Mathematical Quantum Field Theory and Renormalization Theory

The Nishijin Plaza of Kyushu University, Fukuoka, Japan

November 26 2009 —November 29 2009

Preface

This volume of Math-for-Industry Lecture Note Series is dedicated to Professor Izumi Ojima and Professor Kei-ichi Ito on the occasion of their sixtieth birthdays.

Professor Izumi Ojima and Professor Kei-ichi Ito have organized a lot of interesting and advanced conferences, e.g., RIMS conference, on quantum field theory and related topics, and they have encouraged not only young but also senior scientists. We would like to express our hearty gratitude to Professor Izumi Ojima and Professor Kei-ichi Ito for their continuous encouragement to us, stimulating our works, innumerable, unbounded helpful comments to our scientific researches.

This lecture note is collecting several research papers and survey articles contributed by invited speakers of the international conference

Mathematical Quantum Field Theory and Renormalization Theory

held from 26th to 29th, November 2009 at Nishijin Plaza of Kyushu university. Twenty invited speakers including four overseas researchers gave talks and the conference had about 50 participants.

The mathematical analysis of quantum theory and related topics has been largely developed since its foundation, and mathematics itself has been also developed by quantum physics. The organizing committee of the conference considered that it is a good opportunity to organize an international conference in the occasion of sixtieth birthdays of Professor Izumi Ojima and Professor Kei-ichi Ito and the conference was planed to make a bridge between quantum physics and pure mathematics.

A purpose of this conference was also to provide a forum for the discussion of the latest development of mathematical tools used in quantum physics, and then the topics talked in this conference were in particular selected within the purely mathematical research on quantum theory. Twenty invited speakers gave attractive and splendid lectures on their latest researches, and a lot of stimulated discussion on the talks were done.

The conference highlighted some purely mathematical aspects on quantum field theory and renormalization, and were focused on the following topics:

- Operator algebra
- Stochastic analysis
- Rigorous renormalization theory
- Quantum probability
- Spectral analysis and operator theory

We hope that all the participants, including speakers, Professor Izumi Ojima and Professor Kei-ichi Ito, enjoyed this conference and found new discovery, and also that this lecture note is presenting the latest research bringing further development on pure mathematics in quantum physics.

This conference is financially supported by

- Global COE Program-Kyushu University Education and Research Hub for Mathematics - for - Industry Leader: Prof Masato Wakayama
- JSPS Grant-in-Aid for Scientific Research (A) 19204015, Yasuyuki Kawahigashi
- JSPS Grant-in-Aid for Scientific Research (B) 20340032, Fumio Hiroshima

We also would like to express our gratitude to the secretarial staffs of Global COE program for their helping in editing this lecture note.

Takashi Hara
Taku Matsui
Fumio Hiroshima
Fukuoka in December, 2010

P r o g r a m

Nov. 26 (Thu)

- 14:20-15:10 Hal Tasaki (Gakushuin)

Origin of ferromagnetism--- A ``constructive condensed matter physics'' approach

Tea Break

- 15:40-16:30 Ryo Harada (Kyoto)

A unified scheme of measurement and amplification processes

- 16:40-17:30 Herbert Spohn (München)

The retarded van der Waals potential

Nov. 27 (Fri)

- 10:00-10:30 Taku Matsui (Kyushu)

Factorization Lemmas of Hastings and Split Property

- 10:35-11:25 Hiroshi Tamura (Kanazawa)

Random point fields, random measures and Bose-Einstein condensation

Tea Break

- 11:50-12:40 Yasuyuki Kawahigashi (Tokyo)

Superconformal field theory, operator algebras and noncommutative geometry

Lunch Break

- 14:20-15:10 Martin Porrmann (KwaZulu-Natal)

Local Causal Structures-Relating Quantum Field Theories on Different Spacetime Backgrounds

Tea Break

- 15:40-16:30 Tetsuya Hattori (Keio)

Where is my book? --- Burgers equation in an online bookstore ranking

- 16:40-17:30 Raymond Streater (London)

A Theory of Scattering Based on Free Fields

Nov. 28 (Sat)

- 10:00-10:30 Fumio Hiroshima (Kyushu)

Relativistic Pauli-Fierz model in QED by path measures

- 10:35-11:25 Shigeki Aida (Osaka)

Semi-classical limit of the lowest eigenvalue of $P(\Phi)_2$ Hamiltonian on a finite interval

Tea Break

- 11:50-12:40 Akito Suzuki (Kyushu)

Existence and absence of ground state on a pseudo Riemannian manifold

Lunch Break

- 14:20-15:10 Asao Arai (Hokkaido)

Representations of Quantum Phase Spaces

Tea Break

- 15:40-16:30 Masao Hirokawa (Okayama)

Have fun exploring circuit QED with non-commutative oscillators –From mathematics to experimental physics

- 16:40-17:30 Hayato Saigo (Kyoto)

On Generalized Cumulants

Nov. 29 (Sun)

- 10:00-10:50 Sumio Watanabe (Tokyo IT)

A Singular Limit Theorem in Statistical Learning Theory

Tea Break

- 11:00-11:50 Nobuaki Obata (Tohoku)

Quantum White Noise Derivatives and Implementation Problem

- 12:00-12:30 Hajime Moriya (Shibaura)

On supersymmetric states in C^* -systems

Lunch Break

- 13:50-14:40 Takashi Hara (Kyushu)

Critical Behaviour of Stochastic Geometric Models and the Lace Expansion

- 14:50-15:40 Erhard Seiler (Munchen)

The Strange World of Non-amenable Symmetries

Celebrating Professor K. R. Ito and Professor I. Ojima on their 60th birthdays

Tetsuya Hattori (Keio Univ.)

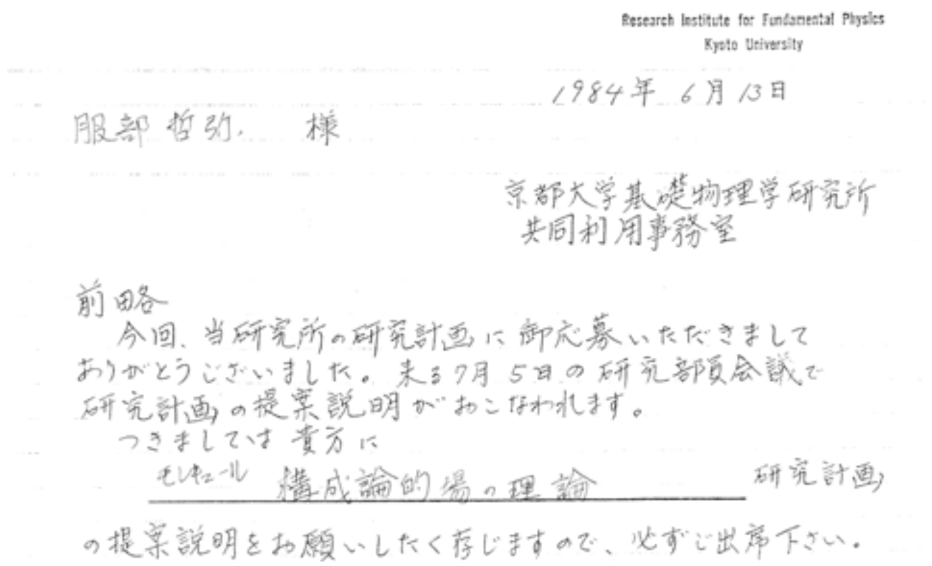
As a piece in the proceedings to the International conference at Kyushu University, Nov. 2009, held in honor of Professor K. R. Ito and Professor I. Ojima, celebrating their 60th birthdays, I would like to leave here a short personal note of how I came to know Professor K. R. Ito and Professor I. Ojima, a quarter of century ago.

Though I found that there are lots of potentially important historical background subjects, both social and scientific, necessary for present (or future) day young readers to understand why certain things could happen in a way recorded here, I gave up going into any depth. I apologize if this note is not clear or if there are misunderstanding on my side, in what follows.

Professor Kei-ichi R. Ito

In the year 1984, in the fourth year out of five years for my graduate study, I was among a group of a few graduate students who were making a research proposal to RIFP, Research Institute for Fundamental Physics, Kyoto University, now officially changed its English name to Yukawa Institute for Theoretical Physics. RIFP was then located at the next building to RIMS, the Research Institute for Mathematical Sciences, where Prof. Ito finished his graduate study.

RIFP had, and as I understand still has, a flexible fund for small size research proposals in physics. This meant something special in 20th century. At those time, Japan was about to have its best time in economy, but national budget for fundamental sciences had still been suppressed, and lack of flexibility made it even difficult to have official financial aid for graduate students. RIFP had been working hard on this problem, and was taking all possible measures to give small financial aid to encourage young physicists and improve their research environment. It was one of such funds from RIFP that we were applying in 1984.



Research Institute for Fundamental Physics
Kyoto University

1984年 6月 13日

服部哲弘　様

京都大学基礎物理学研究所
共同利用事務室

前略
今回、当研究所の研究計画に御応募いただきましてありがとうございました。来る7月5日の研究部員会議で研究計画の提案説明がおこなわれます。
つきましては貴方に
モレキュール構成論的場の理論　研究計画の提案説明をお願いしたく存じますので、必ずご出席下さい。

It was a coincidence that Prof. Ito returned to Japan from Europe in the same year 1984. As the budget situation for research was bad at those times in Japan, so was the job situation desperate for graduate students in physics. Japan was probably so 'advanced' a country in this problem, that we already long had had a Japanese-English word to describe the situation, the 'over-doctor' problem; the problem of large portion of PhDs without research job positions.

After completing graduate course program at RIMS, Prof. Ito remained at RIMS as a JSPS (Japan Society for the Promotion of Science) research fellow and then as a research fellow of the elemetary particle group Japan, before finding a three years SERC fellowship position at Bedford College, London University, where he stayed until 1983, and after visiting ZiF, Bielefeld University, for a year, he returned to Japan. He had to survive with a temporal position at Kyoto University, before he finally succeeded in finding a permanent position at Konan college, where he stayed from 1987 to 1993. He then moved to Setsunan University, where he is now.

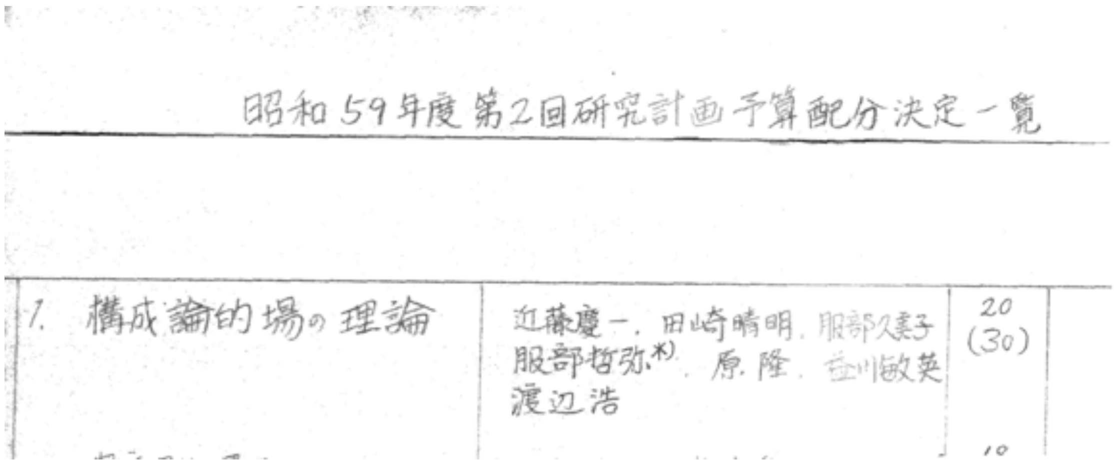
A main 'social' reason for the difficulty in winning permanent academic positions at those times was the 'over-doctor' problem. I would say that there must have been another reason for Prof. Ito's situation. At those time in Japan, a mathematical treatment of quantum field theory and statistical mechanics, or a study in quantum field theory and statistical mechanics as mathematical physics, which is Prof. Ito's main research concern to date, was thought to be mathematics from physicists, and physics from mathematicians. Few people could appreciate the importance of such studies, and very little number of academic positions could be expected for such approaches. I am confident about this, because a few year later, at around 1987 or 1988, I was personally warned from a very prominent senior professor, with such a strong word as 'You are strangling yourself'.

In fact, the situation partly motivated us in applying for a fund at RIFP. Though still graduate students, we somehow felt that not many physicists in Japan would like to be mathematically serious, and therefore, we felt a need to make an appeal to physics society, that we are interested in mathematical physics approach to quantum field theory and statistical mechanics. A particular topic which excited us then were the new approaches brought by people such as J. Fröhlich and M. Aizenman. The new ideas were to represent (Euclidean) quantum fields as random geometrical objects and use mathematically rigorous inequalities to prove physical intuitions on these random objects mathematically, resulting in interesting conclusions from quantum field theoretical point of view, such as the 'triviality' (non-renormalizability) of interacting scalar quantum field theories.

I wrote above that I felt an atmosphere in physics society against serious mathematical studies of physics. I should however add that scientific information was not blocked. In fact, the new results of J. Fröhlich and M. Aizenman were brought quickly to Japan, and the graduate students could know such latest results, even though the mathematical approach had not been appreciated widely. (This is not at all trivial, because we had another decade to go before the internet and web to prevail.) Scientists willingly accept information on new or even exotic ideas, which is good. Scientists are perhaps more conservative in evaluation, which may be reasonable, if it is not biased too much.

At the time we were applying to the RIFP funds, I did not know Prof. Ito: Much less did I know that he was returning to Japan. Anyway, on 5th July 1984, I attended a meeting at RIFP for proposal explanations, and explained our proposal with subject title 'Constructive quantum field theory'.

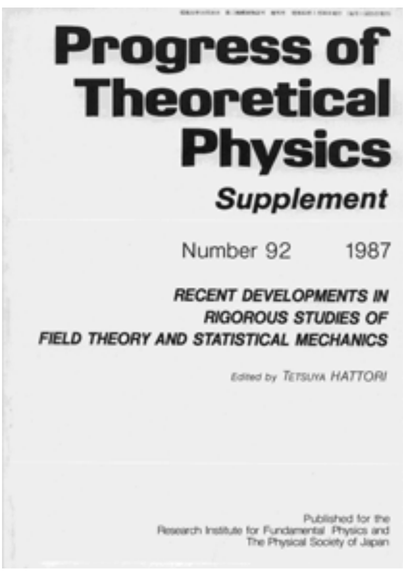
Guessing that the subject would not be welcome, and also having no senior professors joining the application, I anticipated that the proposal would be rejected, or at least, budget would be largely reduced. To my great surprise, the proposal was accepted with great encouragement, and with smallest decrease in rate among the proposals.


昭和59年度第2回研究計画予算配分決定一覧

1. 構成論的場の理論	近藤慶一, 田崎晴明, 服部久美子 服部哲弥*), 原隆, 益川敏英 渡辺浩	20 (30)

Prof. T. Maskawa, a Nobel Laureate of 2008, who was at RIFP, apparently supported our proposal greatly. It turned out that he independently knew the works of J. Fröhlich and was interested in these approaches. The document says that he even ended up in joining our proposal as a member, though I don't remember how this was possible. (I think he was on the board of committee at RIFP, judging the proposals. Perhaps things were very flexible at those times.)

Approved as a RIFP project, we held a meeting at RIFP right away, which Prof. Ito noticed and, he reached us at the meeting room. I remember a tall person coming into the meeting room, with lively smile, as if he knew us for a long time. That was how I met him for the first time.



We applied successfully to the RIFP fund again the next year, this time with Prof. Ito in the list from the beginning. As an output of our two years activities,

昭和60年度第２回

研究計画提案申請書　せつめーつき！

昭和60年5月27日

京都大学基礎物理学研究所長　殿

所　属　東京大学　理学部　物理学教室
（鈴木 研）

氏　名　田崎　晴明

下記のとおり共同利用研究計画を提案します。

記

イ）研究テーマ　構成的場の理論および厳密統計力学

ロ）種　　別　　1.　短期研究計画　　2.　長期研究計画
（○でかこむ）
③　モレキュール型研究計画　　4.　その他

ハ）開催希望時期・日数　'85年10月, '86年2月, 6日 各2,3日　　参加予定者数　10人程度

ニ）予算希望額　　第１希望　L・M・S・SS　　第２希望　L・M・S・SS
（○でかこむ）

ホ）世話人及び提案説明者

氏　名	所属部局名	職　名	氏　名	所属部局名	職　名
伊東恵一	京大・教	非常勤講師	益川敏英	京大・基研	教授
近藤慶一	名大・理	D.C.	渡辺浩	東都大・理	D.C.
鈴木増雄	東大・理	教授			
田崎晴明	〃	D.C.			
服部恭子	〃	D.C.			
服部哲弥	学習院大・理	助手			
原隆	東大・教	D.C.			

連絡責任者氏名　田崎　晴明　　提案説明者氏名　服部　哲弥

せつめいちゅう

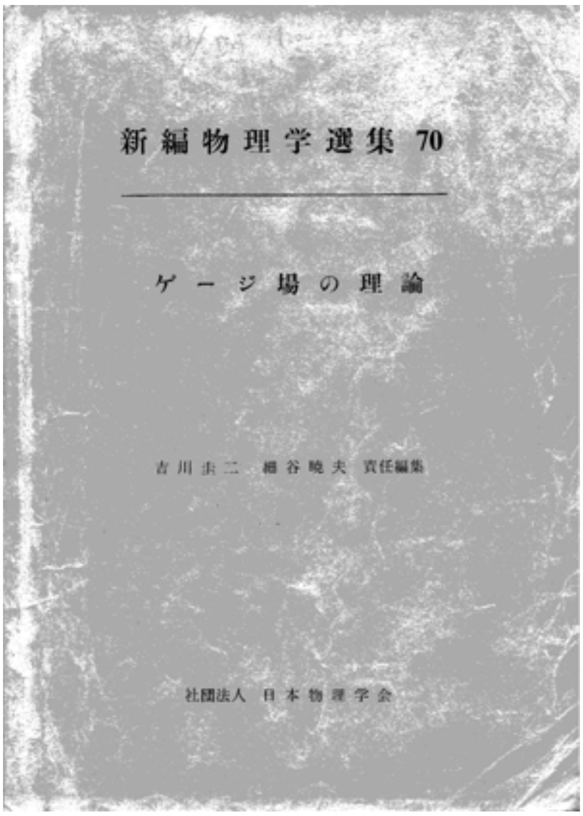
we were invited to publish a volume of Progress of Theoretical Physics Supplement, a review journal published by RIFP. Of course, Prof. K. Ito contributes a paper in the volume: K. R. Ito, 'Renormalization group methods on hierarchical lattices and beyond', Progress of Theoretical Physics Supplement **92** (1987) 46–71.

Prof. Ito has continued to study quantum field theory and statistical mechanics as mathematical physics. In 1970s his main concern was in quantum electro dynamics in 2 space-time dimensions, in 1980s he turned to the renormalization group theories, and in 1990s he focused more on polymer expansions.

Besides these original studies, he continues to organize a series of RIMS Symposium 'Applications of Renormalization Group Methods in Mathematical Sciences' starting in 1999 and held every two years (`http://www.setsunan.ac.jp/mpg/`). In this series of symposium, Prof. Ito invites foreign speakers who he finds at meetings abroad. I suspect that he intentionally does this, as a responsibility of a senior leader of the field in Japan, to keep introducing to Japan up-to-date research progress of the rest of the world, to stimulate younger generation, and hopefully, persuade them to go beyond.

Professor Izumi Ojima

The educational background and job situation for Prof. Ojima is very different from those for most of us. The way how I came to know the name is also very different from how I came to know Prof. Ito.

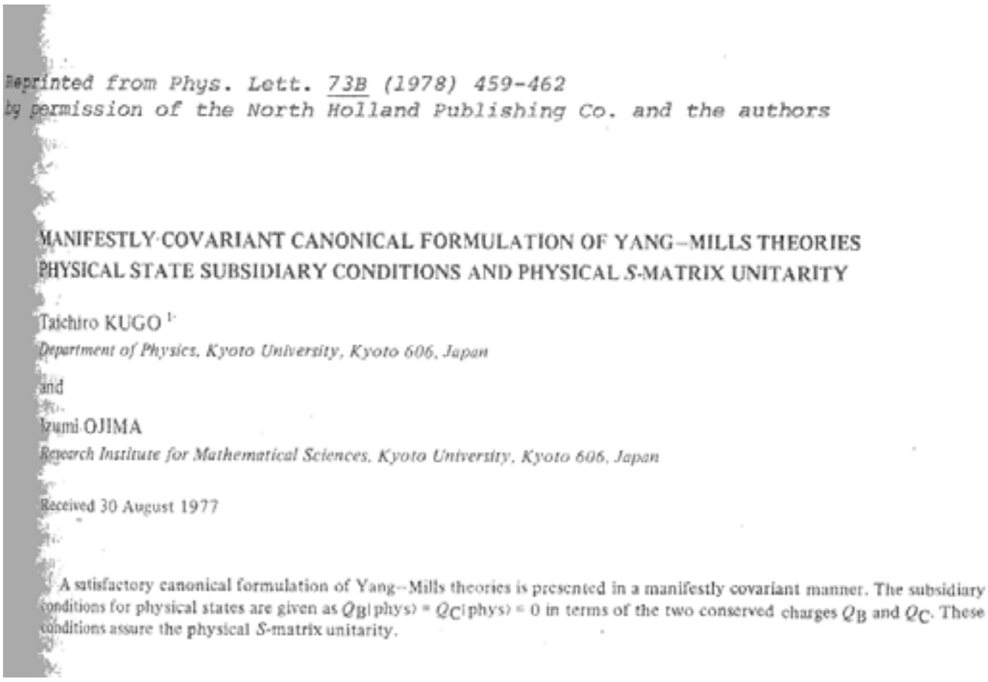


Prof. Ojima graduated Faculty of Medicine, Kyoto University, and has a doctor's licence. He however did not choose the field of medicine as his professional career.

Instead he went to Graduate school of Science, Kyoto University, and received his PhD in 1980.

Prof. Ojima won Nishina Prize in 1980 for a joint study with Prof. T. Kugo: 'Theory of covariant quantization of non-abelian gauge fields', which was accomplished in their graduate studies. I learned their name instantly as I became a graduate student in 1980. In the weekly seminar for graduate students, my thesis adviser chose, as a textbook for the seminar, a volume in the series of collected papers in physics, edited by the Physical Society of Japan.

Volume 70 of the 'new series' is devoted to the theory of gauge fields, and in the volume, a summary of Kugo and Ojima's results published in Physics Letters 73B is included.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MANIFESTLY COVARIANT CANONICAL FORMULATION OF YANG–MILLS THEORIES
PHYSICAL STATE SUBSIDIARY CONDITIONS AND PHYSICAL S-MATRIX UNITARITY

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and

Izumi OJIMA
Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan

Received 30 August 1977

A satisfactory canonical formulation of Yang–Mills theories is presented in a manifestly covariant manner. The subsidiary conditions for physical states are given as $Q_B|\mathrm{phys}\rangle = Q_C|\mathrm{phys}\rangle = 0$ in terms of the two conserved charges Q_B and Q_C. These conditions assure the physical S-matrix unitarity.

Nishina Prize was then the only prize in physics society in Japan, so winning the prize meant being noticed by all the physicists at the time in Japan. It is no wonder that after being in a research position at Princeton IAS for a year, Prof. Ojima won a permanent position at RIMS in 1981, where he is now.

I hear that, nowadays, in the applications for job positions or for promotions especially in smaller universities, one has to compete with candidates from other fields of study of very wide range, and also has to explain one's academic accomplishments. Such a kind of social pressure resulted in creating more and more prizes in physics, with a reason that it makes it easier to explain to non-specialists. In contrast, in the good old days, Japanese physics society preferred to doubt authority, and moreover, kept the physics society itself from being an authority. I heard that it was from these basic idea that the Japanese physics society kept the number of prizes to minimum, in those days when Prof. Ojima won Nishina Prize.

The study of covariant quantization of non-abelian gauge fields, for which Prof. Kugo and Prof. Ojima won Nishina Prize, gives a clear algebraic structure of the reason why the physical states of non-abelian gauge theories, such as QCD, are positive metric, in spite of the fact that in the covariant formulations of the theories, negative metric states are inevitable. The problem was known as the unitarity problem of

$$(P^{(n)})^2 = P^{(n)} = P^{(n)\dagger} \,, \tag{11}$$

$$P^{(n)}P^{(m)} = P^{(m)}P^{(n)} = \delta_{mn} P^{(n)} \,, \tag{12}$$

where $P^{(0)}$ is defined as the projection operator onto the Hilbert subspace $\mathcal{H}_{\text{phys}}$ consisting solely of the physical particles. The eqs. (11) and (12) indicate the following *orthogonal* decomposition of the total space $\mathcal{V}$:

$$\mathcal{V} = \mathcal{H}_{\text{phys}} \oplus \bigoplus_{n \geq 1} (P^{(n)}\mathcal{V}) \,. \tag{13}$$

Next we turn into the most important problem, that is, the subsidiary conditions for selecting the physical state subspace $\mathcal{V}_{\text{phys}}$. We impose two subsidiary conditions:

$$Q_{\text{B}}|\text{phys}\rangle = 0 \,, \tag{14}$$

$$Q_{\text{c}}|\text{phys}\rangle = 0 \,. \tag{15}$$

By the conservation of these charges, the second of the two physical state conditions (1) follows trivially: $\mathcal{V}^{\text{in}}_{\text{phys}} = \mathcal{V}^{\text{out}}_{\text{phys}}$. In what follows we only have to prove the first of conditions (1), the positive semi-definiteness of the metric in $\mathcal{V}_{\text{phys}}$.

A rather lengthy proof using the GLZ formula [8] gives expressions for Q_{B} and Q_{c}:

$$Q_{\text{B}} = \int \mathrm{d}^3x \,:B^{\text{as}}\overleftrightarrow{\partial}_0 c^{\text{as}}: = \mathrm{i} \sum_k (c_k^\dagger B_k - B_k^\dagger c_k) \,, \tag{16}$$

$$Q_{\text{c}} = \int \mathrm{d}^3x \,:\bar{c}^{\text{as}}\overleftrightarrow{\partial}_0 c^{\text{as}}: = -\mathrm{i} \sum_k (c_k^\dagger \bar{c}_k + \bar{c}_k^\dagger c_k) \,, \tag{17}$$

in terms of the asymptotic (in or out) fields, where $Q_{\text{B}} = \tilde{Z}_3^{1/2} Z_{\text{B}}^{-1/2} Q_{\text{B}}^0$. The assumption of asymptotic completeness and the WT identities as results of symmetries generated by Q_{B} and Q_{c}, are the essential ingredients to prove eqs. (16) and (17). Eqs. (16) and (9) say

$$[Q_{\text{B}}, P^{(0)}] = 0 \,, \tag{18}$$

$$Q_{\text{B}}^2 = 0 \,, \tag{19}$$

$$\begin{aligned} &[Q_{\text{B}}, B_k] = \{Q_{\text{B}}, c_k\} = 0 \,, \\ &[Q_{\text{B}}, \chi_k] = -\mathrm{i}c_k \,, \quad \{Q_{\text{B}}, \bar{c}_k\} = \mathrm{i}B_k \,, \end{aligned} \tag{20}$$

and their hermitian conjugates. The physical subspace $\mathcal{V}_{\text{phys}}$, including the Hilbert subspace $\mathcal{H}_{\text{phys}}$ by eq. (18), is orthogonally decomposed corresponding to eq. (13) as

$$\mathcal{V}_{\text{phys}} = \mathcal{H}_{\text{phys}} \oplus \mathcal{V}_0 \,, \quad \mathcal{V}_0 \equiv \bigoplus_{n \geq 1} (\mathcal{V}_{\text{phys}} \cap P^{(n)}\mathcal{V}) . \tag{21}$$

Therefore, the positive semi-definiteness of metric in $\mathcal{V}_{\text{phys}}$ can be shown by proving the zero-norm property of $\mathcal{V}_0$. This is proved by using only the subsidiary condition (14). Noting eq. (18), it is an easy task to show inductively

$$[Q_{\text{B}}, P^{(n)}] = 0 \,, \quad n = 0, 1, 2, \ldots, \tag{22}$$

by the help of eqs. (10) and (20). Next, let $|f_m\rangle$ and $|g_n\rangle$ be arbitrary states containing m and n unphysical particles, respectively, and satisfying the subsidiary condition (14): $Q_{\text{B}}|f_m\rangle = Q_{\text{B}}|g_n\rangle = 0$. Then, by using this and eq.(20),

$$\begin{aligned} \langle f_m|g_n\rangle &= \langle f_m|P^{(n)}|g_n\rangle \\ &= (1/n) \sum_k \langle f_m| -\mathrm{i}\bar{c}_k^\dagger [Q_{\text{B}}, P^{(n-1)}]\chi_k \\ &\quad -\mathrm{i}\chi_k^\dagger [Q_{\text{B}}, P^{(n-1)}]\bar{c}_k|g_n\rangle \\ &\quad - (1/n) \sum_k \langle f_m|\bar{c}_k^\dagger Q_{\text{B}} P^{(n-1)} Q_{\text{B}} \bar{c}_k|g_n\rangle \,. \end{aligned} \tag{23}$$

The first two terms of eq. (23) vanish by eq. (22). The second term also vanishes by eqs. (22) and (19). Thus we have proved $\langle f_m|g_n\rangle = 0$ for $n \geq 1$. This completes the proof of the zero-norm property of $\mathcal{V}_0$, and therefore of the physical S-matrix unitarity:

$$S^\dagger_{\text{phys}} S_{\text{phys}} = S_{\text{phys}} S^\dagger_{\text{phys}} = P^{(0)} \,, \tag{24}$$

where $S_{\text{phys}} \equiv \mathcal{P}^\dagger S \mathcal{P}$ and $\mathcal{P}$ is a projection operator onto $\mathcal{V}_{\text{phys}}$.

Finally we add two comments. (i) If we had adopted the usual hermitian conjugation assignment $\bar{c} = c^\dagger$, we would have failed in proving the zero-norm property of $\mathcal{V}_0$ for lack of the relation $\langle f_m|Q_{\text{B}} = \langle g_n|Q_{\text{B}} = 0$, because we would not have $Q_{\text{B}}^\dagger \propto Q_{\text{B}}$ in the usual assignment [6,9]. In fact this can be seen in a more concrete example. For usual ghosts, denoted by the capital letters C, $\bar{C}$ for distinction, we would have (symbolically)

$$\begin{aligned} &C(x) \sim C_k + \bar{C}_k^\dagger \,, \quad \bar{C}(x) \sim \bar{C}_k + C_k^\dagger \,, \\ &\{\bar{C}_k, \bar{C}_l^\dagger\} = -\{C_k, C_l^\dagger\} = \delta_{kl} \,, \\ &Q_{\text{B}} = \mathrm{i} \sum_k (\bar{C}_k^\dagger B_k - B_k^\dagger C_k) \,, \end{aligned} \tag{25}$$

non-abelian gauge theories and is also related to the (perturbative) renormalizability of the theories. The unitarity problem was first solved by t'Hooft and Veltman using perturbation theories, and the Kugo–Ojima theory gives an algebraic interpretation of why the diagrammatic cancellation worked in t'Hooft–Veltman results: A graded Lie algebra structure of the field operators, now known as the BRST-symmetry, implies the consistency of the restriction to the positive metric states, namely, the consistency of the Kugo-Ojima physical state condition. A page in Kugo and Ojima's Physic Letters paper shows all these theoretical structure in a compact and clear way, just like what one would see in the present day textbooks, which shows the perfectness of Kugo-Ojima theory well ahead of time. The theory is also described in full detail and in an utmost clarity in T. Kugo, I. Ojima, *Local Covariant Operator Formalism of Non-Abelian Gauge Theories and Quark Confinement Problem,* Progress of Theoretical Physics Supplement **66** (1979) 1–130. The name BRST-symmetry is due to an independent work published a year earlier than Kugo and Ojima Physics Letters paper. But the full algebraic theory, including the definition of physical states using the BRST-charge, the Kugo-Ojima condition, and the mechanism that the symmetry implies the consistency of the condition, is the discovery of Kugo and Ojima.

After the accomplishment of his graduate study, Prof. Ojima gradually moved to philosophically deeper problems of mathematical and information theoretical foundation of quantum physics through thermodynamic and statistical physics. Symmetry breaking and micro-macro duality are among the key words of his study. These key words suggest me that Prof. Ojima is trying to give an answer to a question: 'What is the mathematical structure in quantum physics which intellectually attracts people (in particular, its relation with classical physics)?'

Quantum phenomena are very different from macroscopic phenomena explained by classical physics. For example, it is well-known that Albert Einstein, who played a leading role in the construction of quantum physics, could not accept probabilistic interpretation of quantum fields. A classical phenomena, on the other hand, is theoretically a many body problem of the quantum physics, so at least theoretically it is a logical consequence of quantum physics. All these thoughts enchant people. It seems to me that Prof. Ojima is trying to find out precisely which aspect of the mathematical (or intellectual, if 'mathematics' is too restrictive a word) structure in quantum physics attracts people. We say that a theory explains a reality only if we are convinced that the theory reflects some essential aspect of the reality, and we cannot be convinced by an unattractive theory. Therefore a theoretical essence must be at the part where people are intellectually attracted most. That is perhaps what Prof. Ojima is now trying to find out.

Professor Ito and Professor Ojima

I have been a lazy student all my life, trying to learn neither in depth nor in width. I still have lot to learn both from Prof. Ito and Prof. Ojima. But I think I unconsciously learned one common lesson from the two professors: Be absolutely sincere to one's own scientific interest, and be proud enough to stick to unpopular field of research.

The two professors perhaps are examples from good old days when there were people who studied what they thought are important, in spite of (perhaps) authorities' warning that they are wasting their academic talent and career. A new direction, invention or discovery, means important and not standard. We always need a new direction to push forward our intellectual frontier. I hope we continue to have a few, non-zero good young people in the future, following the examples of Prof. Ito and Prof. Ojima, who would stick to their own scientific interest, to surprise the scientific community with new ideas, and eventually persuade the community to move on to new research directions.

A happy 60th birthday to Professor Ito and Professor Ojima!

A Theory of Scattering Based on Free Fields

R.F. Streater
Department of Mathematics
King's College London

16 October 2009

Abstract

In 3 + 1 dimensions, only quasifree fields have been shown to satisfy the Haag-Araki axioms for local algebras of observables; we show from a model in 1 + 1 dimensions that there can be representations in which two ingoing free particles produce a pair of out-going solitons, provided that one chooses to observe this outcome. It is proposed that the same idea will work in 3 + 1 dimensions.

Contents

1 Introduction to Haag Fields

It has been extremely difficult to construct solutions to renormalisable quantum field theories that satisfy the Wightman axioms, in four space-time dimensions, except free fields and generalised free fields. It has been conjectured that quantum electrodynamics does not exist; only theories containing non-abelian gauge fields, it is claimed, could exist and give a non-trivial S-matrix. Similar remarks apply to the C^*-algebraic systems of Haag and Araki.

