THE MATHEMATICAL PUZZLE OF HYPERSONIC FLOW FIELDS: FROM MEASURE-THEORETIC SINGULARITIES TO ISENTROPIC CHARACTERISTICS

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ABSTRACT. This report outlines the mathematical and physical challenges of calculating aerodynamic forces on vehicles in the extreme hypersonic limit ($M \to \infty$). By dissecting the flow field into two distinct physical regimes—the violent compression on the windward surface and the continuous expansion on the leeward surface. We demonstrate how Radon measure solutions resolve the singularity of infinitely thin shock layers, and how to construct approximate solutions by incorporating the strong rarefaction waves and bending walls. Prandtl-Meyer expansion waves.

1. INTRODUCTION

Imagine a vehicle tearing through the atmosphere at an extremely high Mach number ($M \to \infty$). A fundamental engineering question arises: how can we precisely calculate the global lift and drag forces it experiences?

This is a monumental challenge that can only be unlocked through the pure mathematical lens of two-dimensional steady compressible Euler equations. When observing a hypersonic vehicle, we find that its upper and lower surfaces face entirely different physical fates: the windward surface undergoes violent gas compression, while the leeward surface allows the flow to expand and escape. To compute the global aerodynamic forces, we must trace the historical footprints of mathematical fluid dynamics and find the specific tools capable of taming these two extreme physical phenomena.

Let us rigorously formulate the problem. Two-dimensional steady non-isentropic compressible Euler flows are generally governed by the system:

$$\begin{cases} \partial_x(\rho u) + \partial_y(\rho v) = 0, \\ \partial_x(\rho u^2 + p) + \partial_y(\rho uv) = 0, \\ \partial_x(\rho uv) + \partial_y(\rho v^2 + p) = 0, \\ \partial_x(\rho uE) + \partial_y(\rho vE) = 0, \end{cases} \tag{1.1}$$

where the unknowns ρ, E, and (u, v) represent the fluid density, total enthalpy per unit mass, and velocity field, respectively; p is the scalar pressure. The constitutive

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relation (equation of state) is given by

$$p = \frac{\gamma - 1}{\gamma} \rho \cdot \big(E - \frac{1}{2}(u^2 + v^2)\big), \tag{1.2}$$

where $\gamma > 1$ is the adiabatic exponent of a polytropic gas. Let $c \doteq \sqrt{\gamma p/\rho}$ be the local sound speed, and $M \doteq \sqrt{u^2 + v^2}/c$ be the Mach number of the flow. For $M > 1$, the flow is supersonic, and it is well known that system (1.1)–(1.2) then becomes a hyperbolic system of conservation laws.

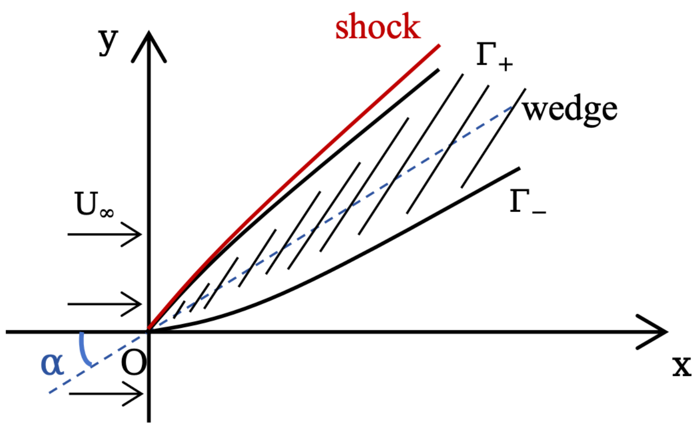

FIGURE 1. Steady Euler flows over a two-dimensional straight wedge with angle of attack α.

This configuration poses an initial-boundary value problem for a hyperbolic system, featuring a strong rarefaction wave on the lower side and an oblique shock wave on the upper side. We study the flow in the upper domain Ω_+ bounded by Γ_+ and the positive y-axis (see Figure 1) within the framework of Radon measure solutions. Conversely, in the lower domain Ω_- bounded by Γ_- and the negative y-axis (see Figure 1), the flow is analyzed by constructing approximate solutions. The flow satisfies the following slip boundary condition:

$$(u, v) \cdot \mathbf{n}_\pm = 0 \quad \text{on } \Gamma_\pm, \tag{1.3}$$

where $\mathbf{n}_\pm$ is the unit normal vector on $\Gamma_\pm$ pointing into $\Omega_\pm$.

