\partial\mathbb{R}^3_+}\frac{\partial\Gamma_{12}}{\partial\boldsymbol{n}_y}(x,y)\boldsymbol{\psi}(y)dS_y, \tag{4.7}
$$

$$
\boldsymbol{v}(x) = \Gamma_{21}f(x) + \Gamma_{22}\boldsymbol{g}(x) - \int_{\partial\mathbb{R}^3_+}\frac{\partial\Gamma_{22}}{\partial\boldsymbol{n}_y}(x,y)\boldsymbol{\psi}(y)dS_y. \tag{4.8}
$$

Since $\frac{\partial}{\partial\boldsymbol{n}_y} = -\partial_{y_3}$ on $\partial\mathbb{R}^3_+ = \{(y', y_3); y' \in \mathbb{R}^2, y_3 = 0\} = \mathbb{R}^2$, we obtain the solution formula (4.2) from (4.7) and (4.8).

5 Application to the Taylor problem

We consider the Navier-Stokes equation

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0, & (5.1)\\ \partial_t \boldsymbol{v} - \nu \Delta \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \frac{1}{\rho_*} \nabla p &= \boldsymbol{0}, & (5.2) \end{aligned}$$

in a domain Ω_{R_1,R_2} between two concentric cylinders with radii R_1 and R_2, $R_1 < R_2$. In the cylindrical coordinates (r, θ, z), the domain Ω_{R_1,R_2} is represented as

$$\Omega_{R_1,R_2} = \{(r, \theta, z) : \, R_1 < r < R_2, \, \theta \in [0, \, 2\pi], \, z \in \mathbb{R}\},$$

and the velocity field $\boldsymbol{v}$ is given by

$$\boldsymbol{v} = v^r \boldsymbol{e}_r + v^\theta \boldsymbol{e}_\theta + v^z \boldsymbol{e}_z$$

with $\boldsymbol{e}_r = {}^\top(\cos\theta, \sin\theta, 0)$, $\boldsymbol{e}_\theta = {}^\top(-\sin\theta, \cos\theta, 0)$, $\boldsymbol{e}_z = {}^\top(0, 0, 1)$. Here v^r, v^θ and v^z are the r, θ and z-components of $\boldsymbol{v}$, respectively. The inner and outer cylinders rotate with constant angular velocities ω_1 and ω_2, respectively. We assume that $\omega_1 > 0$.

The boundary conditions are given by

$$v^r|_{r=R_1,R_2} = v^z|_{r=R_1,R_2} = 0, \; v^\theta|_{r=R_1} = \omega_1 R_1, \; v^\theta|_{r=R_2} = \omega_2 R_2. \qquad (5.3)$$

We impose a periodic boundary condition in z, i.e.,

$$\boldsymbol{v}, \, p \text{ are } \frac{2\pi}{\alpha_*}\text{-periodic in } z, \qquad (5.4)$$

where $\alpha_* > 0$ is a given wave number.

We rewrite the problem (5.1)–(5.4) in a non-dimensional form. We introduce the following non-dimensional variables:

$$\boldsymbol{x} = d\tilde{\boldsymbol{x}}, \; \boldsymbol{v} = \omega_1 R_1 \tilde{\boldsymbol{v}}, \; t = \frac{d^2}{\nu}\tilde{t}, \; p = \frac{\rho_* \nu \omega_1 R_1}{d}\tilde{p},$$

where $d = R_2 - R_1$. Then (5.1) and (5.2) are transformed into

$$\begin{aligned} \operatorname{div}_{\tilde{x}} \tilde{\boldsymbol{v}} &= 0, & (5.5)\\ \partial_{\tilde{t}} \tilde{\boldsymbol{v}} - \Delta_{\tilde{x}} \tilde{\boldsymbol{v}} + \mathcal{R} \tilde{\boldsymbol{v}} \cdot \nabla_{\tilde{x}} \tilde{\boldsymbol{v}} + \nabla_{\tilde{x}} \tilde{p} &= \boldsymbol{0}, & (5.6) \end{aligned}$$

where $\mathcal{R} = \frac{\omega_1 R_1 d}{\nu}$ is the Reynolds number. The domain Ω_{R_1,R_2} is transformed into Ω:

$$\Omega = \{(\tilde{r}, \tilde{\theta}, \tilde{z}) : \frac{\eta}{1-\eta} < \tilde{r} < \frac{1}{1-\eta}, \ \tilde{\theta} \in [0, 2\pi], \ \tilde{z} \in \mathbb{R}\}.$$