The relation between the Wightman axioms and the C^*-algebras is not clear for a general Wightman theory, but for any free boson field a key result due to Slawny [14] suggests a natural way to construct a set of local C^*-algebras which obey the Haag-Kastler axioms. Consider for example a free scalar quantised field of mass $m > 0$. In any Lorentz frame, the free quantised field ϕ and its time derivative π at constant time (say, time zero) can be smeared in the space variable with a continuous function of compact support, to get self-adjoint operators on Fock space. Thus

$$\phi(g) := \int \phi(0,\mathbf{x})g(\mathbf{x})d^3x \tag{1}$$

$$\pi(f) := \int \dot{\phi}(0,\mathbf{x})f(\mathbf{x})d^3x \tag{2}$$

have well-defined exponentials, as do their sums; let $\mathcal{H}$ be the space of real solutions $\varphi(t,\mathbf{x})$ to the wave equation with initial values $\varphi(0,\mathbf{x}) = f(\mathbf{x})$ and $\dot{\varphi}(0,\mathbf{x}) = g(\mathbf{x})$. This is a dense subspace of the one-particle space, a complex Hilbert space. The imaginary part of the scalar product, the symplectic structure of the classical field theory, is the Wronskian B of the two solutions, the Lorentz invariant anti-symmetric bilinear expression

$$B(\varphi_1,\varphi_2) := \int d^3x\,[f_1(\mathbf{x})g_2(\mathbf{x}) - g_1(\mathbf{x})f_2(\mathbf{x})]\,. \tag{3}$$

The expression

$$B(\phi,\varphi) := \phi(g) - \pi(f), \tag{4}$$

the Wronskian between the quantised and the classical solution, is then self-adjoint. Segal uses the operators

$$W(\varphi) := \exp\{iB(\phi,\varphi)\}, \tag{5}$$

and these obey Segal's form of the Weyl relations for the commutation relations of a free quantised field:

$$W(\varphi_1)W(\varphi_2) = W(\varphi_1+\varphi_2)\exp\{-\frac{i}{2}B(\varphi_1,\varphi_2)\}. \tag{6}$$

Eq.(6) gives a product to the vector space defined by symbols $W(\varphi)$ as φ runs over the symplectic space $\mathcal{H}$, irrespective of the representation by operators W. What Slawny [14] did was to prove that the *-algebra obtained by including this product has a unique C^*-norm; this is a norm on the algebra obeying $\|A^*A\| = \|A\|^2$.

We define the Haag field as follows. Let $\mathcal{O}$ be a bounded open set in $\mathbf{R}^4$, of the form of the intersection of the interiors of a forward and backward light cone. The cones themselves intersect in a two-dimensional ellipse. Let

f and g be continuous functions of the points in the interior of the three-dimensional region spanned by this ellipse. Then the local C^*-algebra $\mathcal{A}(\mathcal{O})$ is the completion, in the Slawny norm, of the Segal-Weyl algebra generated by such f and g. The algebra of all observables, $\mathcal{A}$, is then the completion of the inductive limit of all the local algebras. The algebra defined for an arbitrary connected open region of $\mathbf{R}^4$ is the completion of the union of all $\mathcal{A}(\mathcal{O})$, $\mathcal{O}$ being a subset of the region.

This field nearly obeys the Haag-Kastler [8] axioms; Haag and Kastler assumed that the Poincaré group acted on $\mathcal{A}$ norm-continuously, which we do not. The free field satisfies one more, the split property of Doplicher and Roberts [5]. We use the notation $\mathcal{L}$ for the Poincaré group, which is the semi-direct product of the group of space-time translations, $x \mapsto x+a$, where a is a real four-vector, and the Lorentz group $x \mapsto \Lambda x$. Thus $L = (a, \Lambda)$ will denote a general element of $\mathcal{L}$. Then the axioms we use are:

1. There is given an automorphism group τ_L of the Poincaré group; this maps $\mathcal{A}(\mathcal{O})$ onto $\mathcal{A}(L\mathcal{O})$.

2. If two regions $\mathcal{A}_1$ and $\mathcal{A}_2$ are space-like separated, then the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ commute.

3. The vacuum representation: there exists a representation R_0 of $\mathcal{A}$, such that there is a unique vacuum state vector, the Poincaré group is continuously represented by unitary operators, and the spectrum of the energy is bounded below.

4. The split property: if $\mathcal{O}_1^- \subset \mathcal{O}_2$ then there exists a sub-algebra $\mathcal{N}$ of type I such that $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{N} \subset \mathcal{A}(\mathcal{O}_2)$; by type I is meant that the weak closure in the vacuum representation is a von Neumann algebra of type I.

Another possible axiom is Haag duality; this fails to hold in our model in one-plus-one dimensions and we shall not use it.

In their set-up, Haag and Kastler give the following explanation of superselection rules; charged states are not in the state-space containing the vacuum, but are states in some other representation R of the algebra $\mathcal{A}$, which is not equivalent to R_0. We mean the following by equivalence; let $\mathcal{A}$ be a C^*-algebra. Two representations of $\mathcal{A}$, π_1 on a Hilbert space $\mathcal{H}_1$ and π_2 on a Hilbert space $\mathcal{H}_2$, are said to be *equivalent* if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\pi_1(A) = U^{-1}\pi_2(A)U \tag{7}$$

holds for all $A \in \mathcal{A}$. We say that an automorphism σ of $\mathcal{A}$ is *spatial* in a representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ if there exists a unitary operator U_σ on $\mathcal{H}$ such that

$$\sigma(A) = UAU^{-1} \tag{8}$$

holds for all $A \in \mathcal{A}$. We say that U implements the automorphism in this case. Most automorphisms are not spatial.

Haag and Kastler assume that the Poincaré automorphisms are spatial in R, and that the generator of time evolution also has positive spectrum. R is related to the vacuum representation R_0 by an automorphism σ, $A \mapsto \sigma A$ of $\mathcal{A}$; this cannot be a spatial automorphism, since if it were, R and R_0 would be equivalent. Clearly, the representation is given by

$$R_\sigma(A) = R_0(\sigma A), \tag{9}$$

as A runs over $\mathcal{A}$; this acts on the Hilbert space containing the vacuum, but is not equivalent to the representation R_0, since the automorphism σ is not implemented by a unitary operator. We do not expect σ to commute with the space-time translations; thus, the automorphisms of $\mathcal{A}$, $\tau_a\sigma$, $a \in \mathbf{R}^4$, are not the same as $\sigma\tau_a$ in general. Haag showed that one might reveal the existence of Fermions, carrying a charge, by exploring the representations

$$R_n(A) = R_0(\tau_1\sigma \circ \tau_{a_2}\sigma ... \tau_{a_n}\sigma A), \tag{10}$$

which would define the n-particle states. A little later, Doplicher and Roberts [5] generalised this idea; to get a representation of $\mathcal{A}$, one can make do with an *endomorphism* rather than an automorphism; one then gets a reducible representation of the algebra $\mathcal{A}$ by using eq (9). The unitaries of the cummutant of the representation then make up the gauge group. Starting with axioms similar to (1), ..., (4), they [5, 7] find that the gauge group must be a compact Lie group. Now, this holds also for the free field algebra, though Doplicher and Roberts assumed that the given system was not the free field. They are stuck, in that no interacting Wightman theory in four dimensions has yet been constructed.

In this paper, we start with the free field as in [17], and try to find what endomorphisms give rise to new states. We note that it is not obvious that the Lorentz group should be implemented in R even if the space-time automorphisms are; more, the space-time group might acquire non-abelian multipliers. In Sect (2) we show that if every one-parameter space-time translation group with a time-like direction has spectrum that is bounded below, then the four-dimensional translation group is represented by unitaries which have multipliers in the centre of $R(\mathcal{A})''$. This proof uses Borchers's theorem [2] in the form proved in Brattelli and Robinson [1]; it arose from a discussion with G. Morchio. We are then reduced to the suggestion of several authors, that the space-time group might be represented with multipliers in the commutant.

In Sect. (3) we study the case of a free massless field in $1+1$ dimensions, following [18]. This model has been further developed by Ciolli [3]. We show that a soliton pair of states with opposite charges does lie in Fock space, and converges $*$-weakly to an out-going pair in a new representation. The pair is created from a state in Fock space by the very act of asking the question, is a pair present at $t=\infty$?

In Sect.(4) we suggest a programme that might lead to similar results in four space-time dimensions.

2 Reduction to Abelian Multipliers

It is usually required that the endomorphism, denoted by σ above, should be such that the Poincaré group should be spatial in the representation R_σ. However, with particles of zero mass, it might not be true. In any case, we shall just assume that space-time translations are symmetries in R_σ; that is, are each given by an isometric operator with transition probabilities that are measurable functions of the group parameters; then Wigner's analysis can be applied. Now, R_σ is reducible if σ is not an automorphism; thus the commutant $R_\sigma(\mathcal{A})'$ of the representation contains non-commuting unitaries, and so possible multipliers of the group $\mathbf{R}^4$ might be non-abelian [15, 16]. It is well known that a one-parameter group of automorphisms, if spatial in a representation, has only trivial multipliers [9]. It has been suggested that conditions might be such that the multiplier is abelian. Indeed, there does exist a natural condition which ensures this.

Theorem 11 *Let $\mathcal{A}$ be a C^*-algebra and τ_a be a group action of $\mathbf{R}^4$ by automorphisms. Let $A \mapsto R(A)$ be a representation of $\mathcal{A}$. Suppose that for each time-like one-parameter subgroup of $\mathbf{R}^4$, the automorphisms are implemented (in the representation R) by a continuous one-parameter unitary group, whose self-adjoint generator is bounded below. Then the group $\mathbf{R}^4$ is projectively represented by unitary operators with abelian multipliers.*

Proof. Borchers's theorem [2] was modified by Bratteli and Robinson [1] to the form: Let $\mathcal{A}$ be a C^*-algebra on a separable Hilbert space, τ_t a one-parameter group of automorphisms of $\mathcal{A}$, implemented by the continuous one-parameter unitary group $t \mapsto U(t)$. Then there exists a continuous unitary group $t \mapsto V(t)$ in the weak closure of $\mathcal{A}$ which implements τ_t.

We apply this result to four independent one-parameter timelike one-parameter groups of space-time translations. The generators are bounded below, and so can be replaced by unitary operators in the weak closure. The multipliers, which are expressed as

$$\omega(a,b) = U(a)U(b)U(a+b)^{-1}, \qquad a,b \text{ being time-like vectors,} \tag{12}$$

shows that for all time-like vectors a, b we have $\omega(a,b)$ lying in $R(\mathcal{A})''$; but these multipliers also lie in $R(\mathcal{A})'$, so must lie in the centre. Since these group elements generate the group $\mathbf{R}^4$, we have proved the theorem.

3 A Model in One-Plus-One Dimensions

The existence of Wightman theories with interaction in $1+1$-dimensions [6] means that it has not been necessary to consider our idea in this case; however, in view of the difficulty, if not the impossibility, of there existing an interacting Wightman theory in four space-time dimensions, it is worth while pointing out the following model.

Consider the Wightman theory of a scalar massless free field $\phi(x,t)$ in $1+1$ dimensions. This does not exist, but its space-time derivatives, $\phi_\mu := \partial_\mu \phi$, do. We take this derivative, $\mu = 0, 1$, to define the observable Wightman fields. The smeared fields ϕ_μ at time zero, obey a form of the CCR which can be written in Segal form. We [18] get a Haag field, and show that it obeys axioms 1, 2 and 3. We consider new representations of the form

$$\begin{aligned} \partial_x \phi_\sigma &= \partial_x \phi + \partial_x \varphi && (13) \\ \partial_t \phi_\sigma &= \partial_t \phi + \varpi. && (14) \end{aligned}$$

Here, φ and ϖ are real-valued smooth functions, and such that $\partial_x \varphi$ and ϖ have compact support. It is known that the representation obtained this way is equivalent to the Fock representation if and only if the classical solution determined by the initial values φ, ϖ lies in the one-particle space. We showed [18] that there exists a two-parameter family of superselection rules, labelled by "charges" Q, Q' say; these can be any pair of real numbers. If they are both zero, then the automorphism is spatial in the free Fock representation. The allowed set of φ consists of functions such that $\partial_x \varphi \in \mathcal{D}$, and the set of ϖ is $\mathcal{D}$ itself, Schwartz space; this can lead to states not in Fock space. Two representations with different values of either Q or Q^* are inequivalent; it is thus reasonable to put the discrete topology on the set $\mathbf{R}^2$. The dual of this topological space is thus the compact gauge group $U(1) \times U(1)$.

Consider, for example, the choice of $Q = 1,\ Q' = 1$. A general solution to the wave equation can be written as the sum of a left-going and a right-going wave:

$$f(x,t) = f_L(x+t) + f_R(x-t). \tag{15}$$

We see that a left-going wave can have $Q = 1$ and $Q' = 1$ if $f_R = 0$ and $f_L = \varphi$, $\varpi = \partial_x \varphi$, where $\varphi(x) = 1$ if x is sufficiently large, and $\varphi(x) = 0$ for x sufficiently negative. It folllows that there is a state in Fock space, with φ consisting of a right-moving positive bump to the right of space, with $Q = -1$ and $Q' = -1$, and a left-moving negative bump to the left of space, with $Q = 1$ and $Q' = 1$. Let $F(x,t)$ be classical solution with these properties. Then the automorphism is implemented by the unitary operator $W(F) = \exp\{i\left(\phi(\dot{F}) - \pi(F)\right)\}$.

As time goes by, these solitons move as out-going free particles. There is a non-zero probability P that a given two-particle state $|2\rangle$ in Fock space will lead to this configuration:

$$P = |\,\langle 2|W\Psi_0\rangle\,|^2 > 0. \tag{16}$$

It is clear that if we look for the free particles, we will see them; no new particles particles are produced. The charged particles are produced by the setting-up of the procedure to see them.

Further work on this model was done by Ciolli [3]. He proved using Roberts's net cohomology [13] that all possible superselection rules were found in [18].

4 An Attempt in Three + One Dimensions

The electromagnetic field obeys the Maxwell equations

$$\begin{aligned} \operatorname{div}\mathbf{E} &= \rho & (17)\\ \operatorname{div}\mathbf{B} &= 0 & (18)\\ \partial_t\mathbf{B} &= -\operatorname{curl}\mathbf{E} & (19)\\ \partial_t\mathbf{E} &= \operatorname{curl}\mathbf{B} + \mathbf{j} & (20) \end{aligned}$$

The free-field arises when ρ and $\mathbf{j}$ vanish; the classical electromagnetic wave is described by a transverse free $\mathbf{E}, \mathbf{B}$. That is, $\mathbf{E}$ and $\mathbf{B}$ are both orthogonal to the momentum of the wave. There are two states, labelled by the polarisation, for each momentum. The set of such solutions form a real Hilbert space, with a symplectic form and a complex structure. The action of the Poincaré group is unitary, the representation being of mass zero and helicity ± 1. The three components of $\operatorname{curl}\mathbf{E}$ are transverse, even when ρ is not zero. For, the distribution $\operatorname{curl}\mathbf{E}$ has three components. The x-component is $\partial_z E_y - \partial_y E_z$; thus, $\operatorname{curl}\mathbf{E}$, smeared with the three-vector $\mathbf{f}$, is the space of operators

$$\operatorname{curl}\mathbf{E}.(\mathbf{f}) = \mathbf{E}.\operatorname{curl}\mathbf{f}$$

whereas the longitudinal part of the field is of the form $\mathbf{E}.\nabla g$. Since the set curl.$\mathbf{f}$ is disjoint from the set of ∇g except for 0, we have shown that $\operatorname{curl}\mathbf{E}$ is transverse.

Smeared with test functions $\mathbf{f}$ in $\mathcal{D}(\mathbf{R}^3)$, the functions $\operatorname{curl}\mathbf{f}$ are dense in the one-particle space. We define the local C^*-algebra $\mathcal{A}(\mathcal{O})$ using Slawny's theorem, using test-functions in $\mathcal{D}(\mathcal{O})$. The global C^*-algebra $\mathcal{A}$ is the completion of the union of all such algebras for bounded regions in space-time. Let R be the relativistic Fock representation of the transverse electromagnetic field.

We seek an endomorphism σ of $\mathcal{A}$ so that the representation obtained by $R_\sigma(A) = R(\sigma(A))$ is disjoint from the representation R_0. More, we need that the space-time automorphisms of $\mathcal{A}$ should be spatial in R_σ, and that any one-dimensional time-like translation group should be continuous, and that its generator should be bounded below. The dynamics of the operators in R_σ is given by the free automorphism group of the free field. However it is not a trivial dynamics, so we hope. The Hamiltonian is not a bounded operator, and neither are the field operators. So these are not in the C^*-algebra, and their algebraic properties might not be preserved if we change to an inequivalent representation. The commutator of these gives the time evolution of the field operator. However, the Lie algebra of such commutators might not be preserved under the endomorphism: there might be new terms, an induced interaction. This is due to the anomalies that arise in commutators. Another possibility, which changes the equations of motion, is to change coordinates of space-time by a smooth but non-linear map. This might lead to a new representation, but it is not clear that the space-time translations would be spatial in the new representation.

Leyland and Roberts [10] have used the theory of sheaf cohomology to study the possible two-cocycles of some free classsical fields in Minkowski space. They conclude that for the scalar Klein-Gordon real field, the two-cohomolgy group is trivial, while for the free Maxwell field there is a two-parameter family of two-cocycles, labelled by electric and the magnetic charge. They also showed that the classical four-potential, A_μ, obeying the subsidiary condition $\partial^\mu A_\mu = 0$ and the wave equation $(\partial_{tt} - \Delta)A_\mu = 0$, showed a one-parameter family of electric charges. It is not clear from their remarks that this holds in the quantum case, which requires non-commuting operators for the fields; however, it does. As we did in 1 + 1 dimensions, we can add this classical solution to the free quantised field, to generate an automorphism of the free field algebra. When we add a cocycle which is not a coboundary, we get a new representation. Leyland and Roberts do not consider the condition that the Maxwell field should be transverse, nor the requirement that the new representations found should have energy bounded below. The latter condition can be satisfied if we require that the solution should extend to the point at infinity, as in the methods described by Ward and Wells [21]. This is possible only for a subset of the solutions, namely, those with integer charge. Thus, the problem with continuous charge can be solved in this way. We can remove the occurrence of magnetic charge by requiring the existence of a potential A_μ. However, this work leads to sectors with zero mass, since there is no mass-parameter in the model. This leads to doubts that it is an electron.

Of interest is the model of Prasad and Sommerfield [12]. They explicitly construct a smooth solution of a free massive boson field in a non-abelian gauge field, and the electromagnetic part of the gauge field has a magnetic pole as well as an electric pole. The energy of the solution is finite. The rigorous treatment [21] concerns the analytic continuation from Minkowski to Euclidean space $\mathbf{R}^4$. It mostly assumes that the Euclidean gauge field is dual or anti-dual $\mathbf{E} = \pm i\mathbf{H}$, though the book also deals with some non-self-dual electromagnetic fields. Donaldson [4] has pointed out that in four dimensions, in the Euclidean formulation, and in the case of self-dual electromagnetic tensors, the second sheaf cohomolgy group is non-trivial. He remarks that this would furnish $\mathbf{R}^4$ with new differential structures. From the point of view of the second quantised theory, the C^*-algebra of the electromagnetic field reveals the charge in its equations of motion in the corresponding representation.

The book [21] deals with the classical version of this problem. However, for linear fields, this is close to the quantum version, as we saw in [18]; we use the classical solution to get the displaced Fock representations used in [18]. Further, the non-linearity of the gauge field in classical field theory can sometimes be linearised by a suitable change of coordinates. The representations obtained by smooth invertible change of coordinates are generally spatial in Fock space; they would produce unstable particles instead of superselected states. The Euclidean approach of Symanzik [19] and Nelson [11] might be the way to proceed; a coordinate change in Euclidean variables could lead to the correct version of the relation between the Fock and non-Fock representations.

In $4 + 1$ dimensions, Vasilliev has shown that a four-dimensional change of coordinates leads us to the soliton, which obeys a Dirac equation.

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Superconformal field theory, operator algebras and noncommutative geometry

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Abstract

We survey recent progress in the operator algebraic approach to superconformal field theory in connection to the Jones theory of subfactors and noncommutative theory of Connes.

1 General setting

Quantum field theory is a physical theory, but it has had many rich interactions with various topics in mathematics over many years. Theory of operator algebras has been closely connected to quantum physics since its creation by J. von Neumann, and we recently see many interesting developments in the interface between operator algebras and quantum field theory, particularly conformal field theory [1], [11], [15]. We present the current status of such developments here. A more detailed recent review is given in [24].

We make a general description of quantum field theory in the mathematical setting. First, we have a space time, e.g. a Minkowski space. Then we have a spacetime symmetry group, e.g. the Poincaré group of the Minkowski space. Then we have Wightman fields on the spacetime. Mathematically speaking, they are certain type of operator-valued distributions on the spacetime.

We now focus on a specific type of quantum field theory, a conformal field theory. Now the initial spacetime is the two-dimensional Minkowski space $\{(x,t) \mid x,t \in \mathbb{R}\}$, but with a certain "restriction" procedure, we deal with one of the light rays $x = \pm t$, and then we work on its compactification S^1. This is our "spacetime", though the space and time are now combined together. The spacetime symmetry group is now the orientation preserving diffeomorphism group $\mathrm{Diff}(S^1)$ of S^1. We deal with operator-valued distributions acting on a fixed Hilbert space of states having a "vacuum" vector. This setting is called a chiral conformal field theory and we make an operator algebraic axiomatization based on the following idea.

Suppose we have ome family $\{T\}$ of operator-valued distributions on S^1. Fix an interval $I \subset S^1$, and consider $\langle T, \varphi\rangle$ with supp $\varphi \subset I$. Note that $\langle T, \varphi\rangle$ is a (possibly unbounded) operator on H. Bounded linear operators are much easier to handle, so we consider the von Neumann algebra $A(I)$ generated by these (possibly unbounded) operators. We then make the following set of operator algebraic axioms. This is a version of algebraic quantum field theory [20].

Our object is an assignment of a von Neumann algebra $A(I)$ to each interval I contained in S^1. We impose the following conditions.

1. For $I_1 \subset I_2$, we have $A(I_1) \subset A(I_2)$.
2. (Locality) For $I_1 \cap I_2 = \varnothing$, we have $[A(I_1), A(I_2)] = 0$.
3. (Conformal covariance) We have a projective unitary representation U of $\mathrm{Diff}(S^1)$ with $U_g A(I) U_g^* = A(gI)$.
4. Positivity of the energy.
5. Unique existence of the vacuum vector in H.

Such a family $\{A(I)\}$ is called a *local conformal net.* (See [25] for the precise forms of the axioms.)

We also mention that full and boundary conformal field theory can be axiomatized in a similar manner. That is, instead of an interval in S^1, we consider certain bounded regions contained in the 2-dimensional Minkowski space $\{(x,t) \mid x, t \in \mathbb{R}\}$ or its half space $\{(x,t) \mid x, t \in \mathbb{R}, x > 0\}$. We impose a set of axioms similar to the above ones.

We have a restriction procedure of a theory from a full/half 2-dimensional Minkowski space to S^1. We also have a machinery to recover a theory on a full/half 2-dimensional Minkowski space from that on S^1. See [26] and [36] for precise results.

2 Examples and classification

The Virasoro algebra is an infinite dimensional Lie algebra generated by $\{L_n \mid n \in \mathbb{Z}\}$ and a central element c with the following relations.

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}.$$

Consider a power series $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$. In the vacuum representation of the Virasoro algebra, this power series with $z \in \mathbb{C}$, $|z| = 1$, can be interpreted as an operator-valued distribution on S^1.

This single operator-valued distribution produces a local conformal net, the Virasoro net with central charge $c \in \mathbb{C}$.

We also have its super version arising from the $N = 1$ super Virasoro algebras. We consider the infinite dimensional super Lie algebra generated by a central element c, even elements L_n, $n \in \mathbb{Z}$, and odd elements G_r, $r \in \mathbb{Z}$ or $r \in \mathbb{Z} + 1/2$, with the following relations.

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \\
[L_m, G_r] &= (\frac{m}{2} - r)G_{m+r}, \\
[G_r, G_s] &= 2L_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}.
\end{aligned}$$

Depending on $r \in \mathbb{Z}$ or $r \in \mathbb{Z} + 1/2$, we call our super Lie algebra the Ramond algebra or the Neveu-Schwarz algebra.

We now explain its operator algebraic counterpart, a superconformal net. In the vacuum representation of the Neveu-Schwarz algebra, the power series $\sum_r G_r z^{-r-3/2}$ makes sense as an operator-valued distribution called super stress-energy tensor. Together with $L(z)$ (the stress-energy tensor), this gives a superconformal net through test functions supported in an interval. The $\mathbb{Z}_2$-grading of the super Lie algebra passes to the $\mathbb{Z}_2$-grading of the operator algebras. In the operator algebraic axioms, the commutator in the locality axiom is now replaced with graded commutator. (That is, we have an anticommutator for odd elements.)

We have classification results as follows. Up to the central charge value 1, we have a discrete series of possible values. For this part, we have complete classification results for each of chiral, full and boundary conformal field theories in [25], [26], [30]. They are based on [2], [3], [4], [29], [37], [39], [40], [41], [42], [43], [44]. For the superconformal case, the value of the central charge c is in the set

$$\left\{ \frac{3}{2}\left(1 - \frac{8}{m(m+2)}\right) \middle| \, m = 3, 4, 5, \ldots \right\} \cup [3/2, \infty).$$

We also have a complete classification result in [8] for the discrete series. They are labeled with certain pairs of the A-D-E Dynkin diagrams.

3 Vertex operator algebras

We have another mathematical formulation of Wightman fields in conformal field theory, and it is called a (super)vertex operator algebra. A *vertex operator* is a mathematical axiomatization of a Wightman field on S^1.

We have a graded $\mathbb{C}$-vector space $V = \bigoplus_{n \geq 0} V_n$. Its completion is the Hilbert space of the states of the theory. For each state in V, we have a corresponding vertex operator, which is a Fourier expansion of an operator-valued distribution on S^1 acting on V.

All the axioms are purely algebraic, and we have a notion of the automorphism group. See [14], [23] for a precise presentation. Also see [18], [19], [45] for recent progresses. Realization of the Monster group and other sporadic finite simple groups is a deep problem surrounding the Moonshine conjecture [5], [10], [17], [16]. We have operator algebraic counterparts including the *super Moonshine net* [28].

4 Geometric aspects of local conformal nets

We recall some well-known facts in classical differential geometry. Consider the Laplacian Δ on an n-dimensional compact oriented Riemannian manifold. The classical Weyl formula gives an asymptotic expansion

$$\mathrm{Tr}(e^{-t\Delta}) \sim \frac{1}{(4\pi t)^{n/2}}(a_0 + a_1 t + \cdots),$$

for $t \to 0$, where a_0 is the volume of the manifold, and if $n = 2$, then a_1 is (constant times) the Euler characteristic of the manifold. So the coefficients in the asymptotic expansion have a geometric meaning.

We have some analogue for a local conformal net. For a *nice* local conformal net, we have an expansion

$$\log \mathrm{Tr}(e^{-tL_0}) \sim \frac{1}{t}(a_0 + a_1 t + \cdots),$$

where a_0, a_1, a_2 are explicitly given [27].

This gives an analogy of the Laplacian of a manifold and the conformal Hamiltonian L_0 of a local conformal net. A local conformal net has an *infinite dimension* because of log in the expansion, but after some regularization, we could say it has a dimension 2 because of the above expansions. The term a_0 also has some formal similarity to black hole entropy.

We also explain what a nice local conformal net in the above assumption. Being nice is defined by coincidence of two actions of $SL(2, \mathbb{Z})$ on the set of irreducible representations of the local conformal net. One arises from fractional linear transformation on characters and the other arises from the braiding structure of representations [38].

A priori, there are no reasons to expect that the two actions coincide and we have no general proof for the coincidence, but for all explicitly known examples, the two actions are equal. Such local conformal nets are said to be modular.

5 Noncommutative geometry

The slogan in noncommutative geometry is that noncommutative operator algebras are regarded as function algebras on noncommutative spaces. Such spaces should be counterparts of compact Hausdorff spaces or measure spaces, but in geometry, we need *manifolds* rather than compact Hausdorff spaces or measure spaces.

The Connes axiomatization gives a noncommutative compact Riemannian spin manifold in terms of a spectral triple $(\mathcal{A}, H, D)$ as follows [9].

1. The algebra $\mathcal{A}$ is a $*$-subalgebra of $B(H)$, which should the noncommutative smooth function algebra on the noncommutative manifold.

2. The space H is a Hilbert space, the space of L^2-spinors on the noncommutative manifold.

3. The operator D is an (unbounded) self-adjoint operator, the Dirac operator on the noncommutative manifold.

4. We require $[D,x] \in B(H)$ for all $x \in \mathcal{A}$.

A basic example is given s follows. a noncommutative tori A_θ, the noncommutative version of the 2-dimensional torus $\mathbb{T}^2$, is given as the C^*-algebra generated by two unitaries u, v with $uv = e^{2\pi i\theta} vu$, with irrational θ. It has a dense $*$-subalgebra $\mathcal{A}$ of the smooth part. The Hilbert space H is a direct sum of two copies of the L^2-completion of A_θ. The Dirac operator D is given by a certain linear combination of the two standard derivations on A_θ,

$$\delta_1(u) = iu, \delta_1(v) = 0, \quad \delta_2(u) = 0, \delta_2(v) = iv.$$

It has the *dimension* 2.

We now give our construction of spectral triples in the operator algebraic framework of superconformal field theory. We have a construction of a family $(\mathcal{A}(I), H, D)$ of spectral triples parametrized by intervals $I \subset S^1$ from a representation of the Ramond algebra [7] as follows.

Note tha one of the Ramond relations gives $G_0^2 = L_0 - c/24$. We have the following analogy.

$$\begin{array}{ccc} \text{The Laplacian} & \rightarrow & \text{square root} = \text{The Dirac operator} \\ \updownarrow & & \updownarrow \\ L_0 & \rightarrow & \text{square root} = G_0 \end{array}$$

So G_0 should play the role of the Dirac operator. We start with a unitary representation of the Ramond algebra. Its representation space is our Hilbert space H for the spectral triples (without a vacuum vector). The image of G_0 is now the Dirac operator D for the spectral triples. From $L(z) = \sum_n L_n z^{-n-2}$ and $G(z) = \sum_r G_r z^{-r-3/2}$, we have a superconformal net $\{A(I)\}$ of von Neumann algebras without a vacuum vector parametrized by intervals I.

Then we define

$$\mathcal{A}(I) = \{x \in A(I) \mid [D,x] \in B(H)\}.$$

By definition, it is clear that we have a net of spectral triples $\{\mathcal{A}(I), H, D\}$ parametrized by intervals I.

However, we could have $\mathcal{A}(I) = \mathbb{C}$, which is too trivial. In order to show that this does not happen, we use the resolvent method of Buchholz-Grundling [6]. Let f be a test function supported in I with some nice property. Then for real α with sufficiently large $|\alpha|$, we have $G(f)(L(f) + i\alpha)^{-1} \in \mathcal{A}(I)$. This further shows that $\mathcal{A}(I)$ is strongly dense in $A(I)$, which in particular shows $\mathcal{A}(I)$ is nontrivial.

We now have $\mathrm{Tr}(e^{-tD^2}) < \infty$ for all $t > 0$, and this condition is called θ-summability, which gives a nice class of well-behaved *infinite dimensional noncommutative manifolds*. Note that we have a very slow growth of eigenvalues of L_0.

More topics in noncommutative geometry to be studied include the following.

1. Quantum index in the sense of Longo [33].
2. Analogy between an elliptic operator and a DHR sector [12].
3. Analogy between the Fredholm index and the Jones index [22], [31], [32]. (Note we have a direct relations between the two in [8].)
4. Computation of JLO-cocycles [21].

Here is some explanation on the JLO-cocycle in the last entry. The entire cyclic cohomology is a certain cohomology theory for Banach algebras. An entire cocycle is a sequence of certain multilinear functionals on $\mathcal{A}$. Connes considered the Chern character as an entire cocycle. Jaffe-Lesniewski-Osterwalder gave another construction of the Chern character, which is called a JLO cocycle. Its computation in our setting would deepen our understanding of superconformal field theory in the framework of noncommutative geometry.

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Boundedness of Entanglement Entropy, and Split Property in Quantum Spin Systems

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1 Introduction

Area law of entanglement entropy attracted much attention of researchers of quantum information and statistical mechanics recently. This is because the set of matrix product states or projected entangled pair states is dense in state spaces of quantum spin systems and *the condition that certain the density matrix renormalization group method is valid is related to Area law of entanglement entropy of quantum ground states.*

On the other hand, in [9], we considered a relationship between *split property* and symmetry of of translationally invariant pure states for quantum spin chains on an integer lattice $\mathbf{Z}$. The *split property* is a kind of statistical independence of left and right semi-infinite subsystems. Namely, a state of a quantum spin chain on an integer lattice $\mathbf{Z}$ has the split property between left and right semi-infinite subsystems if the state is quasi-equivalent to a product state of these infinite subsystems. The condition holds for Gibbs states of finite range interactions for one-dimensional quantum spin chains. We have shown that the split property cannot hold for translationally invariant pure states of quantum spin chains if the state is SU(2) invariant and the spin S is half-odd integer. Our proof is carried out in a sense that states with split property looks like a generalization of matrix product state.

In this note we will see that boundedness of entanglement entropy for bipartite quantum spin systems implies split property, which is equivalent to the area law of entanglement entropy in one-dimesnional systems. Combined with a result of M.B.Hastings in [6], we see that the presence of the spectral gap between the ground state energy and the rest of spectrum implies the *split property* . We do not assume translational invariance of infinite volume Hamiltonians and that of states but certain boundedness of the norm of local energy operators.

By *area law of entanglement entropy* we mean von Neumann entropy of finite sub-systems on finite connected regions in a pure state of an inifinite quantum system increase in a rate proportional to the size of boundary of the underlying space and for one-dimesional lattice models, it is the boundedness of entropy of finite subsystems.

In [6], M.B. Hastings proved the area law of entanglement entropy for ground states with a spectral gap and his results implies split property, M.B. Hastings assumed that uniqueness of finite volume ground states of finite volume Hamiltonians in [6]. However, uniqueness condition of finite volume ground states may not be satisfied for Hamiltonians for which a pure matrix product state is a ground state. For example, the AKLT model of [2] has a unique infinite volume ground state while the dimension of the finite volume ground state is four. We claim that , for any infinite pure ground state with spectral gap, the split property holds without assuming uniqueness of finite volume ground states. To prove this, we have to reformulate M.B. Hastings' proof of the are law of entanglement entropy in an infinite dimensional setting suitably and for that purpose, we find that an extension of the factorization lemma of M.B. Hasting due to E.Hamza, S.Michalakis, B.Nachtergaele, and R.Sims in [5] is useful. In the proof of the area law of M.B. Hastings, the Lieb-Robinson bound is a crucial mathematical tool and for our infinite dimensional setting a version due to B.Nachtergaele, and R.Sims in [10] is useful.

As a corollary we can show that a gapless excitation is present in any half-odd integer spin SU(2) invariant quantum spin chains and the same result holds in $U(1)$ symmetric spinless fermion models on $\mathbf{Z}$, provided that the ground state is non-trivial. The similar results of gapless excitation in infinite systems are available due to known results of [1], [11] ,[12] however, in these previous works, proof is based on the uniqueness of finite volume ground states, which we do assume here. Instead purity of infinite volume gauge invariant ground states is essential in our argument.

Another application of our result is a no-go theorem in quantum information theory. We see that distillation of infinitely many pairs of maximally entangled stated in one copy of infinite bipartite systems is not possible if the entanglement entropy is bounded.