The uniform oncoming flow state

$$U_\infty = (\rho_\infty, u_\infty, 0, E_\infty)^\top \tag{1.4}$$

is assumed to be supersonic, satisfying

$$p_\infty = \frac{\gamma - 1}{2\gamma} \rho_\infty (2E_\infty - u_\infty^2). \tag{1.5}$$

Main Problem: For the oncoming flow U_∞ given by (1.4), find a solution to (1.1)–(1.2) in the domain $\Omega_+^{curve} := \{(x, y) \in \mathbb{R}^2,\ x \geq 0,\ y > g^+(x)\} \cup \Omega_-^{curve} := \{(x, y) \in \mathbb{R}^2,\ x \geq 0,\ y < g^-(x)\}$ subject to the slip boundary condition (1.3), where

$\Gamma^+ = \{(x, y),\ x \geq 0,\ y = g^+(x)\}$ and $\Gamma^- = \{(x, y),\ x \geq 0,\ y = g^-(x)\}$ denote the upper and lower boundaries, respectively.

2. The Windward Surface: the Radon Measure Framework

Following the impact of the incoming flow, we first encounter the shock wave on the windward surface.

In the study of hypersonic flow passing an obstacle, the primary challenge lies in the appearance of special singularities within the flow field around the obstacle when the Mach number M_∞ of oncoming flow is extremely high. These singularities make classical shock wave theory insufficient to describe the flow field. In fact, it has been observed that when M_∞ approaches infinity, the shock surface becomes very close to the obstacle's surface (i.e., infinite-thin shock layer appears). Consequently, the mass, momentum and energy concentrate on the upwind boundary of the obstacle, causing the state quantities (such as mass density) to become infinite, which means the mass density is no longer a Lebesgue measurable function. Classical integrable weak solutions therefore fail to depict the hypersonic-limit flow field correctly, and one needs to introduce the notion of measure solutions (e.g., Radon measure solutions) to characterize the infinite-thin shock layers generated within hypersonic-limit flow. Then, it is natural to study the hypersonic similarity law within the framework of Radon measure solutions.

Under this framework, the spatial singularities are perfectly justified. By solving a differential system containing Dirac measures, we elegantly derive the wall pressure on the windward surface. This result is not only mathematically self-consistent but also coincides exactly with the renowned **Newtonian-Busemann pressure law** in engineering.(see more in a series work of Prof.Hairong Yuan)

This is the first piece of our mathematical puzzle: taming the singularities of hypersonic flows using measure theory.

3. Leeward Surface: Methodologies for Expansion Waves

Turning our attention to the leeward surface, the physical landscape alters drastically. As the flow passes over a convex curved wall, there are no violent collisions; instead, a continuously diverging Prandtl-Meyer expansion fan is generated.

Historically, this appears to be a fully resolved classical problem. Within this purely isentropic and irrotational simple wave region, the **Riemann invariants** are strictly conserved along characteristic lines (Mach lines). The flow deflection angle θ is tightly bound to the Prandtl-Meyer function $\nu(M)$.

We outline two schemes to calculate the pressure variations across such pure rarefaction waves.

3.1. **Scheme A:.** This scheme is highly efficient for pure isentropic expansion flows over a continuously smooth convex wall $y = g^{-}(x)$. The flow properties are governed by the Riemann invariant, which tightly couples the flow deflection angle θ and the Prandtl-Meyer function $\nu(M)$:

$$\nu(M(x)) - \theta(x) = \nu(M_{\infty}) - \theta_{\infty} \equiv \text{Constant}. \tag{3.1}$$

where $\theta(x) = \arctan\big((g^{-})'(x)\big)$.

The static pressure $p_{-}(x)$ on the leeward wall is explicitly computed using the isentropic relations, ensuring continuous depressurization:

$$p_{-}(x) = p_{\infty}\left[\frac{1 + \frac{\gamma-1}{2}M_{\infty}^{2}}{1 + \frac{\gamma-1}{2}M(x)^{2}}\right]^{\frac{\gamma}{\gamma-1}}. \tag{3.2}$$

3.2. **Scheme B.** For arbitrary convex profiles or numerical computational implementations, the continuous boundary is discretized into N short straight-line segments. At each convex corner k, the wall abruptly turns by a small discrete angle $\Delta\theta_k > 0$, generating a local centered rarefaction fan.

By solving local Riemann problems, the change in velocity magnitude ΔV is evaluated via the differential compatibility equation:

$$\Delta\theta_k \approx \sqrt{M_{k-1}^{2} - 1}\frac{\Delta V}{V_{k-1}}. \tag{3.3}$$

Since the total enthalpy and entropy are globally conserved, the numerical marching algorithm strictly traces the propagation of discrete expansion waves without introducing artificial numerical dissipation, precluding unphysical shock formations.

4. Conclusion and Engineering Implications

This research successfully reconciles theoretical mathematical solutions with practical engineering models. By combining the Radon measure solutions for the windward shock layers with the characteristic solutions for the leeward expansion waves, we obtain the complete pressure distribution encircling the wedge. The global lift (L) and drag (D) forces can then be accurately evaluated.