Here $\eta = \frac{R_1}{R_2}$. The boundary conditions (5.3) and (5.4) become

$$\tilde{v}^{\tilde{r}}|_{\tilde{r}=\frac{\eta}{1-\eta},\frac{1}{1-\eta}} = \tilde{v}^{\tilde{z}}|_{\tilde{r}=\frac{\eta}{1-\eta},\frac{1}{1-\eta}} = 0, \ \tilde{v}^{\tilde{\theta}}|_{\tilde{r}=\frac{\eta}{1-\eta}} = 1, \ \tilde{v}^{\tilde{\theta}}|_{\tilde{r}=\frac{1}{1-\eta}} = \frac{\omega}{\eta}, \tag{5.7}$$

where $\omega = \frac{\omega_2}{\omega_1}$, and

$$\tilde{\boldsymbol{v}}, \ \tilde{p} \text{ are } \frac{2\pi}{\alpha}\text{-periodic in } \tilde{z}, \tag{5.8}$$

where $\alpha > 0$ is a wave number. In what follows we omit the tildes " ˜ " in the non-dimensional quantities $\tilde{\boldsymbol{x}}$, $\tilde{t}$, $\tilde{\boldsymbol{v}}$, $\tilde{p}$, and simply write them as $\boldsymbol{x}$, t, $\boldsymbol{v}$, p.

In the cylindrical coordinates (r, θ, z) and $\boldsymbol{v} = v^r \boldsymbol{e}_r + v^\theta \boldsymbol{e}_\theta + v^z \boldsymbol{e}_z$, the problem (5.5)–(5.8) is written as

$$\begin{cases} \operatorname{div} \boldsymbol{v} &= 0, \\ \partial_t v^r + \mathcal{R}[(\boldsymbol{v}\cdot\nabla)v^r - \frac{1}{r}(v^\theta)^2] &= -\partial_r p + (\Delta v^r - \frac{1}{r^2}v^r - \frac{2}{r^2}\partial_\theta v^\theta), \\ \partial_t v^\theta + \mathcal{R}[(\boldsymbol{v}\cdot\nabla)v^\theta + \frac{1}{r}v^r v^\theta] &= -\frac{1}{r}\partial_\theta p + (\Delta v^\theta - \frac{1}{r^2}v^\theta + \frac{2}{r^2}\partial_\theta v^r), \\ \partial_t v^z + \mathcal{R}(\boldsymbol{v}\cdot\nabla)v^z &= -\partial_z p + \Delta v^z, \end{cases}$$

$$v^r|_{r=\frac{\eta}{1-\eta},\frac{1}{1-\eta}} = v^z|_{r=\frac{\eta}{1-\eta},\frac{1}{1-\eta}} = 0, \ v^\theta|_{r=\frac{\eta}{1-\eta}} = 1, \ v^\theta|_{r=\frac{1}{1-\eta}} = \frac{\omega}{\eta},$$

and

$$\boldsymbol{v}, \ p \text{ are } \frac{2\pi}{\alpha}\text{-periodic in } z.$$

Here $\operatorname{div} \boldsymbol{v} = \frac{1}{r}\partial_r(rv^r) + \frac{1}{r}\partial_\theta v^\theta + \partial_z v^z$, $\boldsymbol{v}\cdot\nabla = v^r\partial_r + \frac{1}{r}v^\theta\partial_\theta + v^z\partial_z$ and $\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \partial_z^2$.

This problem has a stationary solution ${}^\top(p_C, \boldsymbol{v}_C)$, the Couette flow, of the form

$$\boldsymbol{v}_C = v_C^\theta(r)\boldsymbol{e}_\theta, \ v_C^\theta(r) = ar + \frac{b}{r},$$

$$p_C = p_C(r) = \mathcal{R}\int^r \frac{\{v_C^\theta(s)\}^2}{s} ds,$$

where

$$a = \frac{\omega - \eta^2}{\eta(1+\eta)}, \ b = \frac{\eta(1-\omega)}{(1-\eta)(1-\eta^2)}.$$