2 Results

We describe results precisely. We use theory of operator algebras and most of definitions and notions we need here can be found in [3] and [4]. We denote the C^*-algebra of (quasi)local observables by $\mathfrak{A}$. $\mathfrak{A}$ will be UHF C^*-algebras n^∞ or (the C^*-algebraic completion of the infinite tensor product of n by n matrix algebras) the CAR (canonical anti-commutation relations) algebra on bfZ^d or more generally arbitrary graphs. In case of UHF algebra we use the following notations. :

$$\mathfrak{A} = \overline{\bigotimes_{\mathbf{Z}^d} M_n(\mathbf{C})}^{C^*} ,$$

where $M_n(\mathbf{C})$ is the set of all n by n complex matirces. Each component of the tensor product is specified with a lattice site $j = (j_1, j_2, \cdots, j_d) \in \mathbf{Z}^d$. $\mathfrak{A}$ is the totality of quasi-local observables. We denote by $Q^{(j)}$ the element of $\mathfrak{A}$ with Q in the j th component of the tensor product and the identity in any other components :

$$Q^{(j)} = \cdots \otimes 1 \otimes 1 \otimes \underbrace{Q}_{j\text{th component}} \otimes 1 \otimes 1 \otimes \cdots$$

For a subset Λ of $\mathbf{Z}^d$, $\mathfrak{A}_\Lambda$ is defined as the C^*-subalgebra of $\mathfrak{A}$ generated by elements $Q^{(j)}$ with all j in Λ. We set

$$\mathfrak{A}_{loc} = \cup_{\Lambda \subset \mathbf{Z}^d : |\Lambda| < \infty} \mathfrak{A}_\Lambda$$

where the cardinality of Λ is denoted by $|\Lambda|$. We call an element of $\mathfrak{A}_{loc}$ a local observable or a strictly local observable.

By a state φ of a quantum spin chain, we mean a normalized positive linear functional on $\mathfrak{A}$ which gives rise to the expectation value of a quantum state. When φ is a state of $\mathfrak{A}$, the restriction of φ to $\mathfrak{A}_\Lambda$ will be denoted by φ_Λ :

$$\varphi_\Lambda = \varphi|_{\mathfrak{A}_\Lambda}.$$

Suppose that Λ_1 and Λ_2 are subsets of $\mathbf{Z}^d$ satisfying

$$\mathbf{Z}^d = \Lambda_1 \cup \Lambda_2, \quad \Lambda_1 \cap \Lambda_2 = \emptyset.$$

We set

$$\mathfrak{A}_1 = \mathfrak{A}_{\Lambda_1} , \ \mathfrak{A}_2 = \mathfrak{A}_{\Lambda_2} , \ \varphi_1 = \varphi_{\Lambda_1} , \ \varphi_2 = \varphi_{\Lambda_2} \ .$$

By τ_j, we denote the automorphism of $\mathfrak{A}$ determined by

$$\tau_j(Q^{(k)}) = Q^{(j+k)}$$

for any j and k in $\mathbf{Z}^d$. τ_j is referred to as the lattice translation of $\mathfrak{A}$.

Given a state φ of $\mathfrak{A}$, we denote the GNS representation of $\mathfrak{A}$ associated with φ by $\{\pi_\varphi(\mathfrak{A}), \Omega_\varphi, \mathfrak{H}_\varphi\}$ where $\pi_\varphi(\cdot)$ is the representation of $\mathfrak{A}$ on the GNS Hilbert space $\mathfrak{H}_\varphi$ and Ω_φ is the GNS cyclic vector satisfying

$$\varphi(Q) = (\Omega_\varphi, \pi_\varphi(Q)\Omega_\varphi) \qquad Q \in \mathfrak{A}.$$

Let π be a representation of $\mathfrak{A}$ on a Hilbert space. The von Neumann algebra generated by $\pi(\mathfrak{A}_\Lambda)$ is denoted by $\mathfrak{M}_\Lambda$. We set

$$\mathfrak{M}_1 = \mathfrak{M}_{\Lambda_1} = \pi(\mathfrak{A}_1)'', \quad \mathfrak{M}_2 = \mathfrak{M}_{\Lambda_2} = \pi(\mathfrak{A}_2)''.$$

Now we state our results on split property.

Definition 2.1 *Let φ be a state of $\mathfrak{A}$. We say the split property is valid for $\mathfrak{A}_1$ and $\mathfrak{A}_2$ if φ is quasi-equivalent ti $\psi_1 \otimes \psi_2$ where ψ_1 is a state of $\mathfrak{A}_1$ and ψ_2 is that of $\mathfrak{A}_2$.*

Definition 2.2 *Let φ be a state of $\mathfrak{A}$. Suppose Λ is a finite subset of $\mathbf{Z}^d$ and ρ_Λ is the density matrix of φ_Λ. We consider the entropy $s(\varphi_\Lambda) = -tr_N(\rho_\Lambda \ln \rho_\Lambda) = -\varphi(\ln \rho_\Lambda)$ where the trace tr is normalized as $tr(1) = |\Lambda|^{2N+1}$.*

We say the entanglement entropy is bounded if $s(\varphi_{[\Lambda]})$ is bounded ,$s(\varphi_\Lambda) \leq C$ for any finite subset Λ of Λ_1.

If the state φ is pure, and the entanglement entropy is bounded it is possible to show $s(\varphi_\Lambda) \leq C$ for any subset Λ of Λ_2.

Theorem 2.3 *Let φ be a pure state of $\mathfrak{A}$ for which the entanglement entropy is bounded. Then the split property is valid for $\mathfrak{A}_1$ and $\mathfrak{A}_2$.*

We do not have any concrete example of states with split property but not having bounded entanglement entropy though we believe it exists.

The split property is statistical independence of the sub-system 1 and 2. The above result has a significant application when the state is a ground state of Hamiltonian of a one-dimensional integer lattice $\mathbf{Z}$. To explain the application to quantum ground states, we introduce the time evolution of infinite volume systems and the ground state in terms of positive linear functionals. For a while, we consider d=1 systems. By *Interaction* we mean an assignment $\{\Psi(X)\}$ of each finite subset X of $\mathbf{Z}$ to a selfadjoint operator $\Psi(X)$ in $\mathfrak{A}_X$. We say that an interaction is of finite range if there exists a positive number r such that $\Psi(X) = 0$ if that the diameter of X is larger than r. An interaction is translationally invariant if and only if $\tau_j(\Psi(X)) = \Psi(X + j)$ for any $X \subset \mathbf{Z}^d$ and for any $j \in \mathbf{Z}^d$. We consider not necessarily translationally invariant finite range interactions (range = r). We assume the following the condition of boundedness :

$$\sup_{j \in \mathbf{Z}} \sum_{X \ni j} \frac{||\Psi(X)||}{|X|} < \infty, \tag{2.1}$$

where $|X|$ is the cardinality of $X (\subset \mathbf{Z})$. The infinite volume Hamiltonian H is an infinite sum of $\{\Psi(X)\}$,

$$H = \sum_{X \subset \mathbf{Z}} \Psi(X).$$

This sum does not converge in the norm topology, however the following commutator makes sense:

$$\delta(Q) = [H, Q] = \lim_{N\to\infty} [H_N, Q] = \sum_{X\subset\mathbf{Z}} [\Psi(X), Q], \quad Q \in \mathfrak{A}_{loc}$$

where $H_N = \sum_{X\subset[-N,N]} \Psi(X)$.
Then,the following limit exists for any real t in the C^* norm topology:

$$\alpha_t(Q) = \lim_{N\to\infty} e^{itH_N} Q e^{-itH_N}$$

for any element Q of $\mathfrak{A}$. δ is the generator of α_t

We call $\alpha_t(Q)$ the time evolution of Q. It is known that $\alpha_t(Q)$ as a function of t has an extension to an entire analytic function $\alpha_z(Q)$ for any $Q \in \mathfrak{A}_{loc}$.

Definition 2.4 *Suppose the time evolution $\alpha_t(Q)$ associated with an interaction satisfying (2.1) is given. Let φ be a state of $\mathfrak{A}$. φ is a ground state of H if and only if*

$$\varphi(Q^*[H, Q]) = \frac{1}{i}\frac{d}{dt}\varphi(Q^*\alpha_t(Q))|_{t=0} \geq 0 \tag{2.2}$$

for any Q in $\mathfrak{A}_{loc}$.

Suppose that φ is a ground state for α_t . In the GNS representation of $\{\pi_\varphi(\mathfrak{A}), \Omega_\varphi, \mathfrak{H}_\varphi\}$, there exists a positive selfadjoint operator $H_\varphi \geq 0$ such that

$$e^{itH_\varphi}\pi_\varphi(Q)e^{-itH_\varphi} = \pi_\varphi(\alpha_t(Q)), \quad e^{itH_\varphi}\Omega_\varphi = \Omega_\varphi$$

for any Q in $\mathfrak{A}$. Roughly speaking, the operator H_φ is the effective Hamiltonian on the physical Hilbert space $\mathfrak{H}_\varphi$ obtained after regularization via subtraction of the vacuum energy.

The spectral gap of an infinite system is that of H_φ. Note that, in principle, a different choice of a ground state gives rise to a different spectrum.

Definition 2.5 *We say that H_φ has a spectral gap if 0 is a non-degenerate eigenvalue of H_φ and for a positive $M > 0$, H_φ has no spectrum in $(0, M)$,i.e. $Spec(H_\varphi) \cap (0, M) = \emptyset$.*

It is easy to see that H_φ has a spectral gap if and only if there exists a positive constant M such that

$$\varphi(Q^*[H, Q]) \geq M(\varphi(Q^*Q) - |\varphi(Q)|^2). \tag{2.3}$$

In [6], M.B.Hastings proved boundedness of entanglement entropy if the finite volume Hamiltonian H_N has non degenerate ground state and the spectral gap is open uniformly in N. We can extend his argument to infinite volume ground state with spectral gap.

Theorem 2.6 *Let H be a finite range Hamiltonian satisfying the boundedness condition (2.1) and let φ be a ground state of H with spectral gap (2.3) . Then the split property is valid for $\mathfrak{A}_{(-\infty,0]}$ and $\mathfrak{A}_{[1,\infty)}$.*

We combine the above results and those of [9]. We consider half-odd integer spin $SU(2)$ symmetry of quantum spin chains and a U(1) symmetry of spinless Fermion. At this stage we assume translational invariance of Hamiltonians and their ground states.

Let $u(g)$ be the spin S representation of $SU(2)$ and γ_g be the infinite product type action $SU(2)$ on $\mathfrak{A}$ associated with $u(g)$.

$$(\cdots u(g) \otimes u(g) \otimes \cdots)Q(\cdots u(g) \otimes u(g) \otimes \cdots)^{-1} = \gamma_g(Q), \;\; Q \in \mathfrak{A}$$

Theorem 2.7 *Consider the quantum spin chain on* $\mathbf{Z}$ *and the spin at each site is a half-odd integer. Let* H_S *be a translationally invariant ,* $SU(2)$ *gauge invariant finite range Hamiltonian. Suppose that* φ *is a translationally invariant pure ground state of* H_S*. Assume that* φ *is* $SU(2)$ *invariant(* γ_g *invariant for any* g *in* $SU(2)$*). Then, there exists gapless excitation in the sense that* $Spec(H_\varphi) \cap (0, M) \neq \emptyset$ *for any positive* M*.*

Via Wigner-Jordan transform, we have equivalence of spin models and fermion on $\mathbf{Z}$. Next we consider fermions on an integer lattice $\mathbf{Z}$. Due to anti-commutativity we impose parity invariance for states, otherwise the split property cannot be defined. Let c_j^* and c_j be the creation annihilation operators satisfying the standard canonical anti-commutation relations:

$$\{c_i, c_j\} = 0, \; \{c_i^*, c_j^*\} = 0, \; \{c_i, c_j^*\} = \delta_{ij}1 \quad i, j \in \mathbf{Z}$$

By $\mathfrak{A}^F$, we denoted the C^*-algebra generated by c_i^* and c_j. $\mathfrak{A}^F$ is referred to as *CAR algebra*. The subalgebras $\mathfrak{A}^F_{loc}$, $\mathfrak{A}^F_\Lambda$, $\mathfrak{A}^F_1$, $\mathfrak{A}^F_2$ of $\mathfrak{A}^F$ are defined as before. Let Θ, γ^F_θ, and τ^F_k be the automorphisms of $\mathfrak{A}^F$ determined by

$$\Theta(c_i) = -c_i, \; \Theta(c_i^*) = -c_i^*,$$

$$\gamma^F_\theta(c_i^*) = e^{i\theta}c_i^*, \quad \gamma^F_\theta(c_i) = e^{-i\theta}c_i,$$

$$\tau^F_k(c_i) = c_{i+k}, \; \tau^F_k(c_i^*) = c_{i+k}^*$$

γ^F_θ (resp. τ^F_k)) is referred to as the $U(1)$ gauge transformation (resp. translation).

Suppose that φ_{Λ^c} is a state of $\mathfrak{A}^F_{\Lambda^c}$ and that a state φ_Λ of $\mathfrak{A}^F_\Lambda$ is Θ invariant. The product state $\varphi_\Lambda \otimes \varphi_{\Lambda^c}$ of $\mathfrak{A}^F$ specified with

$$\varphi_\Lambda \otimes \varphi_{\Lambda^c}(Q_1 Q_2 = \varphi_\Lambda(Q_1)\varphi_{\Lambda^c}(Q_2) \quad (\, Q_1 \in \varphi_\Lambda, \; Q_2 \in \varphi_{\Lambda^c})$$

can be defined. The split property and the boundedness of entanglement entropy can be introduced for fermion systems as before.

Theorem 2.8
Let φ *be a* Θ *invariant state of* $\mathfrak{A}^F$ *for which the area law of entanglement entropy holds. Then the split property is valid for* $\mathfrak{A}^F_1$ *and* $\mathfrak{A}^F_2$*.*

For fermion systems we consider finite range Hamiltonians satisfying

$$H^F = \sum_{j=-\infty}^{\infty} h_j$$
$$h_j \in \mathfrak{A}^F_{[j-r,j+r]}, \quad \Theta(h_j) = h_j, \quad ||h_j|| \leq C \tag{2.4}$$

Corollary 2.9 *Let H^F be a finite range Hamiltonian satisfying the boundedness condition (2.1) and let φ be a ground state of H^F with spectral gap (2.3) . Then the split property is valid for $\mathfrak{A}^F_L$ and $\mathfrak{A}^F_R$.*

By the standard Fock state we mean the state ψ_F specified by the identity $\psi_F(c_j^* c_j) = 0$ for any j and the standard anti-Fock state is the state ψ_{AF} specified by the identity $\psi_{AF}(c_j c_j^*) = 0$ for any j.

Theorem 2.10 *Consider the spinless Fermion lattice system on $\mathbf{Z}$. Let H_F be a translationally invariant , $U(1)$ gauge invariant finite range Hamiltonian. Suppose that φ is a $U(1)$ gauge invariant , translationally invariant pure ground state of H_F and that $\varphi \neq \psi_F$, $\varphi \neq \psi_{AF}$.*
Then, gapless excitation exists between the ground state energy and the rest of the spectrum of the effective Hamiltonian .

We return to systems in $\mathbf{Z}^d$. It may be interesting to know if the split property is useful for higher dimensional systems. The following result shows that the split property is too restrictive.

Theorem 2.11 *We assume that $2 \leq d$. Set*

$$\Lambda_1 = \{j = (j_1, j_2, \cdots, j_d) \in \mathbf{Z}^d | 1 \leq j_1\}, \Lambda_2 = \{j = (j_1, j_2, \cdots, j_d) \in \mathbf{Z}^d | j_1 \leq 0\}.$$

Let φ be a translationally invariant pure state of $\mathfrak{A}$ on $\mathbf{Z}^d$ If the split property holds for the above choice of Λ_1 Λ_2, φ is a product state in the sense that

$$\varphi = \otimes \varphi_j$$

where φ_j is a pure state on $\Lambda(j) = \{j = (j_1, j_2, \cdots, j_d) \in \mathbf{Z}^d | j_1 = j\}$.

Next we explain a significance of our results from a viewpoint of quantum information theory. A pair of mutually commuting algebras on a Hilbert space is called a bipartite system. In our context, we have in mind, for example, one-dimensional conductor we consider von Neumann algebras $\mathfrak{M}_1 = \pi_\varphi(\mathfrak{A}_1)''$ and $\mathfrak{M}_2 = \pi_\varphi(\mathfrak{A}_2)''$ where $\Lambda_1 = (-\infty, 0]$ and $\Lambda_2 = [1, \infty)$. The maximally entangled qubit we consider here is a tensor product of two commuting matrix algebras $M_n(\mathbf{C})$ with a pure state in which restriction to each tensor component is the tracial state of $M_n(\mathbf{C})$ and one question is whether we can extract maximally entangled qubit pairs using an physical operation T which does not generate entanglement itself. To be specific we introduce some notions we use here.

Definition 2.12 *A local operation between two bipartite systems $M_n(\mathbf{C}) \otimes M_n(\mathbf{C})$ and $\mathfrak{M}_1 \cup \mathfrak{M}_2$ is a unital completely positive map $T : M_n(\mathbf{C}) \otimes M_n(\mathbf{C}) \to (\mathfrak{M}_1 \cup \mathfrak{M}_2)''$ such that*

1. *$T(M_n(\mathbf{C}) \otimes 1) \subset \mathfrak{M}_1$ and $T(1 \otimes M_n(\mathbf{C})) \subset \mathfrak{M}_2$,*
2. *and $T(AB) = T(A)T(B)$ holds for all $A \in M_n(\mathbf{C}) \otimes 1$, $B \in 1 \otimes M_n(\mathbf{C})$.*

Usual distillation protocols describe procedures, to extract a certain amount of entanglement *per system*, if a large (possibly infinite) number of equally prepared systems is available. However, if we study an infinite quantum spin chain, we have already a system consisting of infinitely many particles. Hence one copy of the chain could be sufficient for distillation purposes, and if the total amount of entanglement contained in the system is infinite, it might be even possible to extract infinitely many singlets from it. This idea is the motivation for the following definition.

Definition 2.13 *Consider a state φ of a bipartite system $(\mathfrak{M}_1 \cup \mathfrak{M}_2)''$. The quantity $E_1(\omega) = \log_2(n)$ is called the* one copy entanglement *of φ, if n is the biggest integer $n \geq 2$ which admit for each $\epsilon > 0$ a local operation $T_\epsilon : M_n(\mathbf{C}) \otimes M_n(\mathbf{C}) \to (\mathfrak{M}_1 \cup \mathfrak{M}_2)''$ such that*

$$\varphi(T_\epsilon(|\chi_n\rangle\langle\chi_n)) > 1 - \epsilon, \quad \chi_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} |jj\rangle \tag{2.5}$$

holds. If no such n exists we set $E_1(\omega) = 0$ and if (2.5) holds for all $d \geq 2$ we say that φ contains infinite one copy entanglement *(i.e. $E_1(\varphi) = \infty$).*

Then, combined with results in [7], we obtain the following.

Proposition 2.14 *Let φ be a tanslationally invariant ground state of a finite range Hamiltonian. Consider von Neumann algebras $\mathfrak{M}_1 = \pi_\varphi(\mathfrak{A}_{(-\infty,0]})''$ and $\mathfrak{M}_2 = \pi_\varphi(\mathfrak{A}_{[1,\infty)})''$. (i) If the spectral gap opens it is impossible for φ to contain infinite one copy entanglement (i.e. $E_1(\varphi) < \infty$).*

If the Haag duality $\pi_\varphi(\mathfrak{A}_{(-\infty,0]})'' = \pi_\varphi(\mathfrak{A}_{[1,\infty)})'$ is valid and the spin is half-odd integer, any tanslationally invariant $SU(2)$ invariant pure state contains infinite one copy entanglement. In [8] we claimed this Haag duality is valid for the GNS representaiton of any tanslationally invariant pure state, however , there is a gap in proof and it is not yet known that the duality is valid for general tanslationally invariant pure state,.

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On GNS representation of supersymmetric states in $\mathbf{C}^*$-dynamical systems

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Abstract

We study supersymmetry in an abstract $\mathbf{C}^*$-algebraic setting. This paper is a short review of 'Supersymmetry for infinitely extended $\mathbf{C}^*$-systems' by this author, and was presented at the conference " Mathematical Quantum Field Theory and Renormalization Theory "dedicated to Professor K. R. Ito and Professor I. Ojima.

Key Words.
Supersymmetry in non-compact space. Superderivations.

1 Superderivation in $\mathbf{C}^*$-systems

We study supersymmetry under a general $\mathbf{C}^*$-algebraic setting. We do not restrict our consideration to Fock representations of some boson-fermion algebras. A derivation satisfying the graded Leibniz rule with respect to some non-trivial $\mathbb{Z}^2$-periodic automorphism is referred to as superderivation. Infinitesimal transformations abstractly defined by such superderivation imply the usual sense of supersymmetry between bosons and fermions, and also supersymmetric relations among assembly of sole fermions without bosons.

It is important to start from superderivation rather than fermionic charge operators, since its existence is assured irrespective of broken or unbroken supersymmetry. We consider strongly continuous $\mathbf{C}^*$-dynamical systems. Those typically describe fermion (or quantum spin) lattice models. For continuous quantum field models, we need more careful treatment in formulation of supersymmetric kinematics due to the lack of some properties e.g. (1.12) below, see our paper 'Supersymmetry for infinitely extended $\mathbf{C}^*$-systems'.

We are given a graded $\mathbf{C}^*$-algebra $\mathcal{F} = \mathcal{F}^e \oplus \mathcal{F}^o$, with its grading automorphism γ, where

$$\mathcal{F}^e := \{F \in \mathcal{F} \mid \gamma(F) = F\}, \tag{1.1}$$

$$\mathcal{F}^o := \{F \in \mathcal{F} \mid \gamma(F) = -F\}. \tag{1.2}$$

Supersymmetric transformation is given by *superderivation* δ. Let $\mathcal{D}_\delta$ denote its domain.

It is an *odd* linear map

$$\delta \cdot \gamma = -\gamma \cdot \delta \quad \text{on } \mathcal{D}_\delta, \tag{1.3}$$

satisfying the *graded Leibniz rule*

$$\delta(FG) = \delta(F)G + \gamma(F)\delta(G) \quad \text{for } F, G \in \mathcal{D}_\delta. \tag{1.4}$$

It is easy to see that $\mathcal{D}_\delta$ is a γ-invariant subalgebra of $\mathcal{F}$ and

$$\delta(\mathcal{D}_\delta^e) \subset \mathcal{F}^o, \quad \delta(\mathcal{D}_\delta^o) \subset \mathcal{F}^e,$$

where $\mathcal{D}_\delta^e := \mathcal{D}_\delta \cap \mathcal{F}^e$, $\mathcal{D}_\delta^o := \mathcal{D}_\delta \cap \mathcal{F}^o$. We assume that $\mathcal{D}_\delta$ is $*$-invariant. The conjugate of δ is given by

$$\overline{\delta}(F) := -\delta\big(\gamma(F^*)\big)^*, \quad \text{for} \quad F \in \mathcal{D}_\delta. \tag{1.5}$$

(Note that $\overline{\delta}$ is the conjugate, not the closure of δ.) Let δ_q denote the bounded superderivation for $q \in \mathcal{F}^o$ defined by

$$\delta_q(F) := \begin{cases} [q,\ F] & \text{for } F \in \mathcal{F}^e, \\ \{q,\ F\} & \text{for } F \in \mathcal{F}^o. \end{cases} \tag{1.6}$$

We can immediately check

$$\overline{\delta_q} = \delta_{q^*}. \tag{1.7}$$

Let α_t be a strongly continuous one-parameter group of $*$-automorphisms of $\mathcal{F}$, $t \in \mathbb{R}$, that encodes the time development. It should preserve the grading,

$$\alpha_t \cdot \gamma = \gamma \cdot \alpha_t. \tag{1.8}$$

We assume the commutativity of α_t and δ

$$\delta \cdot \alpha_t = \alpha_t \cdot \delta \text{ for all } t \in \mathbb{R} \quad \text{on } \mathcal{D}_\delta, \tag{1.9}$$

hence $\mathcal{D}_\delta$ is invariant under α_t.

Let ω be a state of $\mathcal{F}$, and let $\big(\pi,\ \mathcal{H},\ \Omega\big)$ denote the GNS triplet for ω satisfying

$$\omega(F) = \Big(\Omega,\ \pi(F)\Omega\Big), \quad \text{for } F \in \mathcal{F}.$$

The von Neumann algebra generated by the GNS representation is denoted $\mathfrak{M} := \pi(\mathcal{F})'' \subset \mathfrak{B}(\mathcal{H})$.

We do not restrict our consideration to particular models, even nor particular $\mathbf{C}^*$-algebras. We have, however, in mind homogeneous fermion (or spin) lattice models with constant finite numbers of degree of freedom at each site of the infinitely extended lattice. Hence the $\mathbf{C}^*$-algebra $\mathcal{F}$ under consideration is typically a CAR-algebra. Recently, several statistical mechanical fermion lattice models with hidden exact supersymmetric relations have been studied. Here we only mention an earlier work by H. Nikolai, 'Supersymmetry and spin systems' in 1976.

Let

$$\delta_0 := -i\frac{d}{dt}\alpha_t\Big|_{t=0} \tag{1.10}$$

denote the generator of α_t and $\mathcal{D}_{\delta_0}$ the domain of δ_0. By (1.8), $\mathcal{D}_{\delta_0}$ is a γ-invariant subalgebra and δ_0 is even, i.e.

$$\delta_0(\mathcal{D}^e_{\delta_0}) \subset \mathcal{F}^e,\ \delta_0(\mathcal{D}^o_{\delta_0}) \subset \mathcal{F}^o.$$

We assume that the domain $\mathcal{D}_\delta$ of δ, and also $\mathcal{D}_{\delta_0}$ of δ_0 are both norm dense in $\mathcal{F}$. Also δ is assumed to be a closed linear map with respect to $\mathbf{C}^*$-norm.

Let $\mathcal{A}_\circ$ be a γ-invariant norm-dense $*$-subalgebra of $\mathcal{F}$, on which we will formulate supersymmetric relations. For this sake, we set the following:

$$\mathcal{A}_\circ \subset \mathcal{D}_\delta \cap \mathcal{D}_{\delta_0}. \tag{1.11}$$

$$\delta(\mathcal{A}_\circ) \subset \mathcal{A}_\circ. \tag{1.12}$$

$$\mathcal{A}_\circ \text{ is core for } \delta. \tag{1.13}$$

$$\mathcal{A}_\circ \text{ is core for } \delta_0. \tag{1.14}$$

The above (1.11) (1.12) and (1.13) for $\overline{\delta}$ in place of δ are obviously valid. Due to this (1.12), one can repeat superderivations δ and $\overline{\delta}$, so that supersymmetric kinematics is formulated.

Definition 1.1. Let δ be a superderivation and δ_0 be a derivation satisfying all the assumptions given above. The following set of kinematical relations is referred to as supersymmetric kinematics:

$$\delta \cdot \delta = \mathbf{0},\quad \overline{\delta} \cdot \overline{\delta} = \mathbf{0} \ \text{ on } \ \mathcal{A}_\circ, \tag{1.15}$$

$$\delta_0 = \delta \cdot \overline{\delta} + \overline{\delta} \cdot \delta \ \text{ on } \ \mathcal{A}_\circ, \tag{1.16}$$

where $\mathbf{0}$ is a zero map.

Remark 1.2. We do not require (even not permit) invariance of $\mathcal{A}_\circ$ under α_t:

$$\alpha_t(\mathcal{A}_\circ) \subset \mathcal{A}_\circ. \tag{1.17}$$

For fermion lattice models, we may choose the set of all subsystems on local finite regions as $\mathcal{A}_\circ$. Clearly, it is not stable under time development for non-trivial interaction. Without (1.17) we may keep out of some pitfall on the domain of superderivation pointed out by Kishimoto-Nakamura 1994.

2 GNS construction of supersymmetric states

The following definition of unbroken supersymmetry is given without reference to time development (α_t or δ_0).

Definition 2.1. If a state ω on $\mathcal{F}$ is invariant under superderivation δ,

$$\omega(\delta(F)) = 0, \text{ for any } F \in \mathcal{D}_\delta, \tag{2.1}$$

then it is said to be supersymmetric.

Let $\big(\pi,\ \mathcal{H},\ \Omega\big)$ be a GNS triplet for a supersymmetric state ω on $\mathcal{F}$. Let

$$\mathcal{H}_\circ := \pi(\mathcal{A}_\circ)\Omega. \tag{2.2}$$

Let

$$\mathcal{H}_\delta := \pi(\mathcal{D}_\delta)\Omega, \tag{2.3}$$

which obviously contains $\mathcal{H}_\circ$. Supercharge operators are given by superderivations in the GNS Hilbert space for any supersymmetric state.

Proposition 2.2 (Buchholz-Ojima)**.** *Assume that ω is a (not necessarily even) supersymmetric state on $\mathcal{F}$ with respect to δ. On the dense subspace $\mathcal{H}_\circ$, let*

$$Q\pi(A)\Omega := \pi\big(\delta(A)\big)\Omega, \quad A \in \mathcal{A}_\circ. \tag{2.4}$$

It defines a closable linear operator satisfying

$$Q\Omega = 0. \tag{2.5}$$

The following operator equalities hold for $A \in \mathcal{A}_\circ$

$$\pi\big(\delta(A)\big) = Q\pi(A) - \pi\big(\gamma(A)\big)Q \quad \text{on } \mathcal{H}_\circ, \tag{2.6}$$

$$\pi\big(\overline{\delta}(A)\big) = Q^*\pi(A) - \pi\big(\gamma(A)\big)Q^* \quad \text{on } \mathcal{H}_\circ, \tag{2.7}$$

where Q^* denotes the adjoint of Q.

Lemma 2.3. *Under the same assumption as in Proposition 2.2, let $\overline{Q}$ denote the closure of Q. Then $\pi(\mathcal{D}_\delta)\Omega \subset \mathrm{Dom}(\overline{Q})$ and*

$$\overline{Q}\pi(A)\Omega = \pi\big(\delta(A)\big)\Omega, \quad A \in \mathcal{D}_\delta. \tag{2.8}$$

From now on, we always consider the closure $\overline{Q}$ and shall denote this by Q for simplicity, since there is no fear of confusion.

3 Supersymmetric kinematics and its realization

In the preceding section, we have considered solely δ not Hamiltonian dynamics or its generator. We shall see how the supersymmetric kinematical relations in terms of (super)derivations in Definition 1.1 are implemented as operator forms in the GNS Hilbert space for unbroken supersymmetry. The following shows the nilpotency of supercharge operators.

Proposition 3.1. *Let ω be a (not necessarily even) supersymmetric state with respect to δ, and Q be the closure of* (2.4) *implementing δ in the GNS representation $\big(\pi,\, \mathcal{H},\, \Omega\big)$ for ω. Then*

$$Q^2 = 0, \quad Q^{*2} = 0, \tag{3.1}$$

as unique extension of densely defined bounded operators.

In the following proposition, we consider symmetrization of superderivations and supercharges.

Proposition 3.2. *Under the same assumption as in Proposition 3.1, take*

$$Q_1 := Q + Q^* \quad Q_2 := i(Q - Q^*) \tag{3.2}$$

on the domain $\mathrm{Dom}(Q_1) = \mathrm{Dom}(Q_2) = \mathrm{Dom}(Q) \cap \mathrm{Dom}(Q^*)$. Then Q_1 and Q_2 are symmetric operators implementing symmetric superderivations

$$\delta_1 := \delta + \overline{\delta}, \quad \delta_2 := i(\delta - \overline{\delta}), \tag{3.3}$$

in the following manner,

$$Q_1\pi(A)\Omega = \pi(\delta_1(A))\Omega, \quad Q_2\pi(A)\Omega = \pi(\delta_2(A))\Omega, \quad A \in \mathcal{D}_\delta. \tag{3.4}$$

On the dense subspace $\mathcal{H}_\circ$, which is included in $\mathrm{Dom}(Q_1) = \mathrm{Dom}(Q_2)$, the following operator equalities are satisfied :

$$Q_1^2 = Q_2^2 = QQ^* + Q^*Q, \tag{3.5}$$

$$\{Q_1,\ Q_2\} = 0. \tag{3.6}$$

Now we can relate the above formula of superderivations to Hamiltonian.

Proposition 3.3. *Let ω be a supersymmetric state with respect to δ. There is a self-adjoint operator H satisfying*

$$H\Omega = 0, \tag{3.7}$$

$$\pi\left(\alpha_t(F)\right) = U(t)\pi(F)U(t)^{-1} \text{ for } F \in \mathcal{F} \text{ with } U(t) := e^{itH}. \tag{3.8}$$

Also the following operator equalities are satisfied on $\mathcal{H}_\circ$:

$$H = QQ^* + Q^*Q = Q_1^2 = Q_2^2, \tag{3.9}$$

$$[H,\ Q] = [H,\ Q^*] = [H,\ Q_1] = [H,\ Q_2] = 0. \tag{3.10}$$

For simplicity of notation, by δ_s and Q_s we will denote *symmetric* superderivation and its symmetric supercharge operator.

We now assume that ω is even,

$$\omega\big(\gamma(F)\big) = \omega(F), \text{ for any } F \in \mathcal{F}. \tag{3.11}$$

(Note that so far we have not used the evenness.) By this invariance, there is a grading automorphism $\tilde{\gamma}$ on $\mathfrak{M}$ such that

$$\tilde{\gamma}\big(\pi(F)\big) = \pi\big(\gamma(F)\big).$$

It can be written as

$$\tilde{\gamma}(x) = \mathrm{Ad}(\Gamma)(x) = \Gamma^* x \Gamma, \quad \text{for } x \in \mathfrak{M},$$

where Γ is a self-adjoint unitary,

$$\Gamma = \Gamma^*,\ \Gamma^2 = 1,$$

and satisfies

$$\Gamma\Omega = \Omega.$$

Theorem 3.4. *Let ω be an even supersymmetric state with respect to symmetric superderivation δ_s. Assume supersymmetric kinematics as in Definition 1.1. Let Q_s denote the closed extension of supercharge operator implementing δ_s, and $\tilde{\gamma}$ the grading automorphism in the GNS Hilbert space for ω. Then Q_s is self-adjoint, and is essentially* *self-adjoint on any core of H. Also the following exact operator equalities are satisfied:*

$$H = Q_s^2, \tag{3.12}$$

$$\tilde{\gamma}(Q_s) = -Q_s. \tag{3.13}$$

The domain of Q_s is $\pi(\mathcal{D}_\delta)$-invariant, namely,

$$\pi(A)\mathrm{Dom}(Q_s) \subset \mathrm{Dom}(Q_s) \quad \textit{for any } A \in \mathcal{D}_\delta. \tag{3.14}$$

The spectrum of realized Hamiltonian for supersymmetric states is characterized in the following way.

Corollary 3.5. *Under the same setting as in Theorem 3.4, for the self-adjoint Hamiltonian H and the self-adjoint supercharge Q_s, let*

$$H = \int_{\mathbb{R}} \lambda dE(\lambda), \quad Q_s = \int_{\mathbb{R}} \lambda dF(\lambda),$$

where $\{dE(\lambda)\}_{\lambda\in\mathbb{R}}$ and $\{dF(\lambda)\}_{\lambda\in\mathbb{R}}$ are uniquely determined projection-valued measures. Then the support of $\{dE(\lambda)\}_{\lambda\in\mathbb{R}}$ is included in $\mathbb{R}_+$, non-negative real numbers, and

$$\Gamma F(\lambda)\Gamma = F(-\lambda), \quad \textit{for} \ \ \lambda \in \mathbb{R}_+, \tag{3.15}$$

$$E(\lambda^2) = F(\lambda) + F(-\lambda), \quad \textit{for} \ \ \lambda > 0, \tag{3.16}$$

$$F(0) = E(0), \tag{3.17}$$

which includes Ω.

The following statement was already obtained by Buchholz. We provide a new proof based on Theorem 3.4, namely the self-adjointness of supercharges. Our proof persist in a more general situation suitable for quantum field theory, which is not covered in the original work by Buchholz. See 'Supersymmetry for infinitely extended $\mathbf{C}^*$-systems' for details.

Theorem 3.6. *Assume the set of supersymmetric kinematical relations in the $\mathbf{C}^*$-algebra $\mathcal{F}$. Then*

(a) *If ω is a supersymmetric state, then it is a ground state.*

(b) *The set of supersymmetric states on $\mathcal{F}$ is face.*

It is well known that Hamiltonian for ground states is *observable* in the sense that its projection-measure belongs to the von Neumann algebra generated by the GNS representation. We shall show that supercharges realized in the GNS Hilbert space for supersymmetric states are also observable.

Theorem 3.7. *Assume supersymmetric kinematics. Let ω be an even supersymmetric state with respect to symmetric superderivation δ_s. Let Q_s denote the self-adjoint supercharge operator implementing δ_s in the GNS Hilbert space for ω. Then Q_s is affiliated to $\mathfrak{M}$, the von Neumann algebra generated by the GNS representation.*

4 Approximately inner superderivation

A strongly continuous one-parameter group of $*$-automorphisms (or a derivation with its norm-dense domain) on a $\mathbf{C}^*$-algebra is called approximately inner, if it can be approximated by inner one-parameter groups of $*$-automorphisms (bounded derivations) pointwisely in $\mathbf{C}^*$-norm. Let us introduce a similar class of superderivations and investigate its properties.

Definition 4.1. A superderivation on a graded $\mathbf{C}^*$-algebra $\mathcal{F}$ is said to be approximately inner, if there exists a norm-dense subalgebra $\mathcal{E}(\subset \mathcal{D}_\delta)$ which is a core for δ, and a sequence $\{q_n\}$ of odd elements in $\mathcal{F}$ satisfying

$$\delta(A) = \lim_n \delta_{q_n}(A) \quad \text{for each} \quad A \in \mathcal{E} \tag{4.1}$$

in the norm topology, where δ_{q_n} is bounded superderivation for q_n defined by (1.6).

Remark 4.2. The above $\mathcal{E}$ and $\mathcal{A}_\circ$ are norm-dense subalgebras in $\mathcal{D}_\delta$ and are both cores for δ, however, not necessarily same. (On $\mathcal{E}$ the pointwise approximation of δ by local superderivations is satisfied, while on $\mathcal{A}_\circ$ supersymmetric relation (1.15, 1.16) is satisfied.)

Remark 4.3. In this and next sections, the condition (1.12) is not needed.

The oddness of $\{q_n\}$ in Definition 4.1 is in fact a consequence.

Proposition 4.4. *Let δ be a superderivation on $\mathcal{F}$ such that there exists a sequence $\{q_n\}$ of (not necessarily odd) elements in $\mathcal{F}$ satisfying* (4.1). *Then it is possible to take a sequence of odd elements $\{q_n^o\}$ satisfying* (4.1).

Similarly, we obtain

Proposition 4.5. *Let δ_s be a symmetric, approximately inner superderivation on $\mathcal{F}$. Then it is possible to take a sequence of self-adjoint odd elements $\{q_n\}$ satisfying* (4.1).

5 Existence of invariant states

We look for a sufficient condition for the existence of invariant states under superderivation δ. This problem is raised by Buchholz. For (even) derivations, there are generically plenty of invariant states, e.g. ground states and equilibrium temperature states at arbitrary temperature. On the other hand, for some superderivation there is no invariant state, namely, supersymmetry is broken. Take simply a bounded superderivation δ_q with q being odd self-adjoint and q^2 being strictly positive.

It is well known and frequently used that if supersymmetry is unbroken for an *arbitrary* finite volume V in a total space, then it remains unbroken in the infinite volume limit. The reason goes as follows. If the ground-state energy $E(V)$ confined in V is zero due to *local* unbroken supersymmetry, then its infinite volume limit $\lim_{V\to\infty} E(V)$ obviously stays zero. Since energy is an order-parameter for supersymmetry, unbroken supersymmetry is concluded.

Then, what if the status of unbroken-broken supersymmetry depends on subsystems imbedded or boundary conditions? There actually occurs such subtle situation in a concrete fermion lattice model. Our answer is that there is no breaking. It follows from the following general statement for approximately inner superderivations. It makes the above (somehow heuristic) argument rigorous.

Theorem 5.1. *For a graded* $\mathbf{C}^*$*-algebra* $\mathcal{F}$*, suppose that there is an increasing sequence* $\mathbf{1} \in M_1 \subset M_2 \cdots \subset M_n \subset \cdots$ *of subalgebras of* $\mathcal{F}$ *such that* $\bigcup_n M_n$ *is norm dense. Let* δ *be an approximately inner superderivation on* $\mathcal{F}$ *such that* $\{q_n\}$ *is a set of odd elements satisfying* (4.1) *on* $\mathcal{E} = \bigcup_n M_n$*, which is assumed to be a core for* δ*. Suppose further that for any* δ_{q_n}*, there is a state* ω_n *on* $\mathcal{F}$ *such that*

$$\omega_n\big(\delta_{q_n}(A)\big) = 0 \quad \textit{for any} \quad A \in M_n. \tag{5.1}$$

Then there is a supersymmetric state on $\mathcal{F}$ *with respect to* δ*.*

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A unified scheme of measurement and amplification processes

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1 Introduction

I present a unified scheme of quantum measurements containing the aspects of amplification processes, or translation to macroscopic levels on the basis of [11], a joint work with Prof. I. Ojima. This formulation is based on micro-macro duality [5] as a mathematical expression ofthe general idea of quantum-classical correspondence. An essential difference between classical physics and quantum physics can be seen in their algebraic structures describing the physical systems. Quantum systems are generally described by non-commutative algebras (C*-algebras, von-Neumann algebras, and so on) consists of observables. The concept of amplification or translation to macro is nothing but how the information of (quantum) non-commutative observables is translated in the words of macroscopic worlds, which are described by (classical) commutative ones. This point of view is essential for understanding quantum measurements. In mathematical words, the transformation from quantum internal degrees of freedom to classical ones is constructed by Kac-Takesaki operator or multiplicative unitary, which plays a fundamental roles in harmonic analysis, and the dual structure between q-numbers and c-numbers can be understood via the Helgason duality controlling the Radon transform (in somehow generalized meanings). A concrete observation for the case of Stern-Gerlach experiment can be seen in [11], and recently, we obtained more general formulation which is applicative to various situation.

2 Sector theory and quantum-classical correspondence

Before explaining a concrete setting based on Micro-Macro duality, let us review the basic idea of sector theory for preparation. Suppose the system we are interested in is described by a C*-algebra $\mathfrak{A}$ of (quantum) observables. Then we can define a *sector* using *quasi-equivalence* of factor (centre-trivial) representation defined as follows: For two representations π_1 and π_2 of a C*-algebra $\mathfrak{A}$, they are quasi-equivalent if and only if they are unitary equivalent up to multiplicity, i.e.,

$$\exists m, n, \quad \pi_1^{\oplus m} \simeq \pi_2^{\oplus n},$$

where $\pi^{\oplus n} := \underbrace{\pi \oplus \cdots \oplus \pi}_{n}$ for a representation π of $\mathfrak{A}$ and $n \in \mathbb{N}$, and then denoted by $\pi_1 \approx \pi_2$.

This condition is equivalent to the following isomorphism of representing von Neumann algebras $\pi_1(\mathfrak{A})'' \simeq \pi_2(\mathfrak{A})''$. Then a sector is defined as a quasi-equivalent classes of factor representations. One of the conditions for disjointness of different sectors is equivalently written down as follows: for the representations π_a, π_b of $\mathfrak{A}$ on $\mathfrak{H}_{\pi_a}$ and $\mathfrak{H}_{\pi_b}$, respectively, for any bounded operator $T : \mathfrak{H}_{\pi_a} \to \mathfrak{H}_{\pi_b}$, $T\pi_a(A) = \pi_b(A)T$ for $\forall A \in \mathfrak{A} \Rightarrow T = 0$ i.e., π_a and π_b has no non-zero intertwiner. Then if π is not a factor representation belonging to a sector, it can be uniquely decomposed into the direct sum (or integral) of sectors, through the spectral decomposition of a non-trivial commutative algebra $\mathfrak{Z}(\pi(\mathfrak{A})'') = \pi(\mathfrak{A})'' \cap \pi(\mathfrak{A})' = \mathfrak{Z}_\pi(\mathfrak{A})$ as the centre of $\pi(\mathfrak{A})''$ admitting a "simultaneous diagonalization". Thus the Gel'fand spectrum $Sp(\mathfrak{Z}_\pi(\mathfrak{A}))$ of the centre parametrizes each sector faithfully and plays the role of *classifying space of sectors* to distinguish different sectors, and (commutative) observables belonging to $\mathfrak{Z}_\pi(\mathfrak{A})$ is regarded as *macroscopic order parameters*. This viewpoint based on factoriality of sectors extends the traditional understanding of sectors to general cases of any pure phases associated with reducible factor representations and mixed states which are common in the thermal and/or local aspects of quantum fields [4].

In the above setting we can construct a clear-cut formulation of *quantum-classical correspondence*: The microscopic quantum world is described by non-commutative elements of algebras of each sector, and, on the other hand, the macroscopic classical levels are described by means of the notion of a sector structure consisting of a family of sectors (or pure phases). We can see a clear "boundary" between microscopic and macroscopic levels as the gap of *inside of sectors* and *intersectorial level* [4, 5].

2.1 Fundamental settings of Micro-Macro duality

In the previous subsection we saw a factor algebra $\pi(\mathfrak{A})''$ generated by C*-algebra $\mathfrak{A}$ (and satisfies $\mathfrak{Z}(\pi(\mathfrak{A})'') = \mathbb{C}1$) describes microscopic quantum structure (the internal structure of a sector), so we set here a von Neumann algebra $\mathcal{M} = \pi(\mathfrak{A})''$ representing the full physical system of our attention. First we can take its (non-trivial) maximal abelian subalgebra (MASA, in short) which satisfies $\mathcal{A} = \mathcal{A}' \cap \mathcal{M}$. This MASA $\mathcal{A}$ means the macroscopic observables of the system, or more precisely, we can read the macroscopic information from $Sp(\mathcal{A})$ (to be introduced later) via experimental observation schemes. For the purpose of describing correspondences between microscopic and macroscopic structures, the unitary group $\mathcal{U}$ which generates MASA $\mathcal{A}$ (i.e., $\mathcal{A} = \mathcal{U}''$) also plays essential roles. We now have the fundamental triple $(\mathcal{M}, \mathcal{A}, \mathcal{U})$ describing the quantum system, then in the next step we construct the *dual pairs* for the latter two objects $\mathcal{A}$ and $\mathcal{U}$, i.e., the algebraic spectrum $Sp(\mathcal{A})$ of $\mathcal{A}$ and the set of group characters $\widehat{\mathcal{U}}$ of $\mathcal{U}$. We remember that $\mathcal{A}$ can be recognized as the fixed subalgebra of $\mathcal{M}$ under the action of $\mathcal{U} \curvearrowright_{\alpha} \mathcal{M}$, i.e., $\mathcal{A} = \mathcal{M}^{\mathcal{U}}$.

Let us see more details of $Sp(\mathcal{A})$ and $\widehat{\mathcal{U}}$. Owing to the relation of $\mathcal{A} \supset \mathcal{U}$, the restriction of algebraic characters $\chi \in Sp(\mathcal{A})$ onto $\mathcal{U}$ is naturally derived, and this gives the canonical embedding $Sp(\mathcal{A}) \subset \widehat{\mathcal{U}}$. Moreover, considering the triple $(\mathcal{M}, \mathcal{U}, \alpha)$ as a W*-dynamical system, we obtain its dual system and the co-action $\widehat{\mathcal{U}} \curvearrowright_{\hat{\alpha}} \mathcal{M} \rtimes_{\alpha} \mathcal{U}$. Under these settings, canonical dynamism of reconstructing quantum (microscopic) systems from the classical (observational, macroscopic) variables is summerized in the two mutually equivalent isomorphisms as below under the semi-duality condition for the action α of $\mathcal{U}$ [9]:

- $\mathcal{M} \rtimes_{\alpha} \mathcal{U} \simeq \mathcal{A} \otimes B(L^{\infty}(\mathcal{U}))$ [: amplification process]
- $(\mathcal{A} \otimes B(L^{\infty}(\mathcal{U})) \rtimes_{\hat{\alpha}} \widehat{\mathcal{U}} \simeq \mathcal{M}$ [: reconstruction]

From this viewpoint we can see that the Fourier duality (especially the role of K-T operator, as seen later) works essentially in order to connect the *full system* $\mathcal{M}$ and the *observed values* $Sp(\mathcal{A})$.

2.2 Kac-Takesaki operator for measurement coupling

In this section we introduce a class of operators called Kac-Takesaki operators (K-T operators, in short), which play central roles in the context of harmonic analysis. For a locally compact group $\mathcal{U}$, the isometry W on $L^2(\mathcal{U} \times \mathcal{U}, d\mu \otimes d\mu)$ called a Kac-Takesaki operator is defined as

$$(W\eta)(u, v) := \eta(v^{-1}u, v) \text{ for } \eta \in L^2(\mathcal{U} \times \mathcal{U}, d\mu \otimes d\mu) \text{ and } u, v \in \mathcal{U},$$

where $d\mu$ is the Haar measure of $\mathcal{U}$. It is well-known that the K-T operator W associated to $\mathcal{U}$ is characterized by the following two relations [3]:

- $W_{12}W_{23} = W_{23}W_{13}W_{12}$ on $L^2(\mathcal{U} \times \mathcal{U} \times \mathcal{U}, d\mu \otimes d\mu \otimes d\mu)$ [: pentagonal relation]
- $W(1 \otimes \lambda_u) = (\lambda_u \otimes \lambda_u)W$ [: intertwining relation]

where λ_u $(u \in \mathcal{U})$ is a regular representation of $\mathcal{U}$.

When the action $\mathcal{M} \underset{\alpha}{\curvearrowleft} \mathcal{U}$ of the measuring system is unitarily implemented, it is given in the form $\alpha_u(M) = U_u M U_u^{-1}$ $(M \in \mathcal{M},\ u \in \mathcal{U})$ via a unitary representation U of $\mathcal{U}$ on the (standard) representation Hilbert space $L^2(\mathcal{M})$ of $\mathcal{M}$. Here the representation $U(W)$ of W corresponding to $\alpha = Ad(U)$ is defined by

$$(U(W)\xi)(u) := U_u(\xi(u)) \quad \text{for } \xi \in L^2(\mathcal{M}) \otimes L^2(\mathcal{U}, d\mu)$$

and the modified intertwining and pentagonal relations $U(W)(1 \otimes \lambda_u) = (U_u \otimes \lambda_u)U(W)$ and $U(W)_{12}W_{23} = W_{23}U(W)_{13}U(W)_{12}$ hold. If we hope to obtain a familiar viewpoint to physicists, we can write down this operation in terms of the Dirac bracket notation;

$$U(W) = \int_{g \in \mathcal{U}(\mathcal{A})} U_g \otimes |g\rangle dg \langle g|.$$

In the next step, we consider the Fourier-transformed K-T operator, denoted by V. As W is a (unitary) action on $L^2(\mathcal{U} \times \mathcal{U})$, we have its Fourier transform $V = (\mathcal{F} \otimes \mathcal{F})W^*(\mathcal{F} \otimes \mathcal{F})^{-1}$, where $(\mathcal{F}\xi)(\gamma) := \int_{\mathcal{U}} \overline{\gamma(u)}\xi(u)d\mu(u)$ for $\xi \in L^2(\mathcal{U}, d\mu)$. Then V is nothing but the K-T operator of the dual group $\widehat{\mathcal{U}}$ with the Plancherel measure $d\hat{\mu}$ satisfying the relations:

$$(V\eta)(\gamma, \chi) = \eta(\gamma, \gamma^{-1}\chi) \qquad \text{for } \eta \in L^2(\widehat{\mathcal{U}}, d\hat{\mu}),$$

$$V_{23}V_{12} = V_{12}V_{13}V_{23},$$
$$V(\lambda_\gamma \otimes 1) = (\lambda_\gamma \otimes \lambda_\gamma)V.$$

Similarly, the Fourier transform of $U(W)$ is given by $\widetilde{U(W)} := (id \otimes \mathcal{F})U(W)(id \otimes \mathcal{F})^{-1}$. Owing to the SNAG theorem due to the abelianness of $\mathcal{U}$, its unitary representation $U_u \in \mathcal{U}(L^2(\mathcal{M}))$ admits the spectral decomposition $\widetilde{U(W)} = \int_{\chi \in Sp(\mathcal{A})} dE(\chi) \otimes \lambda_\chi^* =: \widetilde{U}(V)^*$. Hence the spectral decomposition of $\widetilde{U}(V)$ is given by

$$\widetilde{U}(V) = \int_{\chi \in Sp(\mathcal{A})} dE(\chi) \otimes \lambda_\chi, \tag{1}$$

and in the Dirac's notation, the action of $\widetilde{U}(V)$ on $L^2(\mathcal{M}) \otimes L^2(\widehat{\mathcal{U}})$ is given for $\gamma \in \widehat{\mathcal{U}}$, $\xi \in L^2(\mathcal{M})$, so the decomposition is

$$\widetilde{U}(V)(\xi \otimes |\gamma\rangle) = \int_{\chi \in Sp(\mathcal{A})} dE(\chi)\xi \otimes |\chi\gamma\rangle. \tag{2}$$

We will recall the last two equation in describing our measurement scheme in the next section (see also [5, 9, 7]).

3 Description of Amplification Processes by Instruments

Now we describe a measurement scheme in the above mathematical setting, especially using the typical property of K-T operator given in (2). The basic idea (the interpretation of *probe system*,*neutral position* and *instrument*, and the method for describing amplification) is found in [7].

The settings becomes clear-cut in the case $\widehat{\mathcal{U}}$ is discrete ($\Leftrightarrow \mathcal{U}$ is compact) according to the following description; owing to $\widehat{\mathcal{U}}$ being discrete, we can pick the group identity $\iota \in \widehat{\mathcal{U}}$ and substitute $\gamma = \iota$ and $\xi = \sum_{\chi \in \widehat{\mathcal{U}}} c_\chi \xi_\chi$ (a generic state of the observed system) for (2), then it gives

$$\widetilde{U}(V)(\xi \otimes |\iota\rangle) = \sum_{\chi \in \widehat{\mathcal{U}}} c_\chi \xi_\chi \otimes |\chi\rangle. \tag{3}$$

In other words, the action of $\widetilde{U}(V)$ provides the change of the probe states according to the states of the observed system, or transfer the information of the observed system into the probe system. Thus the K-T operator works to describe the mathematical essence of measurements. (In the case when the identity element $\iota \in \widehat{\mathcal{U}}$ is not represented by a normalized vector in $L^2(\mathcal{U})$ (for non-compact $\mathcal{U}$), the invariant mean $m_{\mathcal{U}}$ over $\mathcal{U}$ physically plays the equivalent role of the neutral position ι.)

3.1 Reformulation of instrument for measurement

On the bases we introduced above, we can define an instrument $\mathcal{I}$ (originally defined by Davies and Lewis [2] as a completely positive operation valued measure, and refined by Ozawa [10]) to unify all the ingredients of measurement scheme. We write down the formula according to [7]: For an initial vector state of the observed system $\omega_\xi = \langle \xi | \cdot | \xi \rangle$ and the indicator function ξ_Δ of a Borel set $\Delta \subset Sp(\mathcal{A})$,

$$\begin{aligned} \mathcal{I}(\Delta|\omega_\xi)(M) &:= (\omega_\xi \otimes m_{\mathcal{U}}) \left(\widetilde{U}(V)^*(M \otimes \chi_\Delta)\widetilde{U}(V) \right) \\ &= (\langle \xi| \otimes \langle \iota|)\widetilde{U}(V)^*(M \otimes \chi_\Delta)\widetilde{U}(V)(|\xi\rangle \otimes |\iota\rangle) \\ &= \int_\Delta d\mu(\gamma) \sqrt{\frac{dE(\gamma)}{d\mu(\gamma)}} M \sqrt{\frac{dE(\gamma)}{d\mu(\gamma)}}, \end{aligned} \tag{4}$$

where $d\mu$ is an arbitrary probability measure satisfying $dE(\gamma) \ll d\mu(\gamma)$ (i.e., $dE(\gamma)$ is absolutely continuous with respect to $d\mu(\gamma)$). This formula is equivalent to the original definition given in [2]. The following proposition is essential for the statistical interpretation in the context of instrument. The probability distribution of measured values of observables in $\mathcal{A}$ to be found in a Borel set $\Delta \subset Sp(\mathcal{A})$ is given by

$$p(\Delta|\omega) = \mathcal{I}(\Delta|\omega)(\mathbf{1})$$

which is directly derived from (4). In addition, the initial state ω is changed into a final state specified in a form as $\mathcal{I}(\Delta|\omega)/p(\Delta|\omega)$ according to the read-out of measured values in Δ [10].

We remember that, in this description of measurement processes, the *microscopic* changes of probe systems such as $|\iota\rangle \to |\xi\rangle$ are directly interpreted as the measured data. To adjust our theoretical descriptions to the realistic experimental situations, however, we need to discuss how these changes of probe systems dynamically propagate into macroscopic changes of a measuring pointer (because of some kinds of problems in realistic experiments; e.g., adiabatic conditions for approximate measurements). This is why we are concerned with the problem of *amplification process*, i.e., process to amplify invisible quantum state changes in the probe system into the macroscopic data registered on some suitable order parameter.

3.2 Dynamical description of amplification

The basic idea introduced in this subsection is, to sum up, considering state changes of N-th tensored power of probe system $|\iota\rangle^{\otimes N} := \underbrace{|\iota\rangle \otimes \cdots \otimes |\iota\rangle}_{N}$ for $\forall N \in \mathbb{N}$ caused by N-th tensored power of K-T operator V according to the following procedure (physically corresponding to such situation as a state change in one unit of a probe system triggers a state change in the next unit in such a way as accumulating eventually the effect of changes) [7]. Let λ be a regular representation of $\widehat{\mathcal{U}}$, then its arbitrary tensor powers $\lambda^{\otimes N} := (\widehat{\mathcal{U}} \ni \gamma \mapsto \underbrace{\lambda_\gamma \otimes \cdots \otimes \lambda_\gamma}_{N} \in U(L^2(\widehat{\mathcal{U}}))^{\otimes N}$ satisfy the property of mutually quasi-equivalence [11]:

$$\lambda^{\otimes m} \approx \lambda^{\otimes n} \quad \text{for} \quad \forall m, n \in \mathbb{N}.$$

Now let us write down a dynamical representation of amplification process with large number of iterations by a K-T operator. First, for convenience, we assume $\widehat{\mathcal{U}}$ is discrete so that the process can be seen in Schrödinger picture as follows:

$$\begin{aligned}
&U_N(\xi \otimes |\iota\rangle^{\otimes N}) := V_{N,N+1} \cdots V_{23}\widetilde{U}(V)_{12}(\xi \otimes |\iota\rangle^{\otimes N}) \\
&= \sum_{\gamma \in Sp(\mathcal{A})} c_\gamma V_{N,N+1} \cdots V_{34} V_{23}(\xi_\gamma \otimes |\gamma\rangle \otimes |\iota\rangle \otimes \cdots \otimes |\iota\rangle) \\
&= \sum_{\gamma \in Sp(\mathcal{A})} c_\gamma V_{N,N+1} \cdots V_{34}(\xi_\gamma \otimes |\gamma\rangle \otimes |\gamma\rangle \otimes |\iota\rangle \otimes \cdots \otimes |\iota\rangle) \qquad (5) \\
&= \cdots\cdots \\
&= \sum_{\gamma \in Sp(\mathcal{A})} c_\gamma \xi_\gamma \otimes |\gamma\rangle \otimes |\gamma\rangle \cdots \otimes |\gamma\rangle,
\end{aligned}$$

with use of the property given by (3). From this calculation, we obtain the following result: For $N \in \mathbb{N}$ and a generic state $\xi = \sum_{\gamma \in Sp(\mathcal{A})} c_\gamma \xi_\gamma$,

$$U_N(\xi \otimes |\iota\rangle^{\otimes N}) = \sum_{\gamma \in \widehat{\mathcal{U}}} c_\gamma \xi_\gamma \otimes \left[|\gamma\rangle^{\otimes N}\right]. \tag{6}$$

This gives us the physical interpretation that *the probability for detecting the state* $|\gamma\rangle^{\otimes N}$ *is equal to* $|c_\gamma|^2$ *for* $N \gg 1$. For the treatment being independent of the discreteness of $\widehat{\mathcal{U}}$, we may rewrite the above scheme in the Heisenberg picture;

$$\begin{aligned}
&T_N(A \otimes f_2 \otimes \cdots \otimes f_{N+1}) \\
&:= \widetilde{U}(V)^*_{12} V^*_{23} \cdots V^*_{N,N+1}(A \otimes f_2 \otimes \cdots \otimes f_{N+1}) V_{N,N+1} \cdots V_{23} \widetilde{U}(V)_{12} \\
&= Ad(\widetilde{U}(V)^*_{12}) Ad(V^*_{23}) \cdots Ad(V^*_{N,N+1})(A \otimes f_2 \otimes \cdots \otimes f_{N+1}) \\
&= Ad(\widetilde{U}(V)^*)(A \otimes Ad(V^*)(f_2 \otimes Ad(V^*)(\cdots \otimes Ad(V^*)(f_N \otimes f_{N+1}))) \cdots) \\
&\qquad\qquad\qquad\qquad\qquad \text{for} \quad A \in \mathcal{M} \text{ and } f_i \in L^\infty(\widehat{\mathcal{U}}),
\end{aligned}$$

in a similar way to Accardi's formulation of quantum Markov chain [1].

According to the general basic idea of "quantum-classical correspondence", a classical macroscopic object can be identified with a condensed state of infinite number of quanta, as well exemplified by the macroscopic magnetization of Ising or Heisenberg ferromagnets described by the aligned states $|\uparrow\rangle^{\otimes\infty}$ of infinite number of microscopic spins. Therefore, the above state $|\gamma\rangle^{\otimes N}$ (with $N \gg 1$) can physically be interpreted as representing a macroscopic position via some order parameter (or direction of pointer in the contexts of measurement), and hence, the above repeated action of the K-T operator V describes a cascade process to amplify a state change at the microscopic end of the apparatus into the macroscopic classical motion of the measuring pointer.

The scheme provides us with the correct value of probability for the recurrence number N being *finite*, so the repetition need not be a real infinity. To be frank, the number N itself is not so critical, except for the necessary condition of *being large enough for detecting macroscopic differences of different states of observed system.* This situation becomes clear after reformulating *instruments* in the following subsection. Moreover, the problem as to whether the situation is completely made classical or not depends highly on the relative configurations among many large or small numbers, which can consistently be described in the framework of the non-standard analysis (see [8], for instance).

In relation to this, it is also notable that the above amplification process is closely related to a Lévy process through its "infinite divisibility" as follows [6]: we can derive $\lambda \approx \lambda^{n/m}$ ($\forall m, n \in \mathbb{N}$) from $\lambda \approx \lambda^n$ ($\forall n \in \mathbb{N}$), which means the infinite divisibility $(Ad(V))^{t+s} \approx (Ad(V))^t (Ad(V))^s$ ($\forall t, s > 0$)

of the process induced by the above transformation. In this way, we see that simple individual measurements with definite measured values are connected without gaps with discrete and/or continuous repetitions of measurements.

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On Generalized Cumulants*

Hayato Saigo (Kyoto university)

Dedicated to K. Ito and I. Ojima
on their 60th birthday

Abstract

A family of quantities known as cumulants, which includes the mean and the variance, characterizes the properties of random variables and the relations between them. We generalize the notion in the context of quantum probability, giving a unified approach to central limit theorems associated to the four notions of "independence", namely, tensor, free, Boolean, and especially, monotone independence. The present paper is based on [6].

1 Introduction

The notions of mean and variance play important roles in understanding the behaviours of random variables. Given k-th moments of a random variable X for $1 \leq k \leq n$, a quantity known as the n-th cumulant can be defined: it gives the mean of X for $n = 1$ and the variance of X for $n = 2$. The n-th cumulant is additive with respect to independent random variables and is homogeneous of n-th order under the scalar multiplication. For the Gaussian distribution, higher n-th cumulants with $n \geq 3$ vanish. In terms of these ingredients, the essense of the central limit theorem (CLT) can easily be grasped. Accordingly it is naturally expected that cumulants will play some crucial roles in understanding the nature of CLT and independence.

In quantum probability as a nonncomutative algebraic version of classical probability, various notions of "independence" have been introduced. Among them, four independences known as tensor, free, Boolean and monotone ones are considered as fundamental. Their corresponding CLT's are very interesting, especially because their limit distributions are given, respectively, by the Gaussian law, the Wigner semi-circle law, the Bernoulli law and the arcsine law. In contrast to the cumulants for commutative, free and Boolean independence, the "cumulants" for the monotone case is not additive for arbitrary independent random variables.

We investigate here the nature of (generalized) CLT's and the meaning of (generalized) independence by axiomatizing the generalized cumulants. According to this generalization of the notion, we can unify four CLT's in a simple idea, which is expected to facilitate for us to attain better understanding of the notion of independence (involving "classical" independence).

*Joint work with T. Hasebe (Kyoto university)

2 From variance to cumulants

As is well known, the variance $V(X)$ of random variable X is defined as follows:

$$V(X) = E(X^2) - E(X)^2 \tag{2.1}$$

Here, $E(X)$ denotes the mean of X.

By definition, the properties below are easily proved.

$$\begin{array}{ll} \text{(K1)} & V(X+Y) = V(X) + V(Y) \quad \text{for } X, Y \text{ independent} \\ \text{(K2)} & V(\lambda X) = \lambda^2 V(X) \end{array}$$

As a generalization of the variance (2nd order cumulants), the n-th cumulant of X is defined as follows:

$$\exp(\sum_{n\geq 1} K_n(X)) := \sum_{n\geq 0} M_n(X), \quad M_n(X) := E(X^n), \tag{2.2}$$

where $M_n(X) := E(X^n)$ denotes the n-th moment.

It is easy to see that $E(X)$ and $V(X)$ is nothing but the first and the second order cumulant, respectively.

From the definition above, the properties below holds:
(K1) Additivity: If X, $Y \in \mathcal{A}$ are independent,

$$K_n(X+Y) = K_n(X) + K_n(Y), \tag{2.3}$$

for any $n \geq 1$.
(K2) Homogeneity: for any λ and any n,

$$K_n(\lambda X) = \lambda^n K_n(X). \tag{2.4}$$

(K3) Polynomiality: For any n, there exists a polynomial Q_n of $n-1$ variables such that

$$M_n(X) = K_n(X) + Q_n(K_1(X), \cdots, K_{n-1}(X)). \tag{2.5}$$

As an application of the notion of cumulants, the essensial structure of the central limit theorem (CLT) are understood as follows. (For simplicity, in this paper we assume that all random variables have finite moments.)

First, by the properties (K1) and (K2), cumulants of the scaling sum $S_N := \frac{X_1+X_2+\cdots X_N}{\sqrt{N}}$ of independent, identically distributed (i.i.d) random variables X_i are calculated as

$$K_n(S_N) = N^{1-\frac{n}{2}} K_n(X_i). \tag{2.6}$$

Hence, when all of X_i are normalized, i.e., $E(X_i) = 0$ and $V(X_i) = 1$,

$$\lim_{N\to\infty} K_n(S_N) = \delta_{n,2}. \tag{2.7}$$

On the other hand, the n-th cumulant of normalized Gaussian is nothing but $\delta_{n,2}$. It is easy to see that cumulants converge if and only if moments converge. Moreover, under certain condition which is satisfied here, convergence in moments is equivalent to weak convergence. Then we have central limit theorem (CLT):

Theorem 2.1. *If $\{X_i\}$ is a family of normalized i.i.d.s (which has all finite moments), $\frac{X_1+X_2+\cdots X_N}{\sqrt{N}}$ weakly coverges to normalized Gaussian when N tends to infinity.*

3 Notions of independence in quantum probability

In quantum probability theory, which is a noncommutative extension of (measure-theoretic) probability theory, many different (analogous) notions of independence are defined. Among them, tensor, free, Boolean and monotone independence are considered as fundamental examples[12, 14, 19, 22]. For tensor, free, Boolean and many other notions of independence, associated cumulants, that is, the quantities or functionals which satisfies (K1):additivity for each independence, (K2):homogeneity and (K3):polynomiality, have been introduced[7, 21, 19]. They are fruitful concept especially for understanding asymptotics of algebraic random variables such as CLTs.

In the present paper we will introduce generalized cumulants which allow us to give an unified approach to CLTs, including the case of monotone independence. Since monotone independence depends on the order of random variables, the additivity of cumulants (K1) fails to hold. Instead, we introduce a weakened condition and prove the uniqueness of generalized cumulants in Section 4. In Section 5, we show the existence of the monotone cumulants and obtain an explicit moment-cumulant formula for monotone independence. In Section 6, we show the (monotone) central limit theorem in terms of the (monotone) generalized cumulants.

Let A be an unital *-algebra over $\boldsymbol{C}$. A linear functional on A is called a state on A if $\varphi(a^*a) \geq 0$ and $\varphi(1_A) = 1$.

Definition 3.1. (Algebraic probability space) An algebraic probability space is a pair (A, φ), where A is an unital *-algebra and φ a state on A.

An element a in A is called an algebraic random variable. For algebraic random variables, quantities such as $\varphi(a_1a_2...a_n)$ is called mixed moments.

The notion of independence in classical probability can be understood as a universal structure which gives a rule for calculating mixed moments, at least from the algebraic point of view. In quantum probability, lots of different notions of independence have been introduced. Among them, four notions mentioned below are known as fundamental examples [12, 14].

Let (A, φ) be an algebraic probability space and $\{A_\lambda; \lambda \in \Lambda\}$ be a family of *-subalgebras of A. In the following four notions of independece are defined as the rules for calculating mixed moments such as $\varphi(a_1a_2...a_n)$, where

$$a_i \in A_{\lambda_i},\ a_i \notin \boldsymbol{C}1,\ \lambda_i \neq \lambda_{i+1},\ 1 \leq i \leq n,\ n \geq 2.$$

Definition 3.2. (Tensor independence). $\{A_\lambda\}$ is tensor independent if

$$\varphi(a_1a_2...a_n) = \varphi(a_1)\varphi(a_2...a_n)$$

holds when $\lambda_1 \neq \lambda_r$ for all $2 \leq r \leq n$, and otherwise, letting r be the least number such that $\lambda_1 = \lambda_r$,

$$\varphi(a_1a_2...a_n) = \varphi(a_2...a_{r-1}(a_1a_r)a_{r+1}...a_n).$$

Tensor independence is nothing but a straight generalization of usual independence. On the other hand, very different and in a sense dual notion of independence, that is, free independence, is well known.

Definition 3.3. (Free independence). $\{A_\lambda\}$ is free independent if

$$\varphi(a_1a_2...a_n) = \varphi(a_1)\varphi(a_2...a_n)$$

holds whenever $\varphi(a_2) = ... = \varphi(a_n) = 0$.

Moreover, the concepts below are also considered as basic examples (or analogues) of independence:

Definition 3.4. (Boolean independence). $\{A_\lambda\}$ is Boolean independent if

$$\varphi(a_1a_2...a_n) = \varphi(a_1)\varphi(a_2...a_n)$$

Definition 3.5. (Monotone independence). Assume that the index set Λ is equipped with a linear order $<$. $\{A_\lambda\}$ is monotone independent if

$$\varphi(a_1...a_i...a_n) = \varphi(a_i)\varphi(a_1...a_{i-1}a_{i+1}...a_n)$$

holds when i satisfies $\lambda_{i-1} < \lambda_i$ and $\lambda_i > \lambda_{i+1}$ (understanding that one of the inequalities is eliminated when $i = 1$ or $i = n$).

A system of algebraic random variables $\{x_\lambda\}$ are called tensor/free/Boolean/monotone independent if $\{A_\lambda\}$ are tensor/free/Boolean/monotone independent, where A_λ denotes the algebra generated by x_λ.

4 Generalized cumulants

We introduce generalized cumulants as functionals which satisfies (K1'), (K2), (K3).