The perturbation ${}^\top(q, \boldsymbol{w}) = (p - p_C, \boldsymbol{v} - \boldsymbol{v}_C)$ is governed by

$$\begin{aligned} \operatorname{div} \boldsymbol{w} = & \ 0, \\ \partial_t \boldsymbol{w} - \Delta \boldsymbol{w} + \mathcal{R}(\boldsymbol{v}_C \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v}_C) + \mathcal{R} \boldsymbol{w} \cdot \nabla \boldsymbol{w} + \nabla q = & \ \boldsymbol{0}, \end{aligned} \tag{5.9}$$

$$\boldsymbol{w}|_{r=\frac{\eta}{1-\eta}, \frac{\eta}{1-\eta}} = \boldsymbol{0}, \tag{5.10}$$

and

$$\boldsymbol{w},\ q \text{ are } \frac{2\pi}{\alpha}\text{-periodic in } z. \tag{5.11}$$

We introduce function spaces. We denote the basic period domain by Ω_α:

$$\Omega_\alpha = \{(r, \theta, z) : \ \frac{\eta}{1-\eta} < r < \frac{1}{1-\eta},\ \theta \in [0, 2\pi],\ -\frac{\pi}{\alpha} < z < \frac{\pi}{\alpha}\}.$$

The symbol C^∞_{per} stands for the space of restrictions to Ω_α of functions in $C^\infty(\overline{\Omega})$ which are $\frac{2\pi}{\alpha}$-periodic in z; and $C^\infty_{0,per}$ denotes the space of restrictions to Ω_α of functions in C^∞ which are $\frac{2\pi}{\alpha}$-periodic in z and vanish near $r = \frac{\eta}{1-\eta}, \frac{1}{1-\eta}$.

We set

$$\begin{aligned} L^2_{per} &= \text{the } L^2(\Omega_\alpha)\text{-closure of } C^\infty_{0,per}, \\ H^k_{per} &= \text{the } H^k(\Omega_\alpha)\text{-closure of } C^\infty_{per}, \\ H^1_{0,per} &= \text{the } H^1(\Omega_\alpha)\text{-closure of } C^\infty_{0,per}. \end{aligned}$$

We note that if $f \in H^1_{0,per}$, then $f|_{z=-\pi/\alpha} = f|_{z=\pi/\alpha}$ and $f|_{r=\frac{\eta}{1-\eta}, \frac{1}{1-\eta}} = 0$. The inner product of $f_j \in L^2_{per}$ $(j = 1, 2)$ is denoted by

$$(f_1, f_2) = \int_{\Omega_\alpha} f_1 \overline{f_2}\, r dr d\theta dz,$$

where $\overline{f}$ denotes the complex conjugate of f. The mean value of a function f over Ω_α is denoted by $\langle f \rangle$:

$$\langle f \rangle = \frac{1}{|\Omega_\alpha|} \int_{\Omega_\alpha} f\, r dr d\theta dz.$$

The set of all $f \in L^2_{per}$ with $\langle f \rangle = 0$ is denoted by $L^2_{per,*}$, i.e.,

$$L^2_{per,*} = \{f \in L^2_{per} : \ \langle f \rangle = 0\}.$$

Furthermore, we set

$$H^k_{per,*} = H^k_{per} \cap L^2_{per,*}.$$

We denote by $C^\infty_{0,per,\sigma}$ the set of all vector fields $\boldsymbol{v}$ in $(C^\infty_{0,per})^3$ with $\operatorname{div} \boldsymbol{v} = 0$. We set

$$L^2_{per,\sigma} = \text{the } L^2(\Omega_\alpha)^3\text{-closure of } C^\infty_{0,per,\sigma}.$$

It is known that $(L^2_{per})^3 = L^2_{per,\sigma} \oplus G^2_{per}$, where $G^2_{per} = \{\nabla p;\, p \in H^1_{per,*}\}$ is the orthogonal complement of $L^2_{per,\sigma}$.