Definition 4.1. A functional K_n is called a nth order cumulant for each independence when it satisfies the following condtions.
(K1') Weakened additivity: If $X^{(1)}, X^{(2)}, \cdots, X^{(N)}$ are independent, identically distributed to X in the sense of moments (i.e., $M_n(X) = M_n(X^{(i)})$ for all n), we have

$$K_n(N.X) := K_n(X^{(1)} + \cdots + X^{(N)}) = NK_n(X). \tag{4.1}$$

Here we understand that $K_n(0.X) := \delta_{n0}$.
(K2) Homogeneity: for any λ and any n,

$$K_n(\lambda X) = \lambda^n K_n(X). \tag{4.2}$$

(K3) Polynomiality: For any n, there exists a polynomial Q_n of $n-1$ variables such that

$$M_n(X) = K_n(X) + Q_n(K_1(X), \cdots, K_{n-1}(X)). \tag{4.3}$$

We show the uniqueness of generalized cumulants with respect to each notion of independence.

Theorem 4.2. *Generalized cumulants satisfying (K1'), (K2) and (K3) are (if exist) unique and the n-th cumulant is given by the coefficient of N in $M_n(N.X)$.*

Proof. By (K3) and (K1'), we obtain

$$\begin{aligned} M_n(N.X) &= K_n(N.X) + Q_n(K_1(N.X), \cdots, M_{n-1}(N.X)) \\ &= NK_n(X) + Q_n(NK_1(X), \cdots, NK_{n-1}(X)). \end{aligned} \tag{4.4}$$

By condition (K2), the polynomial Q_n does not contain linear terms and a constant for any n. Therefore, the coefficient of the linear term N is nothing but $K_n(X)$. The right hand side of (4.4) depends only on the notion of independence and the moments of X, the uniqueness of generalized cumulants holds. □

From now on, we use the word "cumulants" instead of "generalized cumulants" to label K_n above.

5 The monotone cumulants

Proposition 5.1. *For monotone independent random variables X and Y, it holds that*

$$\begin{aligned} M_n(X+Y) &= \sum_{k=0}^{n} \sum_{\substack{j_0+j_1+\cdots+j_k=n-k, \\ 0\le j_l,\ 0\le l\le k}} M_k(X)M_{j_0}(Y)\cdots M_{j_k}(Y) \\ &= M_n(X) + M_n(Y) + \sum_{k=1}^{n-1} \sum_{\substack{j_0+j_1+\cdots+j_k=n-k, \\ 0\le j_l,\ 0\le l\le k}} M_k(X)M_{j_0}(Y)\cdots M_{j_k}(Y). \end{aligned} \tag{5.1}$$

Proof. $(X+Y)^n$ can be expanded as

$$(X+Y)^n = X^n + Y^n + \sum_{k=1}^{n-1} \sum_{\substack{j_0+j_1+\cdots+j_k=n-k, \\ 0\le j_l,\ 0\le l\le k}} Y^{j_0}XY^{j_1}X\cdots XY^{j_k} \tag{5.2}$$

Taking the expectation of the above equality, we obtain (5.1). □

By this formula, we obtain the proposition below.

Proposition 5.2. *$M_n(N.X)$ is a polynomial of degree n w.r.t. N (without a constant term) for any $n \ge 0$.*

Proof. We use induction w.r.t. n. For $n = 1$, it is obvious from linearity of expectation. Suppose the proposition holds for $n \le l$. From the formula above, we obtain

$$\Delta M_{l+1}(N.X) = M_{l+1}(X) + \sum_{k=1}^{l} \sum_{\substack{j_0+j_1+\cdots+j_k=n-k, \\ 0\le j_l,\ 0\le l\le k}} M_k((N-1).X)M_{j_0}(X)\cdots M_{j_k}(X). \tag{5.3}$$

Here, $\Delta M_{l+1}(N.X) := M_{l+1}(N.X) - M_{l+1}((N-1).X)$. Then M_{l+1} is a $(l+1)$-th polynomial w.r.t. N (without a constant term) because $\Delta M_{l+1}(N.X)$ is a l-th polynomial w.r.t.N and $M_{l+1}(0.X) = 0$. □

As the proposition above holds, we may define $m_n(t) = M_n(t.X)$ by replacing N with $t \in \mathbb{R}$. Note that this is a polynomial w.r.t. t and that $m_n(1) = M_n(X)$. Moreover, we easily obtain

$$m_n(t+s) = m_n(t) + m_n(s) + \sum_{k=1}^{n-1} \sum_{\substack{j_0+j_1+\cdots+j_k=n-k, \\ 0\le j_l,\ 0\le l\le k}} m_k(t) m_{j_0}(s) \cdots m_{j_k}(s) \tag{5.4}$$

from the definition of $m_n(t)$ and (5.1) .

Now we come to define the main notion.

Definition 5.3. Let $r_n = r_n(X)$ be the coefficient of N in $M_n(N.X)$ (or the coefficient of t in $m_n(t)$). We call r_n the n-th monotone cumulant of X.

The monotone cumulants satisfy the axioms (K1') and (K2) because $M_n(N.(M.X)) = M_n((NM).X)$ and $M_n(N.(\lambda X)) = M_n(\lambda(N.X))$. For the moment-cumulant formula, we obtain the following proposition.

Proposition 5.4. *The equations below hold:*

$$\begin{aligned} \frac{dm_0(t)}{dt} &= 0, \\ \frac{dm_n(t)}{dt} &= \sum_{k=1}^{n} k r_{n-k+1} m_{k-1}(t) \quad \text{for } n \ge 1, \end{aligned} \tag{5.5}$$

with initial conditions $m_0(0) = 1$ and $m_n(0) = 0$ for $n \ge 1$.

Proof. From (5.4), we obtain

$$m_n(t+s) - m_n(t) = m_n(s) + \sum_{k=1}^{n-1} \sum_{\substack{j_0+j_1+\cdots+j_k=n-k, \\ 0\le j_l,\ 0\le l\le k}} m_k(t) m_{j_0}(s) \cdots m_{j_k}(s) \tag{5.6}$$

By definition,

$$m_i(s) = r_i s + s^2(\cdots) \tag{5.7}$$

holds. Comparing the coefficients of s in (5.6), we obtain the conclusion. □

We show that $\{m_n\}_{n\ge0} := \{M_n(X)\}_{n\ge0}$ and $\{r_n\}_{n\ge1}$ are connected with each other by a formula.

Theorem 5.5. *The following formula holds:*

$$m_n = \sum_{k=1}^{n} \sum_{1=i_0<i_1<\cdots<i_{k-1}<i_k=n+1} \frac{1}{k!} \prod_{l=1}^{k} i_{l-1} r_{i_l - i_{l-1}}. \tag{5.8}$$

Proof. This formula is obtained directly by (5.5). We shall use the equations in the integrated forms

$$\begin{aligned} m_0(t) &= 1, \\ m_n(t) &= \sum_{k=1}^{n} k r_{n-k+1} \int_0^t m_{k-1}(s) ds \quad \text{for } n \ge 1. \end{aligned} \tag{5.9}$$

Then we have

$$
\begin{aligned}
m_n(t) &= \sum_{k_1=1}^{n} k_n r_{n-k_1+1} \int_0^t m_{k_1-1}(t_1)dt_1 \\
&= \sum_{k_1=1}^{n} \sum_{k_2=1}^{k_1-1} k_1 k_2 r_{n-k_1+1} r_{k_1-k_2} \int_0^t dt_1 \int_0^{t_1} dt_2 \ m_{k_2-1}(t_2) \\
&= \sum_{k_1=1}^{n} \sum_{k_2=1}^{k_1-1} \sum_{k_3=1}^{k_2-1} k_1 k_2 k_3 r_{n-k_1+1} r_{k_1-k_2} r_{k_2-k_3} \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \ m_{k_3-1}(t_3)dt_3 \\
&= \cdots .
\end{aligned}
$$

When this calculation ends, we obtain the formula

$$
m_n(t) = \sum_{k=1}^{n} \sum_{1=i_0<i_1<\cdots<i_{k-1}<i_k=n+1} \frac{t^k}{k!} \prod_{l=1}^{k} i_{l-1} r_{i_l - i_{l-1}},
$$

where $i_l := k_{n-l}$. Putting $t = 1$, we have (5.8).

□

Remark 5.6. This formula has been already obtained in the case of the monotone Poisson distribution [1, 2].

Corollary 5.7. *The monotone cumulants $r_n = r_n(X)$ satisfy (K3).*

Hence, we obtain the main theorem.

Theorem 5.8. *r_n are the unique (generalized) cumulants for monotone independence.*

6 Central Limit Theorem

We apply the monotone cumulants to the monotone CLT which has already been obtained by combinatorial arguments on certain kind of partitions in [11] and much simplified by [14, 15]. On the other hand, the argument below is applicable to other independences, without concerning explicit combinatorics. It is an analogue of the proof of usual CLT discussed in section 2.

Theorem 6.1. *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space. Let $X^{(1)}, \cdots, X^{(N)}, \cdots$ be identically distributed, monotone independent self-adjoint random variables with $\phi(X^{(1)}) = 0$ and $\phi((X^{(1)})^2) = 1$. Then the probability distribution of $\frac{X^{(1)}+\cdots+X^{(N)}}{\sqrt{N}}$ converges weakly to the arcsine law with mean 0 and variance 1.*

Proof. It is not difficult to show that $M_n(\frac{X^{(1)}+\cdots+X^{(N)}}{\sqrt{N}})$ converges to some M_n which is characterized by the monotone cumulants $(r_1, r_2, r_3, r_4, \cdots) = (0, 1, 0, 0, \cdots)$. We can calculate the limit moments by (5.8) and obtain $r_{2n-1} = 0$ and $r_{2n} = \frac{(2n-1)!!}{n!}$ for all $n \geq 1$. The limit measure is the arcsine law with mean 0 and variance 1 [11], the moment problem of which is deterministic. Then the distribution of $M_n(\frac{X^{(1)}+\cdots+X^{(N)}}{\sqrt{N}})$ converges to the arcsine law weakly (see Theorem 4.5.5 in [4]). □

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The Strange World of Non-amenable Symmetries [1]

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Abstract

Nonlinear sigma models with non-compact target space and non-amenable symmetry group were introduced long ago in the study of disordered electron systems. They also occur in dimensionally reduced quantum gravity; recently they have been considered in the context of the AdS/CFT correspondence. These models show spontaneous symmetry breaking in *any* dimension, even one and two (superficially in contradiction with the Mermin-Wagner theorem) as a consequence of the non-amenability of their symmetry group. The low-dimensional models show other peculiarities: invariant observables remain dependent on boundary conditions in the thermodynamic limit and the Osterwalder-Schrader reconstruction yields a non-separable Hilbert space. The ground state space, however, under quite general conditions, carries a unique unitary and continuous representation. The existence of a continuum limit in 2D is an open question: while the perturbative Renormalization Group suggests triviality, other arguments hint at the existence of a conformally invariant continuum limit at least for suitable observables.

This talk gives an overview of the work done during the last several years in collaboration first of all with Max Niedermaier, some of it also with Peter Weisz and Tony Duncan [1, 2, 3, 4, 5].

[1] Talk given at the International Conference "Mathematical Quantum Field Theory and Renormalization Theory", Nov. 26 to Nov. 29, 2009, Kyushu University, Fukuoka, Japan

1 Introduction

1.1 What are non-amenable symmetries?

The concept of amenable groups was introduced by J. von Neumann in 1929; it can be described as follows: let $\mathcal{C}(G)$ be the space of continuous bounded functions on G; then a *mean* m is a positive (hence continuous) linear functional on $\mathcal{C}(G)$ (or on $L^\infty(G)$) with $m(\mathbb{1}) = 1$. Put differently: m is a *state* on the commutative C^* algebra $\mathcal{C}(G)$ (or $L^\infty(G)$). Since the group G has a natural left action on functions, it makes sense to speak about invariance of such a mean.

Definition: The group G is non-amenable if there is no mean on $\mathcal{C}(G)$ which is invariant under G.

A well-known fact is that noncompact semisimple Lie groups are nonamenable [6].

The concept can be generalized to homogeneous spaces G/H by using the the algebra $\mathcal{C}(G/H)$ instead of $\mathcal{C}(G)$. One also speaks of (non-)amenable actions of a group G on a general G-space $\mathcal{M}$: the left action of G induces an action on $\mathcal{C}(\mathcal{M})$ and non-amenability means nonexistence of a mean invariant under G on $\mathcal{C}(\mathcal{M})$.

Bekka [7] has extended the definition to that of amenable unitary representations π on a Hilbert space $\mathcal{H}$ as follows: π is called amenable if there is a state on $\mathcal{B}(\mathcal{H})$ which is invariant under π. In this context the following result is important: if G is simple, noncompact, connected, with finite center, rank > 1, the trivial representation is the only amenable one.

1.2 Physics motivations

(1) Nonlinear σ models with hyperbolic target space – the prototype of a non-amenable symmetric space – were introduced 1979 by Wegner [8] to describe the conductor-insulator transition in disordered electron systems. Since then there has been a lot of activity, see for instance [9, 10, 11]. Later Efetov [12, 13] and Zirnbauer [14] introduced the supersymmetric version of that model as a better description of the electron system. This line of research was continued more recently in [15, 16, 17].

(2) Some 'warped' versions of nonlinear σ models with hyperbolic target space arise in dimensionally reduced gravity and its quantization [18, 19, 20].

(3) Not surprisingly, these models also appear in the context of string theory;

string theorists think of hyperbolic space as 'Euclidean Anti-de Sitter space' [21, 22].

2 Quantum Mechanics on hyperbolic spaces

Insight into the peculiarites of non-amenable symmetries is easiest to obtain by studying quantum mechanics on hyperbolic spaces. Hyperbolic space can be described as a hyperboloid imbedded in Minkowski space with the metric induced by the ambient space:

$$\mathbb{H}_N \equiv SO_o(1,N)/SO_o(N) \equiv G/K = \{n \in \mathbb{R}^{N+1} | n \cdot n = 1,\ n_o > 0\}\,, \quad (1)$$

where $n \cdot n' \ = \ n_o n'_o - \vec{n} \cdot \vec{n}'$.

2.1 One particle

Let Δ denote the Laplace-Beltrami operator on $\mathbb{H}_N$. The free one particle Hamiltonian, acting on $L^2(\mathbb{H}_N, d\Omega)$ where $d\Omega$ is an invariant measure on $\mathbb{H}_N$, is then

$$H = -\Delta \geq 0\,. \quad (2)$$

This Hamiltonian is diagonalized using the Mehler-Fock transformation [23]; it reveals that the spectrum of H is absolutely continuous, covering the interval $[(N-1)^2/4,\ \infty)$; there is no spectrum in the interval $[0,(N-1)^2/4)$, even though there are bounded eigenfunctions for every value in that interval (the supplementary series). Introducing a spectral parameter ω running from 0 to ∞, we have the spectral resolution

$$L^2(\mathbb{H}_N, d\Omega) = \int_0^\infty dP(\omega)\mathcal{H}_\omega; \quad H\psi = \int_0^\infty dP(\omega)\left(\frac{1}{4}(N-1)^2 + \omega^2)\psi\right)\,. \quad (3)$$

Only the principal series appears; the spectrum is infinitely degenerate because all representations in that series are infinite dimensional. Of course there is no normalizable ground state vector; instead we have a 'ground state space' corresponding to $\omega = 0$ and spanned by 'generalized ground states' (functions in $\mathcal{C}(\mathcal{M})$ but $\notin L^2$) of the form

$$\mathcal{P}^{1-N/2}_{-1/2}(gn \cdot n^\uparrow), \quad g \in SO_o(1,N)\,, \quad (4)$$

and linear combinations thereof, where $\mathcal{P}^{1-N/2}_{-1/2}$ are Legendre functions.

A different scalar product, produced by the Osterwalder-Schrader (OS) reconstruction makes these ground states normalizable. They then generate a Hilbert space of ground states carrying a special unitary irreducible representation σ_0. But the main point is this:

There is no invariant ground state
Spontaneous symmetry breaking (SSB) takes place!

2.2 ν particles: separation of 'center of mass'

When we consider a ν particle system interacting via translation invariant potentials in Euclidean space, the first step is always to separate out the free center of mass motion. Here we consider ν particles on $\mathbb{H}_N$ with a potential invariant under the symmetry group $G = SO_0(1, N)$, and again we would like to to find a way to extract the rigid motions. This requires some tricks.

Our Hilbert space is now $\mathcal{H} = L^2(\mathcal{M})$ ($\mathcal{M} = \mathbb{H}^\nu_N$) and the Hamiltonian is

$$H = -\sum_{i=1}^{n} \Delta_i + \sum_{i<j} V(n_i \cdot n_j) \equiv H_0 + \mathcal{V}\,. \tag{5}$$

Let $\ell_\mathcal{M}$ be the unitary representation of G on $\mathcal{H}$ induced by the left diagonal action of G, representing rigid motions of the particle system. Clearly

$$[H, \ell_\mathcal{M}(G)] = 0\,, \tag{6}$$

We now turn the left diagonal action on $\mathcal{M}$ into a right action on a different manifold $\mathcal{M}_r$ ins such a way that only one 'center of mass' variable is affected. First we define

$$\tilde{\mathcal{M}}_r \equiv G \times \mathcal{H}^{\nu-1} \tag{7}$$

and an injective but not surjective map $\tilde{\phi} : \mathcal{M} \to \tilde{\mathcal{M}}_r$ given by

$$\tilde{\phi}(n_1, \dots, n_n) = (g_s(n_1)^{-1}, g_s(n_1)^{-1} n_2, \dots g_s(n_1)^{-1} n_\nu)\,, \tag{8}$$

where g_s is a function (global section) $\mathbb{H}_N \to G$ such that $n = g_s(n^\uparrow)$. g_s is obviously only determined up to $g_s \to g_s k^{-1}, k \in K$. Let $d_\ell(K)$ be the left diagonal action of K on $\tilde{\mathcal{M}}_r$ and define

$$\mathcal{M}_r = \tilde{\mathcal{M}}_r / d_\ell(K)\,. \tag{9}$$

$\tilde{\phi}$ projects to a well-defined map $\phi : \mathcal{M} \to \mathcal{M}_r$ and this ϕ does the job of converting the left diagonal action $d_\ell(G)$ on $\mathcal{M}$ into a right action $r(G)$ on $\mathcal{M}_r$ acting only on the first entry:

$$r(g')[(g, n_1, \dots, n_\nu)] = [gg', n_1, \dots, n_\nu)], \tag{10}$$

$$\phi \circ d_\ell = r \circ \phi. \tag{11}$$

ϕ induces a unitary map Φ between the corresponding Hilbert spaces $L^2(\mathcal{M}_r)$ and $L^2(\mathcal{M})$; the latter can be viewed as the subspace of $L^2(\tilde{\mathcal{M}}_r)$ invariant under the unitary map induced by $d_\ell(K)$.

The right action of G on the first entry of $\mathcal{M}_r$ induces a unitary representation $\rho(G)$ of the rigid motions:

$$\rho = \Phi^{-1} \circ \ell_{\mathcal{M}} \circ \Phi \tag{12}$$

and $\rho(G)$ commutes with $\ell_r(K)$.

2.3 The ground state representation

The harmonic analysis of ρ is the analogue of the decomposition according to the center of mass momentum in flat space. The Hilbert space $\mathcal{H} = L^2(\mathcal{M}_r)$ decomposes into a direct integral of irreps

$$\mathcal{H} = \int_{\widehat{G}_r}^{\oplus} d\nu(\sigma)\mathcal{H}(\sigma), \tag{13}$$

where $\widehat{G}_r$ is the restricted dual of G, which is the union of the principal and the discrete series (see [24])

$$\widehat{G}_r = \widehat{G}_p \cup \widehat{G}_d; \tag{14}$$

$d\nu$ arises from the Plancherel measure.

The Hamiltonian on $\mathcal{H}$ is $H_r = \Phi^{-1} \circ H \circ \Phi$; we drop the subscript r from now on. Because H commutes with ρ, it can also be resolved into fiber Hamiltonians

$$H = \int_{\widehat{G}_r}^{\oplus} d\nu(\sigma)h(\sigma), \tag{15}$$

which is analogous to the resolution of a ν particle Hamiltonian according to the c.m. momentum in flat space.

We conjecture that generally $d\nu$ is carried by $\widehat{G}_p$ alone, but we can prove only that

$$\inf_\sigma \inf \text{ spec } h(\sigma) \notin \widehat{G}_d \,. \tag{16}$$

This is seen most easily under a certain compactness condition on the interaction $\mathcal{V}$, namely

$$\text{tr } e^{-t(H_o+\mathcal{V}+\mathcal{V}_1)} \leq \text{tr } (e^{-tH_o} e^{-t(\mathcal{V}+\mathcal{V}_1)}) < \infty \,. \tag{17}$$

This implies that the fiber Hamiltonians $h(.)$ have discrete spectrum; for $\sigma \in \widehat{G}_d$ the ground state of the fiber Hamiltonian would give rise to a (normalizable) eigenfunction of H; because σ is not the trivial representation, this ground state could not be unique. This leads to a contradiction with the Perron-Frobenius theorem. We can show furthermore that the ground state representation is always σ_0, the special representation found for the one particle case. Details can be found in [3], where, however, we deal with a discrete time evolution given by a transfer matrix.

The ground state representation σ_0 is universal, and the fact that it is non-trivial means again that there is SSB.

3 Statistical mechanics / lattice quantum field theory

3.1 Action, Gibbs state

We consider configurations of 'spins' given by mapping each site $x \in \Lambda \subset \mathbb{Z}^d$ to a $n(x) \in \mathbb{H}_N$. The Gibbs measure is formally given by

$$\exp(-\beta S) \prod_x d\Omega(x) \,, \tag{18}$$

with (for instance)

$$S = \sum_{\langle xy \rangle} n(x) \cdot n(y) \,. \tag{19}$$

To make the Gibbs measure normalizable, 'gauge fixing' is needed. The simplest choice is to fix a spin at the boundary of the finite lattice Λ.

3.2 Spontaneous symmetry breaking

If in the thermodynamic limit $\Lambda \nearrow \mathbb{Z}^d$ the Gibbs state is not invariant under G, we speak of SSB. Non-amenability *enforces* SSB, because if there were a symmetric Gibbs measure, it would automatically induce an invariant mean on the functions of a single spin. This holds independent of the dimension d or any other details (type of lattice, action).

SSB is unavoidable!

Note that the Mermin-Wagner theorem is not in conflict with this finding: it forbids SSB only for *compact* symmetry groups in dimensions 1 and 2.

3.3 The hyperbolic spin chain

For $d = 1$ the problem can be solved analytically to a large extent [1]. 'Gauge fixing' is done by fixing the spin at the left hand end of the chain, say

$$n(-L) = n^{\uparrow} \,. \tag{20}$$

As the general considerations require, SSB occurs in the form that the system remembers the orientation of the spin $n(-L)$ even in the limit $L \to \infty$. A concrete 'order parameter' that shows this is

$$T_e(n(0)) := \tanh(n(0) \cdot e)\,, \tag{21}$$

where $e \cdot e = -1$. In [1] it is shown that

$$\lim_{L\to\infty} \langle T_e(n(0))\rangle_{bc} = 1 - \frac{2}{\pi} \cos^{-1}\left(\frac{e \cdot n^{\uparrow}}{\sqrt{1+(e \cdot n^{\uparrow})^2}}\right) \,. \tag{22}$$

Two facts are might be unexpected:
(1) even the expectation values of invariant functions, such as $n(0) \cdot n(x)$ remain dependent on b.c. in the thermodynamic limit.

(2) The OS reconstruction of a Hilbert space from the correlation functions yields a *non-separable* space and discontinuous representations, except for the ground state space. This is related to the fact that the construction always produces a *normalizable* ground state, even though there is none in the original L^2 space, so in some sense the OS reconstruction renormalizes the scalar product.

3.4 Two or more dimensions

Not very much is known rigorously beyond the fact of SSB. The models do not have high temperature expansions; presumably they are massless at all temperatures.

Spencer and Zirnbauer [15] have shown, however, that in dimension 3 or more at low temperature there is a 'stronger version' of SSB that presumably is not true in dimensions 1 or 2 or at high temperatures; namely the quadratic fluctuations away from the mean 'magnetiztion' have a finite expectation value.

In [2] we carried out some detailed numerical simulation of the model in $d = 2$ with a different (translation invariant) gauge fixing. We found

(1) The explicit symmetry violations due to the gauge fixing disappear in the thermodynamic limit; this is seen by verifying Ward identities

(2) SSB is seen by looking at $\langle T(e)\rangle$ as above.

(3) The thermodynamic limit for invariant observables seems to exist.

4 Quantum field theoretic considerations

4.1 Peculiarities of the Osterwalder-Schrader reconstruction

As in the hyperbolic spin chain, the OS reconstruction will presumably always lead to a nonseparable Hilbert space. This is a consequence of the fact that the construction always yields a normalized ground state, even though the spectrum of the transfer matrix is most likely continuous.

This somewhat unphysical feature might be avoided by restricting the space of observables. For instance one might restrict attention to only a certain component of the spins, or maybe a special (horospherical) coordinate and functions of it. In this way one would of course lose the signal of SSB.

4.2 Existence of a continuum limit?

Consideration of the perturbative one loop Renormalization Group [25, 26] yields essentially the Ricci flow, indicating that the model is *infrared asymptotically free*. In the case at hand, however, this is counterintuitive: if the long-distance fluctuations become Gaussian, as infrared asymptototic freedom would predict, this would mean that they doen't feel the curvature of the target manifond. But in the infrared the fluctuations necessarily cover the target space over large distances and therefore should become extremely sensitive to the curvature.

But if the conventional wisdom is right, it would suggest that there is no nontrivial continuum limit of 2D nonlinear σ models whose target space has negative curvature; the situation would be similar to the QED$_4$ or ϕ_4^4 quantum field theories, which are believed to be trivial (i.e. Gaussian) in the continuum limit.

A counterpoint has been provided long ago by Haba [27], who by a formal calculation of the 2D hyperbolic σ model concluded that it corresponded to a conformal quantum field theory with central Virasoro charge $c = 1$, as long as $\beta > 1/3\pi$ (it should be noted, however, that he considered only correlations of a so-called horospherical coordinate on the hyperbolic plane). It would be very interesting to know if this formal calculation can be justified.

4.3 Axiomatic considerations

When considering possible continuum limits, it is worthwhile to pause and think how such limits could possibly look, in agreement with the axiomatic structure of quantum field theory.

One thing becomes clear immediately: it is not possible to have a multiplet of quantum fields ϕ_i transforming under the non-unitary vector representation of $G = SO(1,N)$ with unbroken symmetry (as one would expect naively if the ϕ_i are continuum fields arising from renormalizing the basic spin components n_i).

The reasoning goes like this: an unbroken symmetry means that there is a unitary representation $U(.)$ of the symmetry group G leaving the vacuum state invariant and transforming the fields according to the vector representation, i.e.

$$U(g)^{-1}\phi_i(x)U(g) = \sum_j \left(g^{-1}\right)_{ij} \phi_j(x)\,. \tag{23}$$

This leads to a conflict with the positive metric in Hilbert space when considering orbits of $U(.)\psi$: let $\phi_0(f), \phi_1(f)$ be field components smeared with a test function f. Then by the unitarity of $U(.)$

$$\| \left(\phi_o(f) \text{ ch } t + \phi_1(f) \text{ sh } t\ \Omega\right) \|^2 = \|\phi_o(f)\Omega\|^2 \quad \forall t\,, \tag{24}$$

which is impossible unless all $\phi_i = 0$.

Possible alternatives are:

(1) There is SSB, hence no unitary represntation $U(.)$ of G (see for instance-blot

(2) There is an infinite multiplet of fields, transforming according to a unitary representation of G – this could, however, not correspond to a continuum limit of the lattice model

(3) A Quantum Field Theory arises only for a subset of fields. The symmetry is then not visible. Haba's computation suggests that this might be the right scenario.

5 Conclusions, open questions

(1) We have found a certain universal ground state representation both in Quantum Mechanic and Lattice field theory (in a finite spatial volume).

(2) There is always SSB; the Mermin-Wagner theorem does not apply.

(3) In a potential continuum limit also Coleman's version of the Mermin-Wagner theorem would not apply because the currents needed for this argument don't have thermodynamic and continuum limits.

(4) In $D \geq 3$ for large β there is SSB of the conventional kind: with large fluctuations suppressed[15].

(5) There is probably no mass gap, but a proof is lacking.

(6) In 2D infrared asymptotic freedom is suggested by perturbation theory, but there is no proof; the existence of a continuum limit remains an unsolved question.

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Local Causal Structures
Relating Quantum Field Theories on Different Spacetime Backgrounds

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Dedicated to Professor K. R. Ito and Professor I. Ojima on the occasion of their 60th birthdays

Abstract

This is a report on joint work with Claudio Dappiaggi and Nicola Pinamonti (arXiv:1001.0858v1 [hep-th]). We set up a local version of the bulk-to-boundary correspondence for quantum field theories on curved spacetimes which makes it possible to compare expectation values of local field observables in different spacetimes and to extend the principle of general local covariance. In doing so, we single out a distinguished state on the boundary whose pull-back in the bulk is of Hadamard form and which exhibits properties of a local vacuum state.

1 Introduction

One question that both Izumi Ojima and the present author have found intriguing and that has always been present as a background in our discussions is the following one: *How are the characteristics of spacetime encoded in the mathematical structure of physical measurements, and are we able to deduce the former from the latter?* The foundation for this is our common conviction that spacetime is not a concept pertaining to the way human beings are able to perceive reality but indeed engenders from physical measurements as a means to organise the data taken in a sensible way. An obvious problem related to this area of investigation is the characterisation of inertial frames in terms of physical measurements. A closer look reveals that eventually inertial frames are singled out by the fact that measurements with respect to them are subject to the Poincaré symmetry. But this poses further problems:

- This symmetry which involves the whole Minkowski spacetime is actually checked only locally (in a laboratory).

- Moreover, to be able to characterise inertial frames by their symmetry properties, we have to specify a *suitable physical state*. Otherwise it might happen that the system at hand only appears to exhibit the Poincaré symmetry due to the fact that the physical state selected masks the actual complicated structure of the frame.

- Finally, in a curved background the Poincaré symmetry can only hold approximately; spacetime curvature should manifest itself by measurements that reveal the geodesic deviation equation. This amounts to the need to compare measurements performed in flat Minkowski spacetime with corresponding ones in a curved background.

Our strategy in addressing this circle of questions can be summarised in the following three steps:

1. *Geometry*: Consider a point p in a strongly causal four-dimensional spacetime (M, g). Then there exists a geodesic neighbourhood on which the exponential map acts as a local diffeomorphism. Select a second point q in this set such that $\mathscr{D}(p,q) \doteq I^+(p) \cap I^-(q)$ is a globally hyperbolic spacetime. Then $\mathscr{C}_p^+ \doteq J^+(p) \cap \overline{\mathscr{D}(p,q)}$ is a local null hypersurface. The very same procedure can be repeated for a point p' in a second spacetime M'. Since the exponential map is invertible and the tangent spaces $T_p(M)$ and $T_{p'}(M')$ are isomorphic, it is possible to arrange the geometric data in such a way that the boundaries $\mathscr{C}_p^+$ and $\mathscr{C}_{p'}^+$ are related by a suitable restriction map with the choice of a frame (coordinate system) at p and p', respectively, as the only freedom left.

2. *Quantum Field Theory*: Consider the Borchers-Uhlmann algebra and the extended algebra of observables containing more singular objects like Wick polynomials. Then it is possible to construct on $\mathscr{C}_p^+$ a scalar field theory and, in addition, a natural concept of an extended algebra can also be introduced on this boundary. Moreover, there exists an injective *-homomorphism Π between the bulk and boundary counterparts of this extended algebra.

3. *Natural State*: It is possible to identify a natural state on the boundary which is independent of the choice of the frame at p and whose pull-back in the bulk $\mathscr{D}(p,q)$ via Π turns out to still be invariant with respect to the choice of the frame. Physically speaking, this state is the same for all inertial observers at p. This result of ours provides a potential candidate for a local vacuum in the large class of backgrounds amenable to the above constructions.

2 Geometrical Considerations

The geometric setup for the following investigations can be structured in the form of a sequence of fundamental definitions followed by the use of these concepts in a spacetime endowed with a smooth Lorentzian metric.

Definition 2.1 (Spacetime)**.** A *spacetime* M is a four-dimensional Hausdorff connected smooth manifold of signature $(-,+,+,+)$. Henceforth it is second-countable and paracompact [11, 12].

The common requirement of *global hyperbolicity* (*cf.* [1]) is not presupposed at this point in order to restrict attention only to structures that are actually needed in the constructions.

Definition 2.2 (Frames)**.** To any point $p \in M$ one associates a *linear frame* F_p of the tangent space $T_p(M)$, *i.e.* a non-singular linear mapping $e : \mathbb{R}^4 \to T_p(M)$ or, equivalently, an ordered basis $e_1, \ldots, e_4$ of $T_p(M)$.

Definition 2.3 (Exponential Map)**.** If D_p, $p \in M$, denotes the set of vectors v in $T_p(M)$ such that the geodesic $\gamma_v : [0,1] \to M$ admits v as tangent vector in 0, then the exponential map at p is $\exp_p : D_p \to M$ with $\exp_p(v) \doteq \gamma_v(1)$.

Definition 2.4 (Normal Neighbourhoods)**.** For any point $p \in M$ there always exists a neighbourhood $\tilde{\mathscr{O}}$ of the 0-vector in $T_p(M)$ such that the exponential map is a *diffeomorphism* onto an open subset $\mathscr{O} \subset M$. Whenever $\mathscr{O}$ is star-shaped it is called a *normal neighbourhood* carrying the inverse map $\exp_p^{-1}$.

Now, if the spacetime M is endowed with a smooth Lorentzian metric, the concepts just defined can be combined with the additional metric information to yield the basis for further progress. The properties arising in this case are:

1. Given a linear frame at $p \in M$ one can endow the tangent space $T_p(M)$ with the standard Minkowski metric η.

2. Every point in a Lorentzian manifold admits a *normal neighbourhood* as introduced above [17, Chapter 5, Proposition 7 and Definition 5].

3. There is always a choice of *normal coordinates* such that the pull-back of the metric g under the exponential map is η on the pre-image of the point p.

4. In a Lorentzian manifold the *Gauss Lemma* holds true [17, Chapter 5, Lemma 1] so that, if $\tilde{C} \subset T_p(M)$ denotes the null cone, then $\tilde{C} \cap \tilde{\mathscr{O}}$ is mapped into a null cone in $\mathscr{O} \subset M$ consisting of initial segments of all null geodesics starting at p.

Having the above concepts at one's disposal it is possible to establish the connection between different spacetimes via their respective tangent spaces. Consider two spacetimes (M, g) and (M', g') and two generic points $p \in M$ and $p' \in M'$, together with their normal neighbourhoods $\mathscr{O}_p$ and $\mathscr{O}_{p'}$. If we equip each tangent space with an orthonormal basis via frames, $e_p : \mathbb{R}^4 \to T_p(M)$ and $e_{p'} : \mathbb{R}^4 \to T_{p'}(M')$, we are also free to introduce a map $i_{e,e'} : T_p(M) \to T_{p'}(M')$ which is constructed simply by identifying the elements of the two ordered bases. Then the exponential map is a diffeomorphism (hence invertible) in a geodesic neighbourhood. Therefore we introduce a map $\iota_{e,e'} : \mathscr{O} \to \mathscr{O}'$ such that

$$\iota_{e,e'} \doteq \exp_{p'} \circ i_{e,e'} \circ \exp_p^{-1} . \tag{2.1}$$

Remark 2.5. (i) The required inclusion $i_{e,e'} \circ \exp_p^{-1}(\mathscr{O}_p) \subset \tilde{\mathscr{O}}_{p'}$ can always be satisfied by choosing a smaller $\mathscr{O}_p$ with all the desired properties.

(ii) The map $\iota_{e,e'}$ is not unique, but depends on the orthonormal frames e and e'.

In order to ensure the possibility to assign a well-defined quantum field theory to the background M one has to require additional properties of the spacetime. Being interested only in local quantities here, one can relax the standard assumption of global hyperbolicity to strong causality [2] for the class of spacetimes M to be investigated.

Definition 2.6 (Strong Causality)**.** The spacetime M is said to be *strongly causal* if for every point $p \in M$ and every open neighbourhood $\mathscr{O}_p$ there exists a subset $\mathscr{O}'_p \subset \mathscr{O}_p$ such that no causal curve intersects $\mathscr{O}'_p$ more than once. In other words, $\mathscr{O}'_p$ is itself globally hyperbolic.

With the ultimate goal to compare quantum field theories locally on different spacetimes via a bulk-to-boundary reconstruction, we have to single out a preferred submanifold of codimension one. Our analysis is based on the class of *double cones*,

$$\mathscr{D}(p,q) \doteq I^+(p) \cap I^-(q) \subset M, \tag{2.2}$$

$I^\pm$ denoting the chronological future and past of the points, respectively, while $q \in \mathscr{O}'_p$. $\mathscr{D}(p,q)$ is an open, globally hyperbolic subset of $\mathscr{O}'_p$. Its closure, $\overline{\mathscr{D}(p,q)}$, is a compact set (see for example [20, Chapter 8]) which coincides with $J^+(p) \cap J^-(q) \cup \{p\} \cup \{q\}$ (using a definition of causal future and past, $J^\pm$, such that $p \notin J^+(p)$ and $q \notin J^-(q)$). The image of $\overline{\mathscr{D}(p,q)}$ in $T_p(M)$ under the map $\exp_p^{-1}$, $U(p,q)$, is *not* necessarily the closure of a double cone in $T_p(M) \sim \mathbb{R}^4$.

The very existence and properties of the sets $\mathscr{D}(p,q)$ and $J^+(p)$ under the exponential map suggest to consider $\partial J^+(p) \cap \overline{\mathscr{D}(p,q)}$ as the natural boundary on which to encode data from the bulk field theory. Thus, as the natural boundary of $\mathscr{D}(p,q)$ we take

$$\mathscr{C}_p^+ \doteq \partial J^+(p) \cap \overline{\mathscr{D}(p,q)}. \tag{2.3}$$

$\mathscr{C}_p^+$ is generated by future-directed null geodesics originating from p. These are not complete, since the set we are interested in is restricted to $\mathscr{D}(p,q) \subset \mathscr{O}'_p$. Its image under $\exp_p^{-1}$ in $T_p(M)$ is a portion of a null cone C^+ constructed with respect to the flat metric η, topologically equivalent to $I \times \mathbb{S}^2$, $I \subseteq \mathbb{R}$.

Remark 2.7. (i) The above characteristics are universal properties that do not depend on the specific choice of frame e as does the shape of the image under $\exp_p^{-1}$ or the pull-back of the metric in normal coordinates under $\exp_p^*$.

(ii) The bulk is $\mathscr{D}(p,q)$ which is a globally hyperbolic submanifold of M which can carry a full-fledged quantum field theory.

(iii) In contradistinction to the situation investigated in the relevant literature on the bulk-to-boundary techniques, the situation at hand is much more complicated and does not exhibit any symmetry group.

3 Algebras of Observables

Real scalar field theories on different spacetimes M and M' are known to be comparable in the special case that the spacetimes are either *isometrically* or *conformally* embedded into one another. If the spacetimes are related by diffeomorphisms it is always possible to transplant smooth field configurations from one spacetime to the other. But this is not the natural transformation since the diffeomorphisms do *not* in general preserve the geometric structures of the quantum or classical field theories. *E.g.*, if the equations of motion of a dynamical system are constructed out of the spacetime metric, generic diffeomorphisms do not preserve their form unless they are related to isometries.

Thus, the question arises what then is an appropriate procedure to compare quantum field theories on different spacetime backgrounds. The proposal to be put forward here is based on the fact that information on the existence and properties of physically relevant states (Hadamard states) can be extracted for a wide class of spacetimes by relating a *field theory within the spacetime* to *one on its conformal boundary*, a differentiable null submanifold of codimension one of (a suitable extension) of the underlying spacetime. This construction is both universal and intrinsic and thus allows for the comparison of different field theories on different spacetimes.

For the sake of simplicity, we shall henceforth only deal with a free real scalar field with generic mass m and generic curvature coupling ξ. The standard properties of such a physical system can be summarised along the lines of *e.g.* [21]. Consider a free real scalar field $\phi : \mathscr{D} \to \mathbb{R}$ subject to the equation of motion

$$P\phi \doteq \big(\Box_g + \xi R + m^2\big)\phi = 0, \quad m^2 > 0, \quad \xi \in \mathbb{R}, \tag{3.1}$$

where $\Box_g = -\nabla^\mu\nabla_\mu$ is the D'Alembert wave operator constructed out of the metric g while R is the scalar curvature. Each solution of this second-order hyperbolic partial differential equation can be constructed as the image of the map

$$\Delta : C_0^\infty(\mathscr{D}) \to C^\infty(\mathscr{D}) \tag{3.2}$$

with the *causal propagator* Δ defined as the difference of the advanced and the retarded fundamental solution.

By construction, $\mathscr{D}$ is contained in a larger globally hyperbolic set, $\mathscr{O}'$, so that $\phi_f \doteq \Delta(f)$ allows for a unique extension to a solution of the wave equation on all of $\mathscr{O}'$. Thus we are allowed to consider the restriction of ϕ_f on $\mathscr{C}_p^+$,

$$\phi_f|_{\mathscr{C}_p^+} \in C^\infty\big(\mathscr{C}_p^+\big). \tag{3.3}$$

Based on the above construction of the free real scalar field one introduces a suitable quantisation scheme that yields the quantum algebras on the bulk $\mathscr{D}$.

Definition 3.1. We define $\mathscr{F}_b(\mathscr{D})$ as the subset of sequences with a finite number of elements lying in

$$\bigoplus_{n\geq 0} \otimes_s^n C_0^\infty(\mathscr{D}), \tag{3.4}$$

where $n = 0$ yields $\mathbb{C}$ while $\otimes_s^n$ denotes the n-fold symmetric tensor product. $\mathscr{F}_b(\mathscr{D})$ can be promoted to a topological *-algebra via

- a tensor product $\cdot_S$ such that

  $$(F \cdot_S G)_n = \sum_{p+q=n} \mathcal{S}(F_p \otimes G_q), \tag{3.5}$$

  where $\mathcal{S}$ is the operator which realises total symmetrisation;

- a *-operation via complex conjugation, *i.e.*, $\{F_n\}_n^* = \{\overline{F}_n\}_n$ for all $F \in \mathscr{F}_b(\mathscr{D})$;

- the topology induced by the natural one of $\otimes_s^n C_0^\infty(\mathscr{D})$.

This construction can be cast in a different way developed in [5, 3] by considering $\mathscr{F}_b(\mathscr{D})$ as a set of functionals over the smooth field configurations $C^\infty(\mathscr{D})$. Every $F \in \mathscr{F}_b(\mathscr{D})$ yields a functional $F : C^\infty(\mathscr{D}) \to \mathbb{R}$, using the standard pairing $\langle\ ,\ \rangle$ between $\otimes^n C^\infty(\mathscr{D})$ and $\otimes^n C_0^\infty(\mathscr{D})$, via

$$F(\varphi) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \langle F_n, \varphi^n \rangle. \tag{3.6}$$

Decisive in what follows is a modification of the algebraic product $\cdot_S$ to be replaced by $\star$, which is unambiguously constructed out of the causal propagator Δ according to (3.2),

$$(F \star G)(\varphi) = \sum_{n=0}^{\infty} \frac{i^n}{2^n n!} \big\langle F^{(n)}(\varphi), \Delta^{\otimes n} G^{(n)}(\varphi) \big\rangle, \qquad F, G \in \mathscr{F}_b(\mathscr{D}). \tag{3.7}$$

Direct inspection shows that $F \star G$ still lies in $\mathscr{F}_b(\mathscr{D})$ and, more importantly, that $(\mathscr{F}_b(\mathscr{D}), \star)$ gets the structure of a *-algebra under the operation of complex conjugation.

No use has been made as yet of the equation of motion (3.1). Dividing out the ideal $\mathscr{I}$ generated by those elements of $\mathscr{F}_b(\mathscr{D})$ that are images of the operator P in (3.1), one gets another algebra,

$$\mathscr{F}_{bo}(\mathscr{D}) \doteq \mathscr{F}_b(\mathscr{D}) / \mathscr{I}, \tag{3.8}$$

that inherits the $\star$-operation from $\mathscr{F}_b(\mathscr{D})$. This *on-shell* algebra $\mathscr{F}_{bo}(\mathscr{D})$ is the *Borchers-Uhlmann algebra* commonly used.

Neither $\mathscr{F}_b(\mathscr{D})$ nor its on-shell version $\mathscr{F}_{bo}(\mathscr{D})$ are sufficient to fully analyse the underlying quantum field theory. *E.g.* the components of the stress-energy-momentum tensor are not included as they involve products of fields evaluated at the same spacetime point, a procedure not *a priori* permitted due to the distributional character of the fields. The way out of this problem is the so-called *Hadamard regularisation* [13, 14]. To be more specific, it is impossible to include objects of the form

$$F(\varphi) = \int_{\mathscr{D}} d\mu(g)\, f(x) \varphi^2(x)$$

in $\mathscr{F}_b(\mathscr{D})$, where $d\mu(g)$ is the metric-induced volume form, while f is a test function in $C_0^\infty(\mathscr{D})$ and $\varphi \in C^\infty(\mathscr{D})$. The star product (3.7) applied to such fields involves the ill-defined pointwise product of Δ with itself.

A solution to this problem can be given, following the line of reasoning in [5], by introduction of a new class of functionals, $\mathscr{F}_e(\mathscr{D})$, which have a finite number of non-vanishing derivatives, the n-th of which has to be a symmetric element of the space of compactly supported distributions $\mathcal{E}'(\mathscr{D}^n)$, whose wave front sets satisfy the restriction

$$\mathrm{WF}(F_n) \cap \left\{ \left(\mathscr{D} \times \overline{V}^{\,+}\right)^n \cup \left(\mathscr{D} \times \overline{V}^{\,-}\right)^n \right\} = \emptyset, \tag{3.9}$$

$\overline{V}^{\,\pm}$ the forward and the backward causal cone in tangent space, respectively.

$\mathscr{F}_e(\mathscr{D})$ is a *-algebra if one extends the *-operation of $\mathscr{F}_b(\mathscr{D})$ and endows it with a new product, $\star_H$, whose well-posedness was first proved in [6, 4, 13, 14]. The explicit form is

$$(F \star_H G)(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \big\langle F^{(n)}(\varphi), H^{\otimes n} G^{(n)}(\varphi) \big\rangle, \tag{3.10}$$

where $H \in \mathcal{D}'(\mathscr{D}^2)$ is the so-called *Hadamard bi-distribution* which satisfies the microlocal spectrum condition, hence yields a natural substitute for the notion of positivity of energy out of the wave front set. But it also suffers from an ambiguity as there exists the freedom to add a smooth symmetric function which means that, if H and H' are two Hadamard distributions, then the integral kernel of $H - H'$ is a symmetric element of $C^\infty(\mathscr{D}^2)$. On the level of the algebra this freedom yields an algebraic isomorphism $\mathfrak{i}_{H',H} : (\mathscr{F}_e(\mathscr{D}), \star_H) \to (\mathscr{F}_e(\mathscr{D}), \star_{H'})$ [13, 3],

$$\begin{aligned} \mathfrak{i}_{H',H} &= \alpha_{H'} \circ \alpha_H^{-1}, \\ \alpha_H(F) &\doteq \sum_{n=0}^{\infty} \frac{1}{n!} \big\langle H^{\otimes n}, F^{(2n)} \big\rangle. \end{aligned} \tag{3.11}$$

Also the extended algebra, $\mathscr{F}_e$, has its on-shell counterpart, $\mathscr{F}_{eo}$, constructed from the quotient with the ideal generated by the equation of motion applied to the elements of $\mathscr{F}_e$.

One advantage of the formalism expounded above is that it allows for a presentation in terms of categories, revealing the clear-cut concept of the mathematical structures involved and their interrelations. This idea has first been used in [7] to formulate the principle of general local covariance. The categories to be utilised here are:

- DoCo: Objects are the oriented and time-oriented double cones $\mathscr{D}(p, q)$ with the property that there exists a normal neighbourhood $\mathcal{O}_p \subset M$ centred in p that contains $\mathscr{D}(p, q)$. The morphisms are equivalence classes of the maps $\imath_{e,e'} : \mathcal{O}_p \to \mathcal{O}'_{p'}$ of (2.1) such that $\imath_{e,e'}\big|_{\mathscr{D}}\big(\mathscr{D}(p, q)\big) \subset \mathscr{D}'(p', q')$ with respect to the following equivalence relation $\sim$: $\imath_{e,e'} \sim \imath_{\tilde{e},\tilde{e}'}$ if and only if there exists an element $\Lambda \in SO_0(3, 1)$ such that $\tilde{e} = \Lambda e$ and $\tilde{e}' = \Lambda e'$.

- $\mathsf{DoCo}_{\mathrm{iso}}$: This is the subcategory of DoCo with the same objects as above but the morphisms restricted to isometric embeddings.

- Alg_i: Objects are the unital topological *-algebras $\mathscr{F}_i(\mathscr{D})$, $i = b$, bo, and $i = e$, eo, which in the latter case actually are equivalence classes with respect to the map (3.11). The morphism are injective unit-preserving *-homomorphisms.

As in [7] one associates a suitable algebra to each double cone $\mathscr{D}$ via a functor $\mathscr{F}$ between $\mathsf{DoCo}_{\mathrm{iso}}$ and Alg_i,

$$\mathscr{F} : \mathscr{D} \to \mathscr{F}_i(\mathscr{D}), \quad i = b, bo. \tag{3.12}$$

It is obvious to try to extend the functor (3.12) to the category DoCo with its larger group of morphisms. But this is not possible; the diagram

$$\begin{tikzcd} \mathscr{D} \arrow[r, "\mathscr{F}"] \arrow[d, "\iota_{e,e'}"'] & \mathscr{F}_i(\mathscr{D}) \\ \mathscr{D}' \arrow[r, "\mathscr{F}"'] & \mathscr{F}_i(\mathscr{D}') \end{tikzcd} \tag{3.13}$$

cannot be closed on the right-hand side since $\iota_{e,e'}$ is not an isometry in general and will thus not map solutions of the wave equation in $\mathscr{D}$ into those in $\mathscr{D}'$ (of a different metric). It will also neither preserve the causal propagator spoiling the $\star$-operation nor the singular structure of the Hadamard bi-distribution that only depends on the underlying geometry.

Following the procedure indicated above, one now introduces quantum algebras on the boundary, treating the null cone itself as a carrier of a quantum field theory. But in doing so one has to be careful to make these constructs large enough to contain the images of suitable projections of all the elements in the bulk. To this end, consider three sets in $\mathbb{R} \times \mathbb{S}^2 \subset \mathbb{R}^4$,

$$\mathscr{C}_p^+ \doteq \left\{(V, \theta, \phi) \in \mathbb{R}^2 \,\middle|\, V \in (0, V_0(\theta, \phi)) \subset \mathbb{R}, (\theta, \phi) \in \mathbb{S}^2\right\}, \tag{3.14}$$

where $V_0(\theta, \phi)$ is a bounded smooth function on the sphere, while $\mathscr{C}_p$ and $\mathscr{C}$ are defined by allowing the coordinate V to vary in $(0, \infty)$ and the full real line $\mathbb{R}$, respectively. Hence, $\mathscr{C}_p^+ \subset \mathscr{C}_p \subset \mathscr{C}$.

On the boundary one then defines a space of functions,

$$\mathscr{S}\left(\mathscr{C}_p\right) \doteq \left\{\psi \in C^\infty\left(\mathscr{C}_p\right) \,\middle|\, \psi = hf|_{\mathscr{C}_p}, f \in C_0^\infty\left(\mathbb{R}^4\right), h \in C^\infty\left(\mathscr{C}_p\right)\right\}, \tag{3.15}$$

where, uniformly on $\mathbb{S}^2$, $h \to 0$ for $V \to 0$ and each derivative along V tends to a constant. $\mathscr{S}(\mathscr{C}_p)$ is a symplectic space when furnished with the strongly non-degenerate symplectic form,

$$\sigma_{\mathscr{C}}(\psi, \psi') \doteq \int_{\mathscr{C}_p} \left[\psi \frac{d\psi'}{dV} - \frac{d\psi}{dV}\psi'\right] dV \wedge d\mathbb{S}^2, \quad \psi,\, \psi' \in \mathscr{S}(\mathscr{C}_p), \tag{3.16}$$

where $d\mathbb{S}^2$ is the standard measure on the unit 2-sphere. Based on this space of functions, an algebra of observables on the boundary is introduced in

Definition 3.2. We define $\mathscr{A}_b(\mathscr{C}_p)$ as the space, whose generic element F is a sequence $\{F_n\}_n$ with a finite number of components in

$$\bigoplus_{n \geq 0} \otimes_s^n \mathscr{S}(\mathscr{C}_p), \tag{3.17}$$

where $\otimes_s^n$ again denotes the symmetrised n-fold tensor product and the first term in the sum is $\mathbb{C}$. The space (3.17) is a full topological *-algebra when endowed with

- a *-operation which is the complex conjugation, *i.e.*, $\{F_n\}_n^* = \{\overline{F_n}\}_n$ for all $F \in \mathscr{A}_b(\mathscr{C}_p)$;
- multiplication of elements such that for any F, $G \in \mathscr{A}_b(\mathscr{C}_p)$,

$$(F \cdot_S G)_n = \sum_{p+q=n} \mathcal{S}(F_p \otimes G_q); \tag{3.18}$$

- the topology induced by the natural topology of $\mathscr{S}(\mathscr{C}_p)$, *viz.* the topology of smooth functions on $\mathscr{C}_p$.

This algebra on the cone is not yet suited to be put in correspondence with the one on the bulk. Again, a deformation of the product is needed to achieve this. Let $X \doteq \{\Phi \in C^\infty(\mathscr{C}_p) \,|\, V^{-1}\Phi \in C^\infty(\mathscr{C})\}$, then the pairing $\langle\ ,\ \rangle$ between $\otimes^n X$ and $\otimes^n \mathscr{S}(\mathscr{C}_p)$ allows for $F \in \mathscr{A}_b(\mathscr{C}_p)$ to be considered as a functional $F : X \to \mathbb{R}$,

$$F(\Phi) = \sum_{n=0}^{\infty} \frac{1}{n!}\langle F_n, \Phi^n\rangle, \quad \Phi \in X. \tag{3.19}$$

While there is no equation of motion for the theory on $\mathscr{C}_p$ and, hence, no causal propagator, a new $\star_B$-product on $\mathscr{A}_b(\mathscr{C}_p)$ can be introduced nonetheless,

$$(F \star_B G)(\Phi) \doteq \sum_{n=0}^{\infty} \frac{i^n}{2^n n!}\big\langle F_n(\Phi), \Delta^n_{\sigma_\mathscr{C}} G_n(\Phi)\big\rangle, \quad \Phi \in X, \tag{3.20}$$

where $\Delta_{\sigma_\mathscr{C}}$ is the integral kernel of (3.16), *i.e.*,

$$\Delta_{\sigma_\mathscr{C}}\big((V,\theta,\varphi),(V',\theta',\varphi')\big) = -\frac{\partial^2}{\partial V \partial V'}\,\mathrm{sign}(V-V')\,\delta(\theta,\theta'), \tag{3.21}$$

where $\delta(\theta,\theta')$ stands for $\delta(\theta-\theta')\delta(\varphi-\varphi')$ and the derivatives have to be taken in the weak sense.

To conclude the general strategy outlined in the introduction one has to define a notion of extended algebra also on the boundary. But in this case there is no standard Hadamard state or bi-distribution available. As a replacement we select a natural bi-distribution on $\mathscr{C}_p^+$ that has already been studied in [9, 10, 8, 16, 15, 18, 19].

Definition 3.3. A *natural distinguished boundary state* on $\mathscr{C}^2$ is the weak limit

$$\omega\big((V,\theta,\phi),(V',\theta',\phi')\big) \doteq -\frac{1}{\pi}\lim_{\epsilon\to 0^+}\frac{1}{(V-V'-i\epsilon)^2}\delta(\theta,\theta'),$$

a well-defined bi-distribution in $\mathcal{D}'(\mathscr{C}^2)$, where $\mathscr{C} \sim \mathbb{R}\times\mathbb{S}^2$.

This bi-distribution yields a Gaussian state.

Proposition 3.4. *The Gaussian (quasi-free) state constructed out of the distribution ω has the following properties:*

1. *It is a well-defined algebraic state on $\mathscr{A}_b(\mathscr{C}_p)$ and on $\mathscr{A}_b(\mathscr{C})$.*
2. *It is a vacuum with respect to derivatives in the V-coordinate.*
3. *It is invariant under the change of the local frame.*

Now it is possible to introduce a suitable extended algebra on the boundary. In preparation one needs the following

Definition 3.5. $\mathcal{A}^n$ is the set of elements $F_n \in \mathcal{D}'(\mathscr{C}_p^n)$ that fulfil the following properties:

1. **Compactness**: The F_n are compact towards the future, *i.e.*, the support of F_n is contained in a compact subset of $\mathscr{C}^n \sim (\mathbb{R}\times\mathbb{S}^2)^n$.

2. **Causal non-monotonic singular directions**: The wave front set of F_n contains only causal non-monotonic directions which means that

$$\mathrm{WF}(F_n) \subseteq W_n \doteq \left\{(x,\zeta)\in(T^*\mathscr{C}_p)^n\setminus\{0\}\,|\,(x,\zeta)\notin \overline{V}_n^+\cup\overline{V}_n^-,\,(x,\zeta)\notin S_n\right\}, \tag{3.22}$$

where $(x,\zeta)\equiv(x_1,\dots,x_n,\zeta_1,\dots,\zeta_n)\in\overline{V}_n^+$ if, employing the standard coordinates on $\mathscr{C}_p$, for all $i=1,\,\dots,\,n$, $(\zeta_i)_V>0$ or ζ_i vanishes. The subscript V here refers to the component along the V-direction on $\mathscr{C}_p$. Analogously, $(x,\zeta)\in\overline{V}_n^-$ if every $(\zeta_i)_V<0$ or ζ_i vanishes. Furthermore, $(x,\zeta)\in S_n$ if there exists an index i such that, simultaneously, $\zeta_i\neq 0$ and $(\zeta_i)_V=0$.

3. **Smoothness Condition**: The distribution F_n can be factorised into the tensor product of a smooth function and an element of $\mathcal{A}^{n-1}$ when localised in a neighbourhood of $V=0$, *i.e.*, there exists a compact set $\mathcal{O}\subset\mathscr{C}_p$ such that, if $\Theta\in C_0^\infty(\mathscr{C}_p)$ so that it is equal to 1 on $\mathcal{O}$ and $\Theta'\doteq 1-\Theta$, then for every multi-index P in $\{1,\dots,n\}$ and for every $i\leqslant n$,

$$f\doteq\tilde{F}_n(u_{x_{P_{i+1}},\dots,x_{P_n}})\,\Theta'_{x_{P_1}}\cdots\,\Theta'_{x_{P_i}}\in C^\infty(\mathscr{C}_P^i), \tag{3.23}$$

where $\tilde{F}_n: C_0^\infty(\mathscr{C}_p^{n-i-1})\to\mathcal{D}'(\mathscr{C}_p^i)$ is the unique map from $C_0^\infty(\mathscr{C}_p^{n-i-1})$ to $\mathcal{D}'(\mathscr{C}_p^i)$ determined by F_n using the Schwartz kernel theorem. Furthermore, $u_{x_{P_{i+1}},\dots,x_{P_n}}\in C_0^\infty(\mathscr{C}_p^{n-i})$, $x_{P_{i+1}},\,\dots,\,x_{P_n}$ specifying the integrated variables. For every $j\leqslant i$, $\partial_{V_1}\cdots\partial_{V_j}f$ lies in $C^\infty(\mathscr{C}_p^i)\cap L^2(\mathscr{C}_p^i,dV_{P_1}\wedge d\mathbb{S}^2_{P_1}\cdots dV_{P_i}\wedge d\mathbb{S}^2_{P_i})\cap L^\infty(\mathscr{C}_p^i)$, while the limit of f as V_j tends uniformly to 0.

Based on this set the extended algebra on the boundary can be defined making use of the bi-distribution ω to define the appropriate $\star$-product.

Definition 3.6. The extended algebra on $\mathscr{C}_p$ is defined as

$$\mathscr{A}_e(\mathscr{C}_p)=\bigoplus_{n\geqslant 0}\mathcal{A}_s^n,$$

where $\mathcal{A}_s^n$ is the subset of totally symmetric elements in $\mathcal{A}^n$ introduced in Definition 3.5 and only sequences with a finite number of elements are considered; the first space in the direct sum being $\mathbb{C}$. This set can be given the structure of a *-algebra by introducing the *-operation $\{F_n\}^*\doteq\{\overline{F_n}\}$ for all $F\in\mathscr{A}_e$. The composition law arises from a modification of $\star_B$ by means of the state ω introduced above. In the functional representation the composition law is

$$\begin{aligned}&\star_\omega:\mathscr{A}_e(\mathscr{C}_p)\times\mathscr{A}_e(\mathscr{C}_p)\to\mathscr{A}_e(\mathscr{C}_p),\\&(F\star_\omega G)(\Phi)=\sum_{n=0}^\infty\frac{1}{n!}\left\langle F^{(n)}(\Phi),\omega^n G^{(n)}(\Phi)\right\rangle,\end{aligned} \tag{3.24}$$

for all F, $G\in\mathscr{A}_e(\mathscr{C}_p)$ and all $\Phi\in C^\infty(\mathscr{C}_p)$.

An important result then is

Proposition 3.7. *The operation* (3.24) *is a well-defined product in* $\mathscr{A}_e$.

The relation between the data on the bulk and on the boundary is brought about by the following map.

Definition 3.8. Let $\mathscr{D}$ be a double cone and regard the portion $\mathscr{C}_p^+$ of the boundary as part of a cone $\mathscr{C}_p$. Then we introduce the linear map $\Pi : \mathscr{F}_e(\mathscr{D}) \to \mathscr{A}_e(\mathscr{C}_p^+)$ by setting

$$\Pi_n(F_n) \doteq \sqrt[4]{|g_{AB}|}_1 \cdots \sqrt[4]{|g_{AB}|}_n \, \Delta^{\otimes n}(F_n)\big|_{\mathscr{C}_p^n}, \tag{3.25}$$

where Δ is the causal propagator (3.2), $|_{\mathscr{C}_p^+}$ denotes the restriction on $\mathscr{C}_p^+$ and the subscripts $1, \ldots, n$ entail dependence of the root on the coordinates of the i-th cone.

This map enjoys all the properties needed for its intended use to connect the bulk and boundary.

Proposition 3.9. *The linear map*

$$\Pi : \mathscr{F}_e(\mathscr{D}) \to \mathscr{A}_e(\mathscr{C}_p^+)$$

is injective when restricted to the extended algebra $\mathscr{F}_{eo}(\mathscr{D})$.
Moreover, it gives rise to an Hadamard bi-distribution H_ω *constructed as the pull-back of the distinguished boundary state* ω,

$$H_\omega \doteq \Pi^* \omega.$$

The connection between the bulk and boundary is then given by the map Π according to

Theorem 3.10. *Under the assumptions of Definition 3.8, one has:*

1) Π *induces a unit-preserving* **-homomorphism between the algebras* $\big(\mathscr{F}_e(\mathscr{D}), \star_{H_\omega}\big)$ *and* $\big(\mathscr{A}_e(\mathscr{C}_p^+), \star_\omega\big)$.

2) Π *is an injective* **-homomorphism when acting "on shell" on* $\big(\mathscr{F}_{eo}(\mathscr{D}), \star_{H_\omega}\big)$.

4 General Covariance and Comparison of Theories

One problem indicated in Section 3 in connection with the principle of general covariance has been the fact that it is impossible to close the diagram 3.13 on the right-hand side. Now we introduce the category

- BAlg: Objects are the extended boundary (topological *-) algebras constructed on $\mathscr{C}_p^+$, and the morphisms are the corresponding *-homomorphisms.

Furthermore, $\alpha_{\imath_{e,e'}}$ is the *-homomorphism $\alpha_{\imath_{e,e'}} : \mathscr{A}_e(\mathscr{C}_p^+) \to \mathscr{A}_e(\mathscr{C}_p^{+\prime})$ whose action on the $F \in \mathscr{A}_e(\mathscr{C}_p^+)$ is defined as follows: By means of the push-forward, it is

$$\alpha_{\imath_{e,e'}}(F_n) = \imath_{e,e'}{}_* F_n,$$

on $\left(\imath_{e,e'}(\mathscr{C}_p^+)\right)^n \subset (\mathscr{C}_p^{+\prime})^n$, while $\alpha_{\imath_{e,e'}}(F_n) = 0$ on the complement of $\left(\imath_{e,e'}(\mathscr{C}_p^+)\right)^n$ in $(\mathscr{C}_p^{+\prime})^n$. Then we have

Proposition 4.1. *Consider $\mathscr{A}_e : \mathsf{DoCo} \to \mathsf{BAlg}$, whose action on the objects and morphisms is as follows,*

- *the action of $\mathscr{A}_e$ on the objects of DoCo is such that $\mathscr{A}_e(\mathscr{D}) = \Pi \circ \mathscr{F}_e(\mathscr{D}) = \mathscr{A}_e(\mathscr{C}_p^+)$;*
- *the action of $\mathscr{A}_e$ on the morphisms $\imath_{e,e'}$ is such that $\mathscr{A}_e(\imath_{e,e'}) = \alpha_{\imath_{e,e'}}$.*

Then $\mathscr{A}_e$ is a functor between the two categories.

This result allows for the diagrammatic representation,

$$\begin{tikzcd} \mathscr{D} \arrow[r, "\mathscr{F}"] \arrow[d, "\imath_{e,e'}"'] & \mathscr{F}_e(\mathscr{D}) \arrow[r, "\Pi"] & \mathscr{A}_e(\mathscr{C}_p^+) \arrow[d, "\alpha_{\imath_{e,e'}}"] \\ \mathscr{D}' \arrow[r, "\mathscr{F}"'] & \mathscr{F}_e(\mathscr{D}') \arrow[r, "\Pi'"'] & \mathscr{A}_e(\mathscr{C}_p^{+\prime}) \end{tikzcd} , \tag{4.1}$$

which is the closed version of diagram 3.13.

The comparison of theories on different spacetime backgrounds is performed by pulling a state ω on $\mathscr{A}_e(\mathscr{C}_p')$ back either on $\mathscr{F}_e(\mathscr{D}')$ via Π' or on $\mathscr{F}_e(\mathscr{D})$ via $\alpha_{\imath_{e,e'}} \circ \Pi$. An example for this procedure is the extraction of information about the curvature from measurements. Consider a massless real scalar field minimally coupled to scalar curvature both in Minkowski spacetime and in a Friedman-Robertson-Walker spacetime. Upon choosing a reference state ω and another Gaussian state ω', the difference of the expectation values of the regularised squared scalar fields in these two spacetimes can be expanded in a power series of a suitable local coordinate system and yields, at first order, a contribution dependent on the derivative of the scale factor, $a'(x_0)$, at the point x_0 in the Friedman-Robertson-Walker universe.

5 Summary and Outlook

In the general framework presented it can be established that to double cones strictly contained in a normal neighbourhood one can associate a Borchers-Uhlmann algebra of observables and extend it to contain more singular objects like Wick polynomials. The novel result is the construction of the extended algebra also on the causal boundary, well-posed by its mathematical properties and its relation to the theory on the bulk. Using these constructs one can identify a local state on the double cone which is physically well-behaved and looks the same for all inertial observers. It thus turns out to be a kind of local vacuum on a curved spacetime.

From the above scheme arises the possibility to compare the expectation values of field observables in the bulk of different spacetimes. This procedure is compatible with the standard principle of general local covariance and complements it. In particular, the comparison in the special case where one spacetime is Minkowski space can be used to determine the role and magnitude of geometric quantities.

The special Hadamard state in the bulk has to be studied in more detail. Its relation to states of minimum energy in Friedman-Robertson-Walker spacetimes is an example to be investigated. And it seems safe to expect that the procedure suggested here will be the method of choice to tackle many other basic problems of quantum field theories on curved spacetimes.

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Representations of Quantum Phase Spaces with Infinite Degrees of Freedom

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Abstract

A class of quantum phase spaces with infinite degrees of freedom is introduced and their Hilbert space representations are considered, including Fock space representations. It is shown that, under a suitable condition, there is a one-to-one correspondence between representations of such a quantum phase space and representations of the canonical commutation relations with infinite degrees of freedom. Also irreducibility of the representations is discussed.

Keywords: Quantum phase space, non-commutative phase space, canonical commutation relations, Fock space.

Mathematics Subject Classification 2000: 81D05, 81R60, 47L60, 47N50

1 Introduction

In the previous paper [1], we introduced a quantum phase space (QPS) with n degrees of freedom ($n = 1, 2, \cdots$), i.e., the associative algebra generated by elements $\hat{Q}_j, \hat{P}_j$ ($j = 1, \cdots, n$) and a unit element I obeying commutation relations

$$[\hat{Q}_j, \hat{Q}_k] = i\theta_{jk}I, \quad [\hat{P}_j, \hat{P}_k] = i\eta_{jk}I, \tag{1.1}$$

$$[\hat{Q}_j, \hat{P}_k] = i\delta_{jk}I, \quad j, k = 1, \cdots, n, \tag{1.2}$$

with $\theta = (\theta_{jk})$ and $\eta = (\eta_{jk})$ being $n \times n$ real anti-symmetric matrices, where $[X, Y] := XY - YX$, i is the imaginary unit and δ_{jk} is the Kronecker delta, and considered Hilbert space representations of it, which are unbounded due to (1.2). As is easily seen, (1.1) and (1.2) are deformations of the canonical commutation relations (CCR) with n degrees of freedom which are given by the case $\theta = \eta = 0$ in (1.1) and (1.2). Hence the QPS with n degrees of freedom can be viewed as a deformation of the canonical quantum phase space with the same degrees of freedom.

It was shown in [1] that, under a general condition for the parameter (θ, η) describing the non-commutativity of the QPS, there exists a one-to-one correspondence between representations of relations (1.1), (1.2) and representations of the CCR with n degrees of freedom, provided that domains of linear combinations of operators constituting representations are well behaved. Moreover, a QPS version of the von Neumann uniqueness theorem on Weyl representations of the CCR with n degrees of freedom ([3, Theorem VIII.14], [5]) was established. In this paper we introduce a QPS with *infinite degrees of freedom* as a natural extension of the QPS with the n degrees of freedom and consider its Hilbert space representations. To our best knowledge, this notion of QPS is new. Since the present paper is intended to be a summary of some fundamental results on representations of the QPS, we state almost of the results without proof.

2 Definitions

In general, we denote the inner product and the norm of a Hilbert space by $\langle \cdot \, , \cdot \rangle$ (linear in the second variable) and $\| \cdot \|$ respectively. For a linear operator A on a Hilbert space $\mathcal{K}$, we denote its domain by $\mathfrak{D}(A)$. If A is densely defined, then its adjoint is denoted A^*. We say that A is anti-self-adjoint if A is densely defined and $A^* = -A$. We denote by $\mathfrak{B}(\mathcal{K})$ the set of all bounded linear operators B on $\mathcal{K}$ with $\mathfrak{D}(B) = \mathcal{K}$.