Applying the Helmholtz projection $\mathbb{P}$ to (5.9) we have

$$\frac{d}{dt}\boldsymbol{w} + L_{\mathcal{R},C}\boldsymbol{w} = -\mathcal{R}\mathbb{P}(\boldsymbol{w} \cdot \nabla \boldsymbol{w}).$$

Here $L_{\mathcal{R},C}$ is the linearized operator around $\boldsymbol{v}_C$ which is defined by

$$L_{\mathcal{R},C} :\, L^2_{\sigma,per} \to L^2_{\sigma,per},$$

$$D(L_{\mathcal{R},C}) = (H^2_{per} \cap H^1_{0,per})^3 \cap L^2_{\sigma,per},$$

$$L_{\mathcal{R},C}\boldsymbol{w} = -\mathbb{P}\Delta\boldsymbol{w} + \mathcal{R}\mathbb{P}(\boldsymbol{v}_C \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v}_C) \quad (\boldsymbol{w} \in D(L_{\mathcal{R},C})).$$

The corresponding artificial compressible problem are written as

$$\frac{d}{dt}u + L_{\mathcal{R},C,\epsilon}u = -\mathcal{R}\begin{pmatrix} 0 \\ \boldsymbol{w} \cdot \nabla \boldsymbol{w} \end{pmatrix},$$

where $u = {}^\top(q, \boldsymbol{w})$ and $L_{\mathcal{R},C,\epsilon}$ is the linearized operator around the Couette flow defined by

$$L_{\mathcal{R},C,\epsilon} :\, H^1_{per,*} \times (L^2_{per})^3 \to H^1_{per,*} \times (L^2_{per})^3,$$

$$D(L_{\mathcal{R},C,\epsilon}) = H^1_{per,*} \times (H^2_{per} \cap H^1_{0,per})^3,$$

$$L_{\mathcal{R},C,\epsilon} = \begin{pmatrix} 0 & \frac{1}{\epsilon^2}\operatorname{div} \\ \nabla & -\Delta + \mathcal{R}(\boldsymbol{v}_C \cdot \nabla + {}^\top(\nabla \boldsymbol{v}_C)) \end{pmatrix}.$$

Since the Couette flow and Taylor vortices are axisymmetric, in what follows, we restrict the consideration to axisymmetric functions, namely, we consider vector fields $\boldsymbol{v} = v^r\boldsymbol{e}_r + v^\theta\boldsymbol{e}_\theta + v^z\boldsymbol{e}_z$ and scalar functions q with v^r, v^θ, v^z and q independent of θ. For simplicity, function spaces of axisymmetric ones and the linearized operators restricted to such function spaces are denoted by the same symbols.

The instability of the Couette flow and the occurrence of the formation of Taylor vortex patterns are stated mathematically in the following way. We fix α, ω and η and assume that $\omega \geq 0$.

There exists a critical number $\mathcal{R}_c > 0$ such that when $\mathcal{R} < \mathcal{R}_c$, the Couette flow $\boldsymbol{v}_C$ is asymptotically stable, whereas, when $\mathcal{R} > \mathcal{R}_c$, the Couette flow $\boldsymbol{v}_C$ becomes unstable; in other words, if $\mathcal{R} < \mathcal{R}_c$, then $\rho(-L_{\mathcal{R},C}) \supset \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \geq -b_0\}$ for some positive constant $b_0 = b_0(\mathcal{R})$, whereas, if $\mathcal{R} > \mathcal{R}_c$, then $\sigma(-L_{\mathcal{R},C}) \cap \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda > 0\} \neq \emptyset$.

When $\mathcal{R} > \mathcal{R}_c$, for each $\mathcal{R}$ with $0 < \mathcal{R} - \mathcal{R}_c \ll 1$, there exists an axisymmetric stationary solution (the Taylor vortex) $\boldsymbol{v}_T = \boldsymbol{v}_C + \tilde{\boldsymbol{v}}_T$ with $\partial_z \tilde{\boldsymbol{v}}_T \neq 0$ which bifurcates from $\boldsymbol{v}_C$ at $\mathcal{R} = \mathcal{R}_c$ supercritically. The bifurcating solution is unique for $\mathcal{R} \sim \mathcal{R}_c$ up to translations in z. Furthermore, $\boldsymbol{v}_T$ is orbitally stable, more precisely, the linearized operator $L_{\mathcal{R},T}$ around $\boldsymbol{v}_T$ satisfies $\rho(-L_{\mathcal{R},T}) \supset \{\lambda; \operatorname{Re}\lambda \geq -\tilde{b}_0\} \setminus \{0\}$ for some positive constant $\tilde{b}_0 = \tilde{b}_0(\mathcal{R})$ and 0 is a simple eigenvalue due to the translation invariance in z, the eigenspace is spanned by $\partial_z \boldsymbol{v}_T$.