Let $\mathcal{H}_{\mathrm{real}}$ be a real Hilbert space, and θ and η be anti-self-adjoint operators in $\mathfrak{B}(\mathcal{H}_{\mathrm{real}})$. We set

$$\Lambda := (\theta, \eta). \tag{2.1}$$

Let $\mathcal{V}$ be a dense subspace of $\mathcal{H}_{\mathrm{real}}$ and $\mathrm{QPS}_{\mathcal{V}}(\Lambda)$ be the associative algebra generated by elements $\hat{\phi}(f), \hat{\pi}(f)\,(f \in \mathcal{V})$ and a unit element I, obeying linearity

$$\hat{\phi}(af + bg) = a\hat{\phi}(f) + b\hat{\phi}(g),\ \ \hat{\pi}(af + bg) = a\hat{\pi}(f) + b\hat{\pi}(g), \quad a, b \in \mathbb{R}, f, g \in \mathcal{V}$$

and commutation relations

$$[\hat{\phi}(f), \hat{\phi}(g)] = i \langle f, \theta g\rangle I, \quad [\hat{\pi}(f), \hat{\pi}(g)] = i \langle f, \eta g\rangle I, \tag{2.2}$$

$$[\hat{\phi}(f), \hat{\pi}(g)] = i \langle f, g\rangle I, \quad f, g \in \mathcal{V}, \tag{2.3}$$

We call $\mathrm{QPS}_{\mathcal{V}}(\Lambda)$ the *quantum phase space indexed by $\mathcal{V}$ with parameter* Λ.

Remark 2.1 Relations (2.2) and (2.3) can be written in a concise form. We denote by $\mathcal{H}$ the complexification of $\mathcal{H}_{\mathrm{real}}$. Let

$$\hat{\Phi}(f) := \hat{\phi}(f_1) + \hat{\pi}(f_2), \quad f = f_1 + if_2 \in \mathcal{H}, f_1, f_2 \in \mathcal{H}_{\mathrm{real}}$$

and define $\sigma_\Lambda : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ by

$$\sigma_\Lambda(f, g) := \mathrm{Im}\, \langle f, g\rangle + \langle f_1, \theta g_1\rangle + \langle f_2, \eta g_2\rangle\,, \quad f, g \in \mathcal{H}, \tag{2.4}$$

where, for a complex number $z \in \mathbb{C}$ (the set of complex numbers), $\mathrm{Im}\, z$ denotes the imaginary part of z. Then it is easy to see that (2.2) and (2.3) are equivalent to

$$[\hat{\Phi}(f), \hat{\Phi}(g)] = i\sigma_\Lambda(f, g), \quad f, g \in \mathcal{H}.$$

Note that σ_Λ is a real bilinear form on $\mathcal{H} \times \mathcal{H}$ and anti-symmetric:

$$\sigma_\Lambda(f, g) = -\sigma_\Lambda(g, f), \quad f, g \in \mathcal{H}.$$

If $\theta \neq \eta$, then σ_Λ is non-degenerate and hence is a symplectic form on $\mathcal{H} \times \mathcal{H}$. Thus σ_Λ can be a natural extension of the well known symplectic form $\sigma_0(f, g) := \mathrm{Im}\, \langle f, g\rangle$ on $\mathcal{H} \times \mathcal{H}$ which is σ_Λ with $\Lambda = (0, 0)$.

We now come to a definition of representation of $\mathrm{QPS}_\mathcal{V}(\Lambda)$:

Definition 2.2 Let $\mathcal{H}_{\mathrm{real}}$, $\mathcal{V}$, θ, η and Λ be as above. Let $\mathcal{F}$ be a complex Hilbert space and $\mathcal{D} \neq \{0\}$ be a subspace of $\mathcal{F}$. We say that $(\mathcal{F}, \mathcal{D}, \{\hat{\phi}(f), \hat{\pi}(f) | f \in \mathcal{V}\})$ (or simply $\{\hat{\phi}(f), \hat{\pi}(f) | f \in \mathcal{V}\}$) is a representation of $\mathrm{QPS}_\mathcal{V}(\Lambda)$ if, for all $f \in \mathcal{V}$, $\hat{\phi}(f)$ and $\hat{\pi}(f)$ are symmetric operators on $\mathcal{F}$ and satisfy $\mathcal{D} \subset \cap_{f,g\in\mathcal{V}} \mathfrak{D}(\hat{\phi}(f)\hat{\phi}(g)) \cap \mathfrak{D}(\hat{\pi}(f)\hat{\pi}(g)) \cap \mathfrak{D}(\hat{\phi}(f)\hat{\pi}(g)) \cap \mathfrak{D}(\hat{\pi}(g)\hat{\pi}(f))$ with (2.2) and (2.3) on $\mathcal{D}$.

In particular, if, for all $f \in \mathcal{V}$, $\hat{\phi}(f)$ and $\hat{\pi}(f)$ are self-adjoint, then $\{\hat{\phi}(f), \hat{\pi}(f) | f \in \mathcal{V}\}$ is called a self-adjoint representation of $\mathrm{QPS}_\mathcal{V}(\Lambda)$.

Remark 2.3 Let $\mathcal{H}_{\mathrm{real}}$ be an n-dimensional real Hilbert space and $E = \{e_j\}_{j=1}^n$ be an orthonormal basis of $\mathcal{H}_{\mathrm{real}}$. Take $\mathcal{V} = \mathcal{H}_{\mathrm{real}}$. Then, putting

$$\hat{Q}_j := \hat{\phi}(e_j), \quad \hat{P}_j := \hat{\pi}(e_j), \quad \theta_{jk} := \langle e_j, \theta e_k\rangle\,, \quad \eta_{jk} := \langle e_j, \eta e_k\rangle\,, \quad j, k = 1, \cdots, n,$$

we see that $\{\hat{Q}_j, \hat{P}_j, I\}_{j=1}^n$ satisfies (1.1) and (1.2). Hence $\mathrm{QPS}_\mathcal{V}(\Lambda)$ includes, as special cases, QPS's with finite degrees of freedom.

Remark 2.4 The case where $\theta = \eta = 0$ in (2.2) and (2.3) gives the CCR indexed by $\mathcal{V}$:

$$[\hat{\phi}(f), \hat{\phi}(g)] = 0, \quad [\hat{\pi}(f), \hat{\pi}(g)] = 0, \quad [\hat{\phi}(f), \hat{\pi}(g)] = i\,\langle f, g\rangle\, I, \quad f, g \in \mathcal{V}.$$

Hence, in this case, $\mathrm{QPS}_\mathcal{V}(\Lambda)$ reduces to the canonical algebra (unbounded CCR algebra) indexed by $\mathcal{V}$ [2]. Thus $\mathrm{QPS}_\mathcal{V}(\Lambda)$ is a deformation of the canonical algebra indexed by $\mathcal{V}$.

3 Fock Space Representations

3.1 Boson Fock spaces

Let $\mathcal{K}$ be a complex Hilbert space and $\mathcal{F}_{\mathrm{b}}(\mathcal{K})$ be the boson Fock space over $\mathcal{K}$:

$$\mathcal{F}_{\mathrm{b}}(\mathcal{K}) := \bigoplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^{n} \mathcal{K} = \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} | \Psi^{(n)} \in \otimes_{\mathrm{s}}^{n} \mathcal{K}, n \geq 0, \sum_{n=0}^{\infty} \|\Psi^{(n)}\|^2 < \infty \right\},$$

where $\otimes_{\mathrm{s}}^{n} \mathcal{K}$ is the n-fold symmetric tensor product Hilbert space of $\mathcal{K}$ with $\otimes_{\mathrm{s}}^{0} \mathcal{K} := \mathbb{C}$. For each $u \in \mathcal{K}$, we denote by $a(u)$ the annihilation operator on $\mathcal{F}_{\mathrm{b}}(\mathcal{K})$, i.e., $a(u)$ is a densely defined closed linear operator on $\mathcal{F}_{\mathrm{b}}(\mathcal{K})$ such that, for all $u, v \in \mathcal{K}$, $\alpha, \beta \in \mathbb{C}$ and $\alpha^* a(u) + \beta^* a(v) \subset a(\alpha u + \beta v)$ and

$$(a(u)^* \Psi)^{(0)} = 0, \ (a(u)^* \Psi)^{(n)} = \sqrt{n} S_n (u \otimes \Psi^{(n-1)}), \quad n \geq 1, \ u \in \mathcal{K}, \Psi \in \mathfrak{D}(a(u)^*),$$

where S_n is the symmetrization operator on $\otimes^n \mathcal{K}$ (e.g., [4, §X.7]). It is well known that, for all $u \in \mathcal{K}$, $a(u)^{\#}$ ($a(u)$ or $a(u)^*$) leaves the subspace

$$\mathcal{D}_0 := \{\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_{\mathrm{b}}(\mathcal{K}) | \Psi^{(n)} = 0 \text{ for all sufficiently large } n\}$$

invariant and satisfies commutation relations

$$[a(u), a(v)^*] = \langle u, v \rangle, \quad [a(u), a(v)] = 0, \quad [a(u)^*, a(v)^*] = 0 \quad (u, v \in \mathcal{K}) \tag{3.1}$$

on $\mathcal{D}_0$.

For a closable linear operator A on a Hilbert space, we denote its closure by $\bar{A}$.

For each $f \in \mathcal{K}$, the operator

$$\Phi_{\mathrm{S}}(u) := \frac{1}{\sqrt{2}} \overline{(a(u)^* + a(u))},$$

called the Segal field operator, is self-adjoint and essentially self-adjoint on $\mathcal{D}_0$ [4, Theorem X.41]. It follows from (3.1) that, for all $u, v \in \mathcal{K}$,

$$[\Phi_{\mathrm{S}}(u), \Phi_{\mathrm{S}}(v)] = i \mathrm{Im} \ \langle u, v \rangle \tag{3.2}$$

on $\mathcal{D}_0$, where, for $z \in \mathbb{C}$, $\mathrm{Im}\, z$ denotes the imaginary part of z.

Let $C_{\mathcal{K}}$ be a conjugation on $\mathcal{K}$. Then

$$\mathcal{K}_{\mathrm{real}} := \{u \in \mathcal{K} | C_{\mathcal{K}} u = u\}$$

is a real Hilbert space and $\mathcal{K}$ is the complexification of $\mathcal{K}_{\mathrm{real}}$ with respect to $C_{\mathcal{K}}$.

For all $u \in \mathcal{K}_{\rm real}$, we define $\phi_{\rm F}(u)$ and $\pi_{\rm F}(u)$ by

$$\phi_{\rm F}(u) := \Phi_{\rm S}(u), \quad \pi_{\rm F}(u) := \Phi_{\rm S}(iu) = i\frac{1}{\sqrt{2}}\overline{(a(u)^* - a(u))}.$$

Then, by (3.2), we have for all $u, v \in \mathcal{K}_{\rm real}$,

$$[\phi_{\rm F}(u), \pi_{\rm F}(v)] = i\langle u, v\rangle, \ [\phi_{\rm F}(u), \phi_{\rm F}(v)] = 0, \ [\pi_{\rm F}(u), \pi_{\rm F}(v)] = 0$$

on $\mathcal{D}_0$. Namely $\{\phi_{\rm F}(u), \pi_{\rm F}(u) | u \in \mathcal{K}_{\rm real}\}$ satisfies the CCR indexed by $\mathcal{K}_{\rm real}$ on $\mathcal{D}_0$. This representation of the CCR indexed by $\mathcal{K}_{\rm real}$ is called the Fock representation on $\mathcal{F}_{\rm b}(\mathcal{K})$.

3.2 A class of self-adjoint representations of $\mathrm{QPS}_{\mathcal{V}}(\Lambda)$ on a boson Fock space

We say that $\Lambda = (\theta, \eta)$ is *normal* if there exist densely defined linear operators A, B, C and D from $\mathcal{H}_{\rm real}$ to $\mathcal{K}_{\rm real}$ such that

$$\mathcal{V} \subset \mathfrak{D}(A^*B) \cap \mathfrak{D}(B^*A) \cap \mathfrak{D}(C^*D) \cap \mathfrak{D}(D^*C) \cap \mathfrak{D}(A^*D) \cap \mathfrak{D}(D^*A) \cap \mathfrak{D}(B^*C) \cap \mathfrak{D}(C^*B)$$

and

$$A^*B - B^*A = \theta, \quad C^*D - D^*C = \eta, \quad A^*D - B^*C = I \quad \text{on } \mathcal{V}. \tag{3.3}$$

In this case we define an operator matrix

$$G := \begin{pmatrix} A & C \\ B & D \end{pmatrix}, \tag{3.4}$$

a linear operator from $\mathcal{H}_{\rm real} \oplus \mathcal{H}_{\rm real}$ to $\mathcal{K}_{\rm real} \oplus \mathcal{K}_{\rm real}$ with $\mathfrak{D}(G) = [\mathfrak{D}(A) \cap \mathfrak{D}(B)] \oplus [\mathfrak{D}(C) \cap \mathfrak{D}(D)]$. We call it a *generating operator matrix* of Λ. It is easy to see that (3.3) is equivalent to

$$G^* J G = K(\Lambda) \quad \text{on } \mathcal{V}, \tag{3.5}$$

where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad K(\Lambda) = \begin{pmatrix} \theta & I \\ -I & \eta \end{pmatrix}.$$

Remark 3.1 Let $\mathcal{H}$ be the complexification of $\mathcal{H}_{\rm real}$ and σ_Λ be the real bilinear form on $\mathcal{H} \times \mathcal{H}$ given by (2.4) (Remark 2.1). The operator G can be extended to a *real linear* operator on $\mathcal{H}$ by

$$\begin{aligned} G_{\mathbb{C}} f := Af_1 + Cf_2 \ &+ \ i(Bf_1 + Df_2), \\ &f = f_1 + if_2 \in \mathcal{H}, f_1 \in \mathfrak{D}(A) \cap \mathfrak{D}(B), f_2 \in \mathfrak{D}(C) \cap \mathfrak{D}(D). \end{aligned}$$

It follows from (3.5) that

$$\sigma_0(G_{\mathbb{C}} f, G_{\mathbb{C}} g) = \sigma_\Lambda(f, g), \quad f, g \in \mathcal{V}_{\mathbb{C}},$$

where $\mathcal{V}_{\mathbb{C}}$ is the complexification of $\mathcal{V}$. This shows the deformation structure of σ_0 to σ_Λ in terms of $G_{\mathbb{C}}$. In particular, if $\Lambda = (0,0)$, then $G_{\mathbb{C}}$ is a symplectic transformation with respect to the symplectic form σ_0. But, for $\Lambda \neq (0,0)$, $G_{\mathbb{C}}$ is not a symplectic transformation with respect to σ_Λ. Indeed we have

$$\sigma_\Lambda(G_{\mathbb{C}}f, G_{\mathbb{C}}g) = \sigma_\Lambda(f,g) + S_\Lambda(G_{\mathbb{C}}f, G_{\mathbb{C}}g), \quad f,g \in \mathcal{V}_{\mathbb{C}},$$

where

$$S_\Lambda(f,g) := \langle f_1, \theta g_1 \rangle + \langle f_2, \eta g_2 \rangle, \quad f = f_1 + if_2, g = g_1 + ig_2 \in \mathcal{H}.$$

Let Λ be normal with generating operator matrix G given by (3.4) and

$$\hat{\phi}_{\mathrm{F}}(f) := \Phi_{\mathrm{S}}(Af + iBf) = \overline{\phi_{\mathrm{F}}(Af) + \pi_{\mathrm{F}}(Bf)}, \tag{3.6}$$

$$\hat{\pi}_{\mathrm{F}}(f) := \Phi_{\mathrm{S}}(Cf + iDf) = \overline{\phi_{\mathrm{F}}(Cf) + \pi_{\mathrm{F}}(Df)}, \quad f \in \mathcal{V}. \tag{3.7}$$

Then we have:

Proposition 3.2 $(\mathcal{F}_{\mathrm{b}}(\mathcal{K}), \mathcal{D}_0, \{\hat{\phi}_{\mathrm{F}}(f), \hat{\pi}_{\mathrm{F}}(f) | f \in \mathcal{V}\})$ *is a self-adjoint representation of* $\mathrm{QPS}_{\mathcal{V}}(\Lambda)$.

We call the representation $(\mathcal{F}_{\mathrm{b}}(\mathcal{K}), \mathcal{D}_0, \{\hat{\phi}_{\mathrm{F}}(f), \hat{\pi}_{\mathrm{F}}(f) | f \in \mathcal{V}\})$ the *Fock representation* of $\mathrm{QPS}_{\mathcal{V}}(\Lambda)$ with generating operator matrix G given by (3.4).

Remark 3.3 Note that

$$\hat{\phi}_{\mathrm{F}}(f) = \Phi_{\mathrm{S}}(G_{\mathbb{C}}f), \quad \hat{\pi}_{\mathrm{F}}(f) = \Phi_{\mathrm{S}}(G_{\mathbb{C}}(if)), \quad f \in \mathcal{V}.$$

Example 3.4 We consider the case where $\mathcal{K}_{\mathrm{real}} = \mathcal{H}_{\mathrm{real}}$ so that $\mathcal{F}_{\mathrm{b}}(\mathcal{K}) = \mathcal{F}_{\mathrm{b}}(\mathcal{H})$. Let Γ be a bounded anti-self-adjoint operator on $\mathcal{H}$ leaving $\mathcal{H}_{\mathrm{real}}$ invariant and satisfying $\Gamma^2 = -I$. For $a > 0$ and $b > 0$, we define

$$\theta_\Gamma := \xi^2 a \Gamma, \quad \eta_\Gamma := \xi^2 b \Gamma$$

with

$$\xi := \frac{1}{\sqrt{1 + \frac{ab}{4}}}.$$

Put $\Lambda_\Gamma := (\theta_\Gamma, \eta_\Gamma)$. Then Λ_Γ is normal with generating operator matrix

$$G_\Gamma := \begin{pmatrix} \xi I & -\frac{\xi b}{2}\Gamma \\ \frac{\xi a}{2}\Gamma & \xi I \end{pmatrix}.$$

In this case, $\hat{\phi}_{\mathrm{F}}(f)$ and $\hat{\pi}_{\mathrm{F}}(f)$ $(f \in \mathcal{H}_{\mathrm{real}})$ take the following form respectively:

$$\hat{\phi}_\Gamma(f) := \Phi_{\mathrm{S}}\left(\xi f + i\frac{\xi a}{2}\Gamma f\right), \quad \hat{\pi}_\Gamma(f) := \Phi_{\mathrm{S}}\left(i\xi f - \frac{\xi b}{2}\Gamma f\right).$$

Thus $(\mathcal{F}_{\mathrm{b}}(\mathcal{H}), \mathcal{D}_0, \{\hat{\phi}_\Gamma(f), \hat{\pi}_\Gamma(f)\}_{f \in \mathcal{H}_{\mathrm{real}}})$ is a self-adjoint representation of $\mathrm{QPS}_{\mathcal{H}_{\mathrm{real}}}(\Lambda_\Gamma)$.

3.3 Reconstruction of the Fock representation of the CCR

We next consider representing $\phi_{\rm F}(u)$ and $\pi_{\rm F}(u)$ ($u \in \mathcal{K}_{\rm real}$) in terms of $\hat{\phi}_{\rm F}(f)$ and $\hat{\pi}_{\rm F}(f)$ ($f \in \mathcal{V}$) defined by (3.6) and (3.7) respectively. For this purpose, we introduce a class of Λ defined by (2.1).

Definition 3.5 We say that a parameter Λ is *semi-regular* if it is normal and it has a generating operator matrix G such that there exists a densely defined linear operator F from $\mathcal{K}_{\rm real} \oplus \mathcal{K}_{\rm real}$ to $\mathcal{V} \oplus \mathcal{V}$ with domain $D(F) = \mathcal{W} \oplus \mathcal{W}$ ($\mathcal{W}$ is a dense subspace of $\mathcal{K}_{\rm real}$) satisfying

$$GF \subset I. \tag{3.8}$$

Let Λ be semi-regular with generating operator matrix G given by (3.4) and satisfying (3.8). We write F as

$$F = \begin{pmatrix} F_1 & F_3 \\ F_2 & F_4 \end{pmatrix}, \tag{3.9}$$

where $F_j : \mathcal{W} \to \mathcal{V}$ is a linear operator ($j = 1, 2, 3, 4$). Then (3.8) is equivalent to the following relations:

$$AF_1 + CF_2 \subset I, \quad BF_1 + DF_2 \subset 0$$
$$AF_3 + CF_4 \subset 0, \quad BF_3 + DF_4 \subset I.$$

The next theorem shows that the Fock representation $\{\phi_{\rm F}(u), \pi_{\rm F}(u) | u \in \mathcal{W}\}$ of the CCR indexed by $\mathcal{W}$ can be recovered from the representation $\{\hat{\phi}_{\rm F}(f), \hat{\pi}_{\rm F}(f) | f \in \mathcal{V}\}$ of $\mathrm{QPS}_{\mathcal{V}}(\Lambda)$:

Theorem 3.6 *Suppose that Λ is semi-regular and let F be as above. Then, for all $u \in \mathcal{W}$,*

$$\phi_{\rm F}(u) = \hat{\phi}_{\rm F}(F_1 u) + \hat{\pi}_{\rm F}(F_2 u), \quad \pi_{\rm F}(u) = \hat{\phi}_{\rm F}(F_3 u) + \hat{\pi}_{\rm F}(F_4 u)$$

on $\mathcal{D}_0$.

Example 3.7 We consider Example 3.4. Suppose that

$$\chi := 1 - \frac{ab}{4} \neq 0. \tag{3.10}$$

Let

$$F_\Gamma := \frac{1}{\xi\chi} \begin{pmatrix} I & \frac{1}{2}b\Gamma \\ -\frac{1}{2}a\Gamma & I \end{pmatrix}.$$

Then we have

$$G_\Gamma F_\Gamma = I$$

Hence, under condition (3.10), Λ_Γ is semi-regular, Therefore, by Theorem 3.6, we have

$$\phi_{\rm F}(u) = \frac{1}{\xi\chi}\left\{\hat{\phi}_\Gamma(u) - \frac{a}{2}\hat{\pi}_\Gamma(\Gamma u)\right\}, \quad \pi_{\rm F}(u) = \frac{1}{\xi\chi}\left\{\frac{b}{2}\hat{\phi}_\Gamma(\Gamma u) + \hat{\pi}_\Gamma(u)\right\}, \quad u \in \mathcal{K}_{\rm real}$$

on $\mathcal{D}_0$.

4 Construction of a representation of $\mathrm{QPS}_{\mathcal{V}}(\Lambda)$ from a representation of the CCR indexed by $\mathcal{V}$

The method described in Subsection 3.2 can be extended to constructing a representation of $\mathrm{QPS}_{\mathcal{V}}(\Lambda)$ from a representation of the CCR indexed by a real vector space.

Let $\mathcal{W}$ be a dense subspace of $\mathcal{K}_{\mathrm{real}}$ and $(\mathcal{F}, \mathcal{D}, \{\phi(u), \pi(u)\}_{u\in\mathcal{W}})$ be a representation of the CCR indexed by $\mathcal{W}$, i.e., $\mathcal{F}$ is a Hilbert space, $\mathcal{D}$ is a dense subspace of $\mathcal{F}$ and $\phi(u)$ and $\pi(u)$ $(u \in \mathcal{W})$ are symmetric operators on $\mathcal{F}$ such that $\mathcal{D} \subset \cap_{u,v\in\mathcal{W}}\mathfrak{D}(\phi(u)\phi(v)) \cap \mathfrak{D}(\pi(u)\pi(v)) \cap \mathfrak{D}(\phi(u)\pi(v)) \cap \mathfrak{D}(\pi(v)\phi(u))$ and

$$[\phi(u), \phi(v)] = 0, \quad [\pi(u), \pi(u) = 0, \quad [\phi(u), \pi(v)] = i\,\langle u, v\rangle\,, \quad u, v \in \mathcal{W}$$

on $\mathcal{D}$.

Let Λ be normal with generating operator matrix G given by (3.4) and

$$\hat{\phi}(f) := \phi(Af) + \pi(Bf), \quad \hat{\pi}(f) := \phi(Cf) + \pi(Df), \quad f \in \mathcal{V}. \tag{4.1}$$

Then we have:

Theorem 4.1 *The set* $\left(\mathcal{F}, \mathcal{D}, \{\hat{\phi}(f), \hat{\pi}(f)\}_{f\in\mathcal{V}}\right)$ *is a representation of* $\mathrm{QPS}_{\mathcal{V}}(\Lambda)$.

5 Construction of a representation of CCR from a representation of $\mathrm{QPS}_{\mathcal{V}}(\Lambda)$

In this section we generalize the method of Subsection 3.3. Throughout this section we assume the following:

Hypothesis I. The parameter Λ defined by (2.1) is semi-regular with generating operator matrix G such that (3.4) and (3.8) hold with (3.9).

Lemma 5.1 *Under Hypothesis I, the following relations hold on* $\mathcal{W}$*:*

$$F_1^*\theta F_1 + F_2^*\eta F_2 + F_1^* F_2 - F_2^* F_1 = 0, \tag{5.1}$$

$$F_3^*\theta F_3 + F_4^*\eta F_4 + F_3^* F_4 - F_4^* F_3 = 0, \tag{5.2}$$

$$F_1^*\theta F_3 + F_2^*\eta F_4 + F_1^* F_4 - F_2^* F_3 = I. \tag{5.3}$$

.

Proof. It follows from (3.5) and (3.8) that $J \subset F^*K(\Lambda)F$, which implies (5.1)–(5.3). ∎

Let $\left(\mathcal{F}, \mathcal{D}, \{\hat{\phi}(u), \hat{\pi}(u)\}_{f\in\mathcal{V}}\right)$ be a representation of $\mathrm{QPS}_{\mathcal{V}}(\Lambda)$.

Theorem 5.2 *Let Hypothesis I be satisfied and, for each $u \in \mathcal{W}$, define $\phi(u)$ and $\pi(u)$ by*

$$\phi(u) := \hat{\phi}(F_1 u) + \hat{\pi}(F_2 u), \quad \pi(u) := \hat{\phi}(F_3 u) + \hat{\pi}(F_4 u) \tag{5.4}$$

on $\mathcal{D}$. Then $(\mathcal{F}, \mathcal{D}, \{\phi(u), \pi(u)\}_{u \in \mathcal{W}})$ is a representation of the CCR indexed by $\mathcal{W}$.

6 Conditions for Unitary Equivalences

Let $(\mathcal{F}_j, \mathcal{D}_j, \{\phi_j(u), \pi_j(u)\}_{u \in \mathcal{W}})$, $j = 1, 2$, be two self-adjoint representations of the CCR indexed by $\mathcal{W}$ and assume that Λ is normal with generating operator matrix G given by (3.4). Then, by Theorem 4.1, we have two representations $\left(\mathcal{F}_j, \mathcal{D}_j, \{\hat{\phi}_j(f), \hat{\pi}_j(f)\}_{f \in \mathcal{V}}\right)$, $j = 1, 2$, of $\mathrm{QPS}_{\mathcal{V}}(\Lambda)$, where

$$\hat{\phi}_j(f) := \phi_j(Af) + \pi_j(Bf), \quad \hat{\pi}_j(f) := \phi_j(Cf) + \pi_j(Df), \quad f \in \mathcal{V}.$$

In this context, it is interesting to investigate conditions for these two representations to be unitarily equivalent.

Theorem 6.1 *Assume Hypothesis I. Suppose that, for each $j = 1, 2$ and all $u \in \mathcal{W}$, $\phi_j(u)$ and $\pi_j(u)$ are essentially self-adjoint on $\mathcal{D}_j$. Then $\{\hat{\phi}_1(f), \hat{\pi}_1(f)\}_{f \in \mathcal{V}}$ is unitarily equivalent to $\{\hat{\phi}_2(f), \hat{\pi}_2(f)\}_{f \in \mathcal{V}}$ if and only if $\{\phi_1(u), \pi_1(u)\}_{u \in \mathcal{W}}$ is unitarily equivalent to $\{\phi_2(u), \pi_2(u)\}_{u \in \mathcal{W}}$.*

Remark 6.2 There exist infinitely many unitarily *inequivalent* self-adjoint representations of CCR with infinite degrees of freedom (e.g., [4, Theorem X.46]). Hence, Theorem 6.1 shows that there may exist infinitely many unitarily inequivalent representations of a QPS with infinite degrees of freedom.

7 Weyl Representations of $\mathrm{QPS}_{\mathcal{V}}(\Lambda)$

Let $\{\phi(u), \pi(u)\}_{u \in \mathcal{W}}$ be a set of self-adjoint operators on $\mathcal{F}$. The set $\{\phi(u), \pi(u)\}_{u \in \mathcal{W}}$ is called a *Weyl representation* of the CCR indexed by $\mathcal{W}$ if it obeys the Weyl relations

$$e^{i\phi(u)} e^{i\pi(v)} = e^{-i\langle u, v\rangle} e^{i\pi(v)} e^{i\phi(u)},$$
$$e^{i\phi(u)} e^{i\phi(v)} = e^{i\phi(v)} e^{i\phi(u)}, \quad e^{i\pi(u)} e^{i\pi(v)} = e^{i\pi(v)} e^{i\pi(u)}, \quad u, v \in \mathcal{W}.$$

In analogy with Weyl representations of CCR with infinite degrees of freedom, we can define Weyl representations of $\mathrm{QPS}_{\mathcal{V}}(\Lambda)$ as follows:

Definition 7.1 Let $\{\hat{\phi}(f), \hat{\pi}(f)\}_{f\in\mathcal{V}}$ be a set of self-adjoint operators on a Hilbert space. We say that $\{\hat{\phi}(f), \hat{\pi}(f)\}_{f\in\mathcal{V}}$ is a *Weyl representation* of $\mathrm{QPS}_{\mathcal{V}}(\Lambda)$ if

$$e^{i\hat{\phi}(f)}e^{i\hat{\pi}(g)} = e^{-i\langle f,g\rangle}e^{i\hat{\pi}(g)}e^{i\hat{\phi}(f)},$$
$$e^{i\hat{\phi}(f)}e^{i\hat{\phi}(g)} = e^{-i\langle f,\theta g\rangle}e^{i\hat{\phi}(g)}e^{i\hat{\phi}(f)}, \quad e^{i\hat{\pi}(f)}e^{i\hat{\pi}(g)} = e^{-i\langle f,\eta g\rangle}e^{i\hat{\pi}(g)}e^{i\hat{\pi}(f)}, \quad f,g\in\mathcal{V}.$$

Theorem 7.2 *Assume that Λ be normal. Let $\{\phi(u),\pi(u)\}_{u\in\mathcal{W}}$ be a Weyl representation of the CCR indexed by $\mathcal{W}$ and $\hat{\phi}(f)$, $\hat{\pi}(f)$ $(f\in\mathcal{V})$ be defined by (4.1). Suppose that, for all $f\in\mathcal{V}$, $\hat{\phi}(f)$ and $\hat{\pi}(f)$ are essentially self-adjopint. Then $\{\overline{\hat{\phi}(f)},\overline{\hat{\pi}(f)}\}_{f\in\mathcal{V}}$ is a Weyl representation of* $\mathrm{QPS}_{\mathcal{V}}(\Lambda)$.

We can also construct a Weyl representation of CCR from a Weyl representation of a QPS:

Theorem 7.3 *Assume Hypothesis I. Let $\hat{\phi}(f)$, $\hat{\pi}(f)$ $(f\in\mathcal{V})$ be Weyl representation of* $\mathrm{QPS}_{\mathcal{V}}(\Lambda)$. *Let $\phi(u)$ and $\pi(u)$ be defined by (5.4). Suppose that, for all $u\in\mathcal{W}$, $\phi(u)$ and $\pi(u)$ are essentially self-adjopint. Then $\{\overline{\phi(u)},\overline{\pi(u)}\}_{u\in\mathcal{W}}$ is a Weyl representation of the CCR indexed by $\mathcal{W}$.*

8 Irreducibility

Let A be a linear operator on a Hilbert space $\mathcal{H}$. We say that A *strongly commutes* with $B\in\mathfrak{B}(\mathcal{H})$ if $BA\subset AB$ (i.e., for all $\psi\in D(A)$, $B\psi\in D(A)$ and $BA\psi=AB\psi$).

For a set $\mathfrak{A}$ of (not necessarily bounded) linear operators on $\mathcal{H}$, we define

$$\mathfrak{A}' := \{B\in\mathfrak{B}(\mathcal{H})|BA\subset AB, \forall A\in\mathfrak{A}\}. \tag{8.1}$$

We call $\mathfrak{A}'$ the *strong commutant* of $\mathfrak{A}$.

We say that $\mathfrak{A}$ is *strongly irreducible* if $\mathfrak{A}'=\{cI|c\in\mathbb{C}\}$. In the case where $\mathfrak{A}\subset\mathfrak{B}(\mathcal{H})$, we say that $\mathfrak{A}$ is *irreducible* if $\mathfrak{A}'=\{cI\}$.

Remark 8.1 It is easy to see that, if $\mathfrak{A}$ is strongly irreducible, then it cannot be decomposed into a non-trivial direct orthogonal sum.

Example 8.2 The Fock representation $\{\phi_{\mathrm{F}}(u),\pi_{\mathrm{F}}(u)|u\in\mathcal{W}\}$ of the CCR indexed by a subspace $\mathcal{W}$ dense in $\mathcal{K}$ is strongly irreducible. This follows from the irreducibility of the set $\{e^{i\phi_{\mathrm{F}}(u)},e^{i\pi_{\mathrm{F}}(u)}|u\in\mathcal{W}\}$ of unitary operators (e.g., [4, Appendix to X.7]).

Theorem 8.3 *Assume Hypothesis I. Let $\left(\mathcal{F},\mathcal{D},\ \{\phi(u),\ \pi(u)\}_{u\in\mathcal{W}}\right)$ be a strongly irreducible, self-adjoint representation of the CCR indexed by $\mathcal{W}$ such that $\mathcal{D}$ is a core of $\phi(u)$ and $\pi(u)$ for all $u\in\mathcal{W}$. Then the representation $\left(\mathcal{F},\mathcal{D},\{\hat{\phi}(f),\hat{\pi}(f)\}_{f\in\mathcal{V}}\right)$ given by (4.1) is strongly irreducible.*

Corollary 8.4 *Under Hypotheis I, the Fock representation* $(\mathcal{F}_{\rm b}(\mathcal{K}), \mathcal{D}_0, \{\hat{\phi}_{\rm F}(f),\ \hat{\pi}_{\rm F}(f) \,|f \in \mathcal{V}\})$ *of* $\mathrm{QPS}_{\mathcal{V}}(\Lambda)$ *is strongly irreducible.*

Proof. This follows from Example 8.2 and Theorem 8.3. ∎

As for irreducibility of Weyl representations of a QPS, we have the following:

Theorem 8.5 *Let* $\{\hat{\phi}(f), \hat{\pi}(f)\}_{f\in\mathcal{V}}$ *be a Weyl representation of* $\mathrm{QPS}_{\mathcal{V}}(\Lambda)$*. Then,* $\{\hat{\phi}(f), \hat{\pi}(f)\}_{f\in\mathcal{V}}$ *is strongly irreducible if and only if* $\{e^{it\hat{\phi}(f)}, e^{it\hat{\pi}(f)}|t \in \mathbb{R}, f \in \mathcal{V}\}$ *is irreducible.*

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Random Point Fields, Random Measures and Bose-Einstein Condensations

Dedicated to professor Keiichi R. Ito on his 60th birthday

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Abstract

The position distributions of quantum statistical mechanical models of Boson gases are described by the random point fields. We apply this old idea to obtain typical random point field which describe the position distributions of boson gas in Bose-Einstein condensation. Precise informations, such as large deviation property of the random point field obtained in this way are given.

Key words: Boson Random Point Field; Bose-Einstein Condensation; Mean Field Interacting Bose Gas; Large Deviation Principle

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1 Introduction

The boson (as well as fermion) random point fields were derived by taking the thermodynamic limit of the canonical quantum statistical system of finite number of bosons in bounded boxes in $\mathbb{R}^d$ [TI1], based on the framework of [ShTa]. In [TI2], boson point fields corresponding to boson gases in Bose-Einstein condensation are derived. Paraparticle gases are considered in [TI3]. A model of mean field interacting boson gas trapped by a weak harmonic potential is considered by the boson random point fields method [TZ1]. In those models of boson gases, we got two typical random point fields; the usual boson random point field for the systems of low density boson gases, and another random point field for the systems of large density boson gas, i.e., one which describes the position distribution of boson gas in the state of Bose-Einstein condensation. In [TZ2], the large scale properties of the latter random point field were examined and compared to the corresponding known properties of the usual boson random point field.

The purpose of this note is to summarize how above two typical random point fields appear in those simple quantum statistical models of boson gases, and how the limit properties of them differ from each other. For the details, we refer to [TI1, TI2, TZ2]. It is interesting to glance the situation of the other models such as the free boson gas constructed from the grand canonical ensemble and the free mean field boson gas, because mixed random point fields may appear there.

2 Ideal Boson Gas in terms of Canonical Ensemble and Thermodynamic Limit

Consider $L^2(\Lambda_L)$ on $\Lambda_L = [-L/2, L/2]^d \subset \mathbb{R}^d$ with the Lebesgue measure. Let $\triangle_L$ be the Laplacian in $\mathcal{H}_L = L^2(\Lambda_L)$ satisfying periodic boundary conditions at $\partial\Lambda_L$. In this section, we regard

$$H_L = -\triangle_L$$

as the quantum mechanical Hamiltonian of a single free particle. The usual factor $\hbar^2/2m$ is set as unity. For $k \in \mathbb{Z}^d$, $\varphi_k^{(L)}(x) = L^{-d/2}\exp(i2\pi k \cdot x/L)$ is an eigenfunction of $\triangle_L$, and $\{\,\varphi_k^{(L)}\,\}_{k\in\mathbb{Z}^d}$ forms an CONS of $\mathcal{H}_L$. In the following, we use the operator $G_L = \exp(\beta\triangle_L)$, which has the kernel

$$G_L(x,y) = \sum_{k\in\mathbb{Z}^d} e^{-\beta|2\pi k/L|^2}\varphi_k^{(L)}(x)\overline{\varphi_k^{(L)}(y)} \tag{2.1}$$

for $\beta > 0$. We put $g_k^{(L)} = \exp(-\beta|2\pi k/L|^2)$ for the eigenvalue of G_L of the eigenfunction $\varphi_k^{(L)}$.

Consider the system consists of N identical particles which obey Bose-Einstein statistics in a finite box Λ_L. The space of the quantum mechanical states of the system is given by

$$\mathcal{H}_{L,N}^B = \{\, S_N f \mid f \in \otimes^N \mathcal{H}_L \,\},$$

where

$$S_N f(x_1, \cdots, x_N) = \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} f(x_{\sigma(1)}, \cdots x_{\sigma(N)}) \qquad (\, x_1, \cdots, x_N \in \Lambda_L \,)$$

is symmetrization in the N indices. Using the CONS $\{\, \varphi_k^{(L)} \,\}_{k \in \mathbb{Z}^d}$ of $L^2(\Lambda_L)$, we define the element

$$\Psi_k^{(L)}(x_1, \cdots, x_N) = \frac{1}{\sqrt{N! n(k)}} \sum_{\sigma \in \mathcal{S}_N} \varphi_{k_1}^{(L)}(x_{\sigma(1)}) \cdot \cdots \cdot \varphi_{k_N}^{(L)}(x_{\sigma(N)}) \tag{2.2}$$

of $\mathcal{H}_{L,N}^B$ for $k = (k_1, \cdots, k_N) \in \mathbb{Z}^d$, where $n(k) = \prod_{l \in \mathbb{Z}^d} (\sharp\{\, n \in \{\, 1, \cdots, N \,\} \mid k_n = l \,\}!)$. Let us introduce the subset $(\mathbb{Z}^d)^N_{\prec} = \{\, (k_1, \cdots, k_N) \in (\mathbb{Z}^d)^N \mid k_1 \prec \cdots \prec k_N \,\}$ of $(\mathbb{Z}^d)^N$, then $\{\, \Psi_k \,\}_{k \in (\mathbb{Z}^d)^N_{\prec}}$ forms a CONS of $\mathcal{H}_{L,N}^B$.

From the basic postulate of quantum mechanics and statistical physics, the probability density function for the position distribution of the equilibrium state of the system at the inverse temperature β is given by

$$\begin{aligned} & p_{L,N}(x_1, \cdots, x_N) = \\ & Z_{L,N}^{-1} \sum_{k \in (\mathbb{Z}^d)^N_{\prec}} \Big(\prod_{j=1}^{N} e^{-\beta |2\pi k_j / L|^2} \Big) |\Psi_k^{(L)}(x_1, \cdots, x_N)|^2 \\ & = \frac{\mathrm{per}\, \{G_L(x_i, x_j)\}_{1 \leqslant i,j \leqslant N}}{\int_{\Lambda_L^N} \mathrm{per}\, \{G_L(x_i, x_j)\}_{1 \leqslant i,j \leqslant N}\, dx_1 \cdots dx_N}, \end{aligned}$$

where per denotes the permanent:

$$\mathrm{per}\, A = \sum_{\sigma \in \mathfrak{S}_n} \prod_{j=1}^{n} A_{j\sigma(j)} \qquad \text{for } n \times n \text{ matrix} \quad A.$$

2.1 Brief review of Random Point Field

We would like to study the position distribution of the constituent particles of the gases in terms of random point fields[RPFs]. In this subsection, we try to give an brief introduction to the theory of RPFs.

Let $Q(\mathbb{R}^d)$ be the set of all the locally finite subsets of $\mathbb{R}^d$, i.e., the space of all the subsets of $\mathbb{R}^d$ which have no accumulation points. A probability measure on $Q(\mathbb{R}^d)$ is called a random point field on $\mathbb{R}^d$. We can identify the set of points $\{x_1, x_2, \cdots x_n, \cdots\}$ with the point measure $\sum_j \delta_{x_j} = \xi$. Then, $Q(\mathbb{R}^d)$ is considered as the space of all the integer valued Radon measures on $\mathbb{R}^d$. In this scheme, we may introduce the natural functionals on $Q(\mathbb{R}^d)$:

$$\langle f, \xi \rangle = \sum_j f(x_j)$$

for bounded function f on $\mathbb{R}^d$ of compact support. By using this functional, various quantities concerning RPFs are described. For examples,

$$\langle \chi_A, \xi \rangle = \sum_j \chi_A(x_j) = \#\{\, x_j \in A \,\}$$

represents the number of points in the intersection of A and the set identified by ξ, and so on. Especially, a RPF μ on $\mathbb{R}^d$ is characterized by its generating (or Laplace) functional

$$\int_{Q(\mathbb{R}^d)} e^{-\langle f, \xi \rangle} \, d\mu$$

for $f \in C_0(\mathbb{R}^d), f \geqslant 0$. Moreover the weak convergence of any sequence of RPF is established if the point-wise convergence of corresponding sequence of generating functionals is shown. For details, see e.g., [DV].

Now let us see how to represent RPFs on $\mathbb{R}^d$ which describe the position distribution of our system and to calculate their generating functionals. Note that our system has only finite number of particles before the thermodynamic limit. So, our strategy is to construct first RPFs for finite number of particles, and then get an infinite RPF which describes boson gas in the whole space $\mathbb{R}^d$ through thermodynamic limit. In this respect, we use the point-wise convergence of the generating functionals.

A finite RPF is determined if the exclusion measures are given. That is to say,

$$\mathrm{Prob}\left\{ \begin{array}{c} \text{The total number of points is equal to } n \\ \text{and each point is contained in a} \\ d\text{-dimensional rectangle } \ (x_j, x_j + dx_j] \\ = \prod_{k=1}^d (x_j^{(k)}, x_j^{(k)} + dx_j^{(k)}]), \ (j = 1, \cdots, n) \end{array} \right\} \equiv J_n(x_1, \cdots, x_n) dx_1 \cdots dx_n,$$

where $x_j = (x_j^{(k)})_{k=1}^d$. The partition function $Z_{L,N}$ suggests that the position distribution of constituent particles of the present system is given by

$$\begin{aligned} J_n(x_1, \cdots, x_n) &= \delta_{n,N} \,\mathrm{per}\, \big\{ G_L(x_i, x_j) \big\}_{1 \leqslant i,j \leqslant N} / Z_{L,N}. \\ &= p_{L,N}(x_1, \cdots, x_N) \, \delta_{n,N} \end{aligned}$$

Then the resulting RPF $\nu_{L,N}$ has the generating functional

$$\begin{aligned} \int_{Q(\mathbb{R}^d)} d\nu_{L,N}(\xi) e^{-\langle f, \xi \rangle} &= \sum_{n=0}^{\infty} \int_{(\mathbb{R}^d)^n} e^{-\sum_j f(x_j)} \frac{J_n(x_1, \cdots, x_n)}{n!} dx_1 \cdots dx_n \\ &= \frac{\mathrm{Tr}_{\otimes_{sym}^N L^2(\mathbb{R}^d)} [(\otimes^N G_{L,N})(\otimes^N e^{-f})]}{\mathrm{Tr}_{\otimes_{sym}^N L^2(\mathbb{R}^d)} [(\otimes^N G_{L,N})]}. \end{aligned} \tag{2.3}$$

Recall that a RPF is characterized by its generating functional as the theory of RPFs shows. As f, it is enough to consider any non-negative continuous function of compact support. See the arguments in [TI1, TI2, TI3] for detail.

Now let us consider the thermodynamic limit

$$L, N \to \infty, \quad N/L^d \to \rho.$$

Theorem 2.1 *(1) For the normal case $\rho < \rho_c = \Gamma(d/2)/(4\pi\beta)^{d/2} = \lim_{r\to 1} K_r(x,x)$,*

$$\nu_{L,N} \to \nu_r \qquad \textit{weakly,}$$

where ν_r is the so called boson process characterized by the generating functional

$$\int_{Q(\mathbb{R}^d)} \big[e^{-<f,\xi>}\big]\,\nu_r(d\xi) = \mathrm{Det}\,[1 + \sqrt{1-e^{-f}}K_r\sqrt{1-e^{-f}}]^{-1},$$

where

$$K_r = rG(1-rG)^{-1} \quad \textit{with} \quad G = e^{\beta\triangle} \quad \textit{the heat operator on } L^2(\mathbb{R}^d),$$

and $r \in (0,1)$ is the unique solution of

$$\rho = \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d}\frac{re^{-\beta|p|^2}}{1-re^{-\beta|p|^2}} = K_r(x,x),$$

(2) For the BEC case $\rho > \rho_c$,

$$\nu_{L,N} \to \nu_\rho^{(B)} \qquad \textit{weakly.}$$

The limit RPF $\nu_\rho^{(B)}$ has the generating functional:

$$\int_{Q(\mathbb{R}^d)} \big[e^{-<f,\xi>}\big]\,\nu_\rho^{(B)}(d\xi) \quad = $$

$$\frac{\exp\big(-(\rho-\rho_c)(\sqrt{1-e^{-f}},[1+K_f]^{-1}\sqrt{1-e^{-f}})\big)}{\mathrm{Det}\,[1+K_f]}$$

where

$$K_f = \sqrt{1-e^{-f}}G(1-G)^{-1}\sqrt{1-e^{-f}}$$
$$= ((G(1-G)^{-1})^{1/2}\sqrt{1-e^{-f}})^*(G(1-G)^{-1})^{1/2}\sqrt{1-e^{-f}}.$$

RemarkF $\nu_\rho^{(B)}$ is the convolution of two RPFs $\nu_{\rho-\rho_c}^{BEC} * \nu_{r=1}$. In other words, the total point measure ξ is a sum

$$\xi = \xi^{(N)} + \xi^{(C)}$$

and $\xi^{(N)}$ and $\xi^{(C)}$ are independent point measures obeying $\nu_{r=1}$ and ν^{BEC}, respectively. Thus, $\xi^{(N)}$ describes the "normal particles" obeying $\nu_{r=1}$, whose generating functional is the denominator $\mathrm{Det}\,[1+K_f]^{-1}$. While, $\xi^{(C)}$ describes the "condensed particles" obeying $\nu_{\rho-\rho_c}^{BEC}$, whose generating functional is the numerator $\exp\big(-(\rho-\rho_c)(\sqrt{1-e^{-f}},[1+K_f]^{-1}\sqrt{1-e^{-f}})\big)$.

For the details, see [TI1, TI2].

3 Ideal Boson Gas in terms of Grand Canonical Ensemble and Thermodynamic Limit

In this section, we consider the Grand Canonical Ensemble of free bosons in a bounded box $\Lambda_L = [-L/2, L/2]^d \subset \mathbb{R}^d$, and then take the thermodynamic limit to get the corresponding RPF.

As in the previous section, let $\mathcal{H}_L = L^2(\Lambda_L)$ be the one particle state space, $H_L = -\triangle_L$ the one particle Hamiltonian, $G_L = e^{\beta\triangle_L}$, and $\hbar^2/2m = 1$.

The grand partition function is

$$\Xi_L(\mu) = \sum_{n=0}^{\infty} e^{n\beta\mu} \mathrm{Tr}_{\otimes_{sym}^n \mathcal{H}_L}[\otimes^n G_L],$$

where μ is the chemical potential, $\otimes_{sym}^n \mathcal{H}_L$ is the n-th symmetric tensor product of $\mathcal{H}_L$.

Let us recall the formula [ShTa, V]

$$\begin{aligned} &\mathrm{Det}\,(1-J)^{-1} = \\ &\quad \sum_{n=0}^{\infty} \frac{1}{n!} \int_{R^n} \quad \mathrm{per}\,\{J(x_i, x_j)\}_{i,j=1}^n \lambda^{\otimes n}(dx_1 \cdots dx_n), \end{aligned}$$

where J is any trace class integral operator on $L^2(R, \lambda)$, satisfying $\|J\| < 1$. Then we have

$$\begin{aligned} \Xi_L(\mu) &= \sum_{n=0}^{\infty} \frac{1}{n!} e^{n\beta\mu} \int_{\Lambda_L^n} \mathrm{per}\,\{G_L(x_i, x_j)\}_{1\leqslant i,j\leqslant n}\, dx_1 \cdots dx_n \\ &= \mathrm{Det}\,[1 - e^{\beta\mu} G_L]^{-1} \end{aligned}$$

for $\mu < 0$. The condition makes the right hand side well defined, since $\|G_L\| = 1$. In this case, the corresponding RPF $\nu_{L,\mu}$ is given by setting the exclusive measures

$$J_n(x_1, \cdots, x_n) = e^{\beta\mu n} \mathrm{per}\,\{G_L(x_i, x_j)\}_{1\leqslant i,j\leqslant n} \big/ \Xi_L(\mu).$$

The generating functional of $\nu_{L,\mu}$ is

$$\begin{aligned} &\int_{Q(\mathbb{R}^d)} e^{-\langle f, \xi\rangle} \nu_{L,\mu}(d\xi) \\ &= \sum_{n=0}^{\infty} \int_{(\mathbb{R}^d)^n} e^{-\sum_j f(x_j)} \frac{J_n(x_1, \cdots, x_n)}{n!} dx_1 \cdots dx_n \;=\; \frac{\tilde{\Xi}_L(\mu)}{\Xi_L(\mu)}, \end{aligned}$$

where

$$\tilde{\Xi}_L(\mu) = \sum_{n=0}^{\infty} e^{\beta\mu n} \mathrm{Tr}_{\otimes_{sym}^n \mathcal{H}_L}[(\otimes^n e^{-f})(\otimes^n G_L)] = \mathrm{Det}\,[1 - e^{-f} e^{\beta\mu} G_L]^{-1}.$$

Then we get

$$\frac{\tilde{\Xi}_L(\mu)}{\Xi_L(\mu)} = \frac{\mathrm{Det}\,[1 - e^{\beta\mu}G_L]}{\mathrm{Det}\,[1 - e^{-f}e^{\beta\mu}G_L]} = \frac{1}{\mathrm{Det}\,[1 + (1 - e^{-f})e^{\beta\mu}G_L(1 - e^{\beta\mu}G_L)^{-1}]}$$

$$\longrightarrow \frac{1}{\mathrm{Det}\,[1 + (1 - e^{-f})e^{\beta\mu}G(1 - e^{\beta\mu}G)^{-1}]} \qquad \text{as } L \to \infty.$$

Theorem 3.1 *In the thermodynamic limit $L \to \infty$ for fixed $\mu < 0$, we have*

$$\nu_{L,\mu} \to \nu_r \qquad \textit{weakly},$$

where $r = e^{\beta\mu} < 1$ and ν_r is the same one as in the canonical ensemble case.

The RPF ν_r has the density :

$$\rho = K_r(x, x) < \rho_c = K_1(x, x).$$

For $\mu \geqslant 0$, the system is not stable. To get BEC state ($\rho > \rho_c$), we must tune μ like

$$\rho - \rho_c = \frac{1}{L^d}\frac{e^{\beta\mu}}{1 - e^{\beta\mu}} \qquad \text{i.e., } \mu = -\frac{1 + o(1)}{\beta(\rho - \rho_c)L^d}.$$

Theorem 3.2 *We have*

$$\nu_{L,\mu} \to \nu_\rho^{gce} \qquad \textit{weakly as} \quad L \to \infty,$$

where ν_ρ^{gce} has the generating functional

$$\int_{Q(\mathbb{R}^d)} e^{-\langle f,\xi\rangle} d\nu_\rho^{gce}(\xi) =$$

$$(1 + (\rho - \rho_c)(\sqrt{1 - e^{-f}}, (1 + K_f)^{-1}\sqrt{1 - e^{-f}}))^{-1}\mathrm{Det}\,[1 + K_f]^{-1}.$$

RemarkF ν_ρ^{gce} is the convolution of two RPFs: $\nu_\rho^{gce} = \nu_{\rho-\rho_c}^{bec} * \nu_{r=1}$. $\nu_{r=1}$ is the same as in §2 and Theorem 3.1. By inspection, we get

$$\nu_{\rho-\rho_c}^{bec} = \int_0^\infty \nu_{t(\rho-\rho_c)}^{BEC} e^{-t}\, dt,$$

that is, the BEC states for grand canonical ensemble is a mixture of BEC states of §2 for various densities.

4 Mean Field Model of Boson Gas

Mean field model of boson gas is a simplified model of quantum statistical mechanics of boson gas, where constituent particles interact each other by homogeneous repulsive force. The grand partition function is given by

$$\Xi_L(\lambda, \mu) = \sum_{n=0}^{\infty} e^{n\beta\mu - n^2\beta\lambda/2|\Lambda_L|} \mathrm{Tr}_{\otimes_{sym}^n \mathcal{H}_L}[\otimes^n G_L],$$

where μ is the chemical potential and λ represents strength of the mean field interaction Using the kernel $G_L(x, y)$, we get

$$\Xi_L(\lambda, \mu) = \sum_{n=0}^{\infty} \frac{1}{n!} e^{n\beta\mu - n^2\beta\lambda/2|\Lambda_L|} \int_{\Lambda_L^n} \mathrm{per}\, \{G_L(x_i, x_j)\}_{1\leqslant i,j\leqslant n} dx_1 \cdots dx_n,$$

We can introduce the corresponding RPF $\nu_{L,\lambda,\mu}$. This model is not trivial like GCE to handle, but the stability of the system holds for $\mu \in \mathbb{R}$ and $\lambda > 0$. The thermodynamic limit $L \to \infty$ is taken for fixed λ, μ. We need not tune the chemical potential μ artificially to get the BEC states as in the previous section. However, let us content ourselves with giving the result.

Theorem 4.1 *(1) For $\mu/\lambda < \rho_c$,*

$$\nu_{L,\lambda,\mu} \to \nu_r \qquad \textit{weakly},$$

where $r \in (0, 1)$ satisfies

$$\frac{\mu}{\lambda} = \frac{\log r}{\beta\lambda} + \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} \frac{re^{-\beta|p|^2}}{1 - re^{-\beta|p|^2}}.$$

The density ρ of the RPF ν_r satisfies

$$\mu/\lambda < \rho < \rho_c$$

(2) For $\mu/\lambda > \rho_c$,

$$\nu_{L,\lambda,\mu} \to \nu_\rho^{(B)} \qquad \textit{weakly},$$

where $\nu_\rho^{(B)}$ is the same as in the canonical ensemble case

$$\nu_\rho^{(B)} = \nu_{\rho-\rho_c}^{BEC} * \nu_{r=1}$$

with the density

$$\rho = \mu/\lambda > \rho_c.$$

Remark: RPF ν_r and the critical density ρ_c are the same as in the CE case.

5 Limit properties of ν_ρ

In this final section, we summarize the limit theorems of RPF ν_ρ for $\rho > \rho_c$, i.e., BEC states, and compare them to those for $\rho < \rho_c$.

Theorem 5.1 (Law of Large Numbers) *For $\rho > \rho_c$, the limit*

$$\frac{1}{\kappa^d}\langle f(\cdot/\kappa), \xi\rangle \longrightarrow \rho \int_{\mathbb{R}^d} f(x)\,dx$$

holds in $L^2(Q(\mathbb{R}^d), \nu_\rho)$ as $\kappa \to \infty$.[TZ2]

Theorem 5.2 (Central Limit Theorem) *For $\rho > \rho_c$, the distribution of the random variable*

$$Z_\kappa := \frac{\langle f(\cdot/\kappa), \xi\rangle - \kappa^d \rho \int_{\mathbb{R}^d} f(x)\,dx}{\sqrt{2(\rho - \rho_c)}\,\|(-\beta\Delta)^{-1/2} f\|_2\ \kappa^{(d+2)/2}}$$

converges to the standard normal distribution:

$$\lim_{\kappa\to\infty} \int_{Q(\mathbb{R}^d)} e^{itZ_\kappa} \nu_\rho(d\xi) = e^{-t^2/2} \quad .$$

Theorem 5.3 (Large Deviation Principle) *There exists a certain (good) rate convex function $I : \mathbb{R} \mapsto [0, +\infty]$, such that the large deviation principle*

$$\limsup_{\kappa\to\infty} \frac{1}{\kappa^{d-2}} \log \nu_\rho\Big(\frac{1}{\kappa^d}\big\langle f\big(\,\cdot\,/\kappa\big), \xi\big\rangle \in F\Big) \leqslant -\inf_{s\in F} I(s) \qquad \text{for any closed } F \subset \mathbb{R}\ ,$$

$$\liminf_{\kappa\to\infty} \frac{1}{\kappa^{d-2}} \log \nu_\rho\Big(\frac{1}{\kappa^d}\big\langle f\big(\,\cdot\,/\kappa\big), \xi\big\rangle \in G\Big) \geqslant -\inf_{s\in G} I(s) \qquad \text{for any open } G \subset \mathbb{R}$$

hold for $\rho > \rho_c$.

To compare our results for the case: $\rho > \rho_c$ (BEC) to the corresponding results for the case $\rho < \rho_c$ (normal phase without condensation), we would like to refer [LLS, GLM, ShTa] for the latter.

Let us put $K_z := zG^\beta(1 - zG^\beta)^{-1}$ with $z \in (0, 1)$, which satisfies $\rho = K_z(x, x)$ and $\nu_\rho = \mu^{(det)}_{K_z}$. Then we have :

Theorem 5.4 (The law of large number) *For $\rho < \rho_c$. the limit*

$$\frac{1}{\kappa^d}\langle f(\cdot/\kappa), \xi\rangle \longrightarrow \rho \int_{\mathbb{R}^d} f(x)\,dx$$

holds in $L^2(Q(\mathbb{R}^d), \nu_\rho)$ as $\kappa \to \infty$.

Theorem 5.5 (The central limit theorem) *For $\rho < \rho_c$. the distribution of the random variable*

$$Z_\kappa = \frac{\langle f(\cdot/\kappa), \xi\rangle - \kappa^d \rho \int_{\mathbb{R}^d} f(x)\,dx}{\sqrt{K_z(x,x) + K_z^2(x,x)}\|f\|_2 \kappa^{d/2}},$$

converges to the standard normal distribution:

$$\lim_{\kappa\to\infty} \int_{Q(\mathbb{R}^d)} e^{itZ_\kappa} \nu_\rho(d\xi) = e^{-t^2/2} \ .$$

Theorem 5.6 (Large deviation principle) *There exists a certain good rate convex function $I' : \mathbb{R} \mapsto [0, +\infty]$ such that*

$$\limsup_{\kappa\to\infty} \frac{1}{\kappa^d} \log \nu_\rho\Big(\frac{1}{\kappa^d}\big\langle f\big(\cdot/\kappa\big), \xi\big\rangle \in F\Big) \leqslant -\inf_{s\in F} I'(s) \qquad \text{for any closed } F \subset \mathbb{R}$$

and

$$\liminf_{\kappa\to\infty} \frac{1}{\kappa^d} \log \nu_\rho\Big(\frac{1}{\kappa^d}\big\langle f\big(\cdot/\kappa\big), \xi\big\rangle \in G\Big) \geqslant -\inf_{s\in G} I'(s) \qquad \text{for any open } G \subset \mathbb{R}$$

hold for $\rho < \rho_c$.

The behavior of the random variable

$$D_\kappa = \frac{1}{\kappa^d}\langle f(\cdot/\kappa), \xi\rangle,$$

under ν_ρ for large ρ and small ρ are different as follows.
(i) The random variable D_κ converges for $\kappa \to \infty$ to its expectation value $m = \rho \int_{\mathbb{R}^d} f(x)\,dx$ in mean for both cases.
(ii) For large ρ, the law of $\kappa^{(d-2)/2}(D_\kappa - m)$ converges to normal distribution as $\kappa \to \infty$. While, for small ρ, $\kappa^{d/2}(D_\kappa - m)$ does.
(iii) For large ρ, the law of D_κ obeys a large deviation principle with parameter κ^{d-2}, while for small ρ, it does with κ^d.

The comparison shows that there are differences in deviation of density fluctuation between the BEC and the non-BEC states of ideal boson gases, which reminds the large deviation properties for two-phase classical systems, for example lattice spin models, see e.g. [P].

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Semi-classical limit of the lowest eigenvalue of $P(\quad)_2$ Hamiltonian on a finite interval

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1 Introduction

Spatially cut-off $P(\quad)_2$-Hamiltonian is used to construct a non-trivial scalar quantum eld [23, 12]. The Hamiltonian contains a small physical parameter which is called the Planck constant $\hbar$. The classical eld equation which is associated with the $P(\quad)_2$ quantum eld is a non-linear Klein-Gordon equation. It is natural to guess that the asymptotics of spectrum of the spatially cut-off $P(\quad)_2$-Hamiltonian is determined by the corresponding classical system. In this paper, we discuss the semi-classical limit of the lowest eigenvalue of the $P(\quad)_2$-Hamiltonian in the case where the space is $[\quad l/2, l/2]$, where $l > 0$. The contents of this report is based on [2].

2 De nitions and Basic properties

Let $I = [\quad l/2, l/2]$. Let $\Delta = \frac{d^2}{dx^2}$ be the Laplace-Bertlami operator on $L^2(I, dx)$ with periodic boundary condition, where dx denotes the Lebesgue measure. Note that all functions and function spaces in this paper are real-valued ones. Set $e_0(x) = \sqrt{\frac{1}{l}}$ and $e_k(x) = \sqrt{\frac{2}{l}}\cos\left(\frac{2\ k}{l}x\right)$, $e\ \ _k(x) = \sqrt{\frac{2}{l}}\sin\left(\frac{2\ k}{l}x\right)$ for positive integer k. $\{e_n \mid n = 0, \pm 1, \pm 2, \ldots\}$ are eigenfunctions of Δ and constitutes a complete orthonormal system of $L^2(I, dx)$. Since the boundary condition is periodic one, we may consider our function spaces are de ned on a circle with the length l. Let us x a positive number $m > 0$. Let μ be the Gaussian measure on $\mathcal{D}'(I)$ such that for any $h \in C^\infty(S^1(l))$

$$\int_{\mathcal{D}(I)'} \langle h, w\rangle^2 d\mu(w) = \left((m^2 \quad \Delta)^{\ \ 1/2} h, h\right)_{L^2(I,dx)}$$

where $\mathcal{D}'(I)$ denotes the space of Schwartz distributions. Let $\mathfrak{F}C^\infty$ be the set of smooth cylindrical functions such that $f(w) = F(\varphi_1(w), \ldots, \varphi_k(w))$, where $F \in C_b^\infty(\mathbb{R}^k)$,$\varphi_i(w) = \langle h_i, w\rangle$, $h_i \in C^\infty(S^1(l))$. For $f(w) = F(\varphi_1(w), \ldots, \varphi_k(w))$, de ne

$$(\quad f)(w) = \sum_{i=1}^{k} (\partial_i F)(\varphi_1(w), \ldots, \varphi_k(w)) h_i \in L^2(I, dx) \tag{2.1}$$

and

$$\mathcal{E}(f,f)=\int_{\mathcal{D}'(I)}\| \ f(w)\|^2_{L^2(I,dx)}d\mu(w).$$

Let $L(\ 0)$ be the generator of (the closure of) $\mathcal{E}$ which is called the free Hamiltonian.

We introduce potential functions. Let g be a bounded measurable function on I and let

$$\begin{aligned} V_\lambda(w) &= \lambda : V \ \frac{w}{\sqrt{\lambda}} \ :, \\ : V \ \frac{w}{\sqrt{\lambda}} \ : &= \int_I : P \ \frac{w(x)}{\sqrt{\lambda}} \ : g(x)dx, \end{aligned}$$

where $\lambda > 0$ and $P(u) = \sum_{k=0}^{2N} a_k u^k$ is a polynomial. $: P(w(x)) :$ denotes the Wick polynomial with respect to μ. Assume that $a_{2N} > 0$ and $g \in C^\infty$ and $g(x+l) = g(x)$, $g(x) > 0 \ \forall x$. The operator $(\ L + V_\lambda, \mathfrak{F}C^\infty)$ is essentially self-adjoint in $L^2(\mu)$ ([26]) and we denote the self-adjoint extension by $\ L+V_\lambda$. [22, 23, 26, 12] are basic references to this operator. $\ L+V_\lambda$ is called a $P(\ )_2$-Hamiltonian. It is a representation of the quantization of the Hamiltonian whose classical eld equation is the non-linear Klein-Gordon equation with space-time dimension 2:

$$\frac{\partial^2}{\partial t^2}w(t,x) \quad \frac{1}{4}\frac{\partial^2}{\partial x^2}w(t,x)+\frac{m^2}{4}w(t,x)+\frac{1}{2}P'\left(w(t,x)\right)g(x)=0 \qquad (t,x)\in\mathbb{R} \quad I. \tag{2.2}$$

We determine the limit of the lowest eigenvalue $E_0(\lambda)$ of $\ L+V_\lambda$ when $\lambda\to\infty$. In this study, we use the Schilder type large deviation results for $V_\lambda(w)$. It is convenient to study this problem in the setting of abstract Wiener space.

To this end, let $\tilde{A}=(m^2 \quad \Delta)^{1/4}$. We de ne Sobolev spaces:

$$H^s \ = \ \left\{h\in \mathrm{D}(\tilde{A}^{2s}) \mid \|h\|_{H^s} := \|\tilde{A}^{2s}h\|_{L^2(I,dx)}\right\}. \tag{2.3}$$

Let $H=H^{1/2}$. For the separable Hilbert space H, the abstract Wiener space (W,H,μ) is uniquely de ned [13]. μ is the Gaussian measure on W which is the same measure we already de ned on $\mathcal{D}(I)'$. Actually, we can take $W = H^{\ s_0} \quad \mathcal{D}(I)'$, where s_0 is any positive number. That is, the norm is given by $\|w\|_W^2 := \|(m^2 \quad \Delta)^{\ s_0/2}w\|^2_{L^2}$. We de ne a self-adjoint operator A on H by

$$\begin{aligned} Af &= (m^2 \quad \Delta)^{1/4}f \qquad (2.4)\\ \mathrm{D}(A) &= \mathrm{D}((m^2 \quad \Delta)^{1/2}) \quad H. \qquad (2.5) \end{aligned}$$

Let $S=A^{\ 2\gamma}$, where $\gamma=1+2s_0$. Then S is a trace class operator on H and $\|h\|_W^2 = \|\sqrt{S}h\|_H^2$ holds. The H-derivative D and are related as follows:

$$\|ADf(w)\|_H^2 \ = \ \| \ f(w)\|^2_{H^0}. \tag{2.6}$$

In our study, the following function U is important.

De nition 2.1. Let $U(h) = \frac{1}{4}\|Ah\|_H^2 + V(h)$ for $h \in \mathrm{D}(A)$ and $U(h) = +\infty$ for $h \notin \mathrm{D}(A)$. Here $V(h) = \int_I P(h(x))g(x)dx$ and $h \in H$.

We can rewrite U.

Lemma 2.2. *$h \in \mathrm{D}(A)$ is equivalent to $h \in H^1$ and it holds that for any $h \in H^1$*

$$U(h) \quad = \quad \frac{1}{4}\int_I \left(h'(x)^2 + m^2 h(x)^2\right) dx + \int_I P(h(x))g(x)dx.$$

The following results are very well-known.

Lemma 2.3. *For any $p > 1$, $H^{1/2} \quad L^p(I, dx)$ and the embedding is compact. Also $H^{1/2} \ / \ L^\infty(I, dx)$.*

We summarize basic properties of the Hamiltonian and the semi-group.

Theorem 2.4. *For any $f \in \mathfrak{F}C^\infty$, it holds that*

$$\int_W f^2 \log \quad f^2/\|f\|_{L^2(\mu)}^2 \quad d\mu \qquad 2\int_W \|Df(w)\|_H^2 d\mu. \tag{2.7}$$

Theorem 2.5 (GNS(=Glimm-Nelson-Segal) bound)**.** *Let $V \in L^1(W, d\mu)$ be a bounded measurable function. Then for any $f \in \mathfrak{F}C^\infty$, we have*

$$\int_W \|Df(w)\|_H^2 d\mu(w) + \int_W V(w)f(w)^2 d\mu(w) \qquad \frac{1}{2}\log \quad \int_W e^{\ \ 2V} d\mu \quad \|f\|_{L^2(\mu)}^2. \tag{2.8}$$

Actually the logarithmic Sobolev inequality (2.7) is equivalent to that (2.8) holds for all bounded measurable V. See [14]. As to $\quad L + V_\lambda$, we have the following (see [26, 4]).

Theorem 2.6. (1) *($\quad L + V_\lambda, \mathfrak{F}C_A^\infty(W))$ is essentially self-adjoint.*
(2) *For any $\quad > 0$,*

$$\int_W \exp\left(\quad V_\lambda(w)\right) d\mu(w) < \infty.$$

(3) *$E_0(\lambda) = \inf \quad (\quad L + V_\lambda) > \quad \infty$ and $E_0(\lambda)$ is an eigenvalue with multiplicity 1 and the eigenfunction is almost surely positive or negative.*
(4) *L^2-semigroup $T_t = e^{t(L \quad V_\lambda)}$ is a trace class operator. In particular $\quad L + V_\lambda$ has discrete spectrum only.*

Remark 2.7. (1) Arai [4] studied the semi-classical limit of the partition function:

$$\lim_{\lambda\to\infty} \frac{\mathrm{tr} e^{t(L \quad V_\lambda)/\lambda}}{\mathrm{tr} e^{tL/\lambda}}.$$

(2) Spatially cut-off $P(\ )_2$-Hamiltonians are de ned, by replacing I by $\mathbb{R}$ and g by a smooth non-negative function with a compact support. $(1), (2), (3)$ above hold for spatially cut-off $P(\ )_2$-Hamiltonians, too. Also it is known that the Hamiltonians have discrete spectrum only in $[E_0(\lambda), E_0(\lambda) + m)$ by Simon and Hoegh-Krohn. However (4) does not hold any more for them and the set $[E_0(\lambda) + m, \infty)$ may be included in the set of the continuous