The existence of the bifurcating branch of Taylor vortices $\boldsymbol{v}_T$ was shown by Velte [21], Iudovich [12], and Kirchgässner and Sorger [16]; and the supercritical bifurcation of $\boldsymbol{v}_T$ and spectral properties of $L_{\mathcal{R},T}$ are shown by numerical computations for a certain range of values of α, η, ω and $\mathcal{R}$ when $\omega \geq 0$ (even when ω is slightly negative); see the book [4] by Chossat and Iooss. Hereafter we assume that the above mentioned instability of $\boldsymbol{v}_C$, supercritical bifurcation of $\boldsymbol{v}_T$ and the spectral properties of $L_{\mathcal{R},T}$ hold true.

A direct computation gives $\operatorname{Re}((\mathbb{Q}\boldsymbol{w}) \cdot \nabla \boldsymbol{v}_C, \mathbb{Q}\boldsymbol{w}) = 0$ for axisymmetric $\boldsymbol{w}$, i.e., $\boldsymbol{w} = w^r(r,z)\boldsymbol{e}_r + w^\theta(r,z)\boldsymbol{e}_\theta + w^z(r,z)\boldsymbol{e}_z$, since $\mathbb{Q}\boldsymbol{w} = \partial_r\varphi(r,z)\boldsymbol{e}_r + \partial_z\varphi(r,z)\boldsymbol{e}_z$ for such $\boldsymbol{w}$ and $\boldsymbol{v}_C = v_C^\theta(r)\boldsymbol{e}_\theta$. Applying Theorem 2.1 and the instability result [15, Theorem 3.2] to $L_{\mathcal{R},C}$, we have the following result.

Theorem 5.1. *If $\mathcal{R} < \mathcal{R}_c$, then $\rho(-L_{\mathcal{R},C,\epsilon}) \supset \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \geq -b_1\}$ with some positive constant $b_1 = b_1(\mathcal{R})$ for sufficiently small $\epsilon > 0$; and if $\mathcal{R} > \mathcal{R}_c$, then $\sigma(-L_{\mathcal{R},C,\epsilon}) \cap \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda > 0\} \neq \emptyset$ for sufficiently small $\epsilon > 0$.*

We next consider the stability of the Taylor vortex $\boldsymbol{v}_T$. Since $\boldsymbol{v}_T = \boldsymbol{v}_C + \tilde{\boldsymbol{v}}_T$ bifurcates from $\boldsymbol{v}_C$ at $\mathcal{R}_c$, we see that $\|\nabla \tilde{\boldsymbol{v}}_T\|_\infty \to 0$ as $\mathcal{R} \to \mathcal{R}_c$. We then find that

$$\frac{\operatorname{Re}((\mathbb{Q}\boldsymbol{w}) \cdot \nabla \boldsymbol{v}_T, \mathbb{Q}\boldsymbol{w})}{\|\nabla \mathbb{Q}\boldsymbol{w}\|_2^2} = \frac{\operatorname{Re}((\mathbb{Q}\boldsymbol{w}) \cdot \nabla \tilde{\boldsymbol{v}}_T, \mathbb{Q}\boldsymbol{w})}{\|\nabla \mathbb{Q}\boldsymbol{w}\|_2^2} \geq -C\|\nabla \tilde{\boldsymbol{v}}_T\|_\infty \geq -\delta_0$$

for $\mathcal{R}$, $0 < \mathcal{R} - \mathcal{R}_c \ll 1$. We apply Theorem 2.1 (cf., Remark 2.2) to obtain the following result.

Theorem 5.2. *If $\mathcal{R} > \mathcal{R}_c$ and $\mathcal{R} - \mathcal{R}_c \ll 1$, then $\rho(-L_{\mathcal{R},T,\epsilon}) \supset \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \geq -\tilde{b}_1\} \setminus \{0\}$ with some positive constant $\tilde{b}_1 = \tilde{b}_1(\mathcal{R})$ for sufficiently small $\epsilon > 0$ and 0 is a simple eigenvalue with $\mathrm{Ker}\,(-L_{\mathcal{R},T,\epsilon}) = \mathrm{span}\,\{\partial_z u_T\}$. Here $u_T = {}^\top(p_T, \boldsymbol{v}_T)$ with p_T being the pressure of the Taylor vortex.*

Acknowledgements. Y. Kagei was partly supported by JSPS KAKENHI Grant Number 24340028, 15K13449, 24224003, 16H03947; T. Nishida was partly supported by JSPS KAKENHI Grant Number 26400163.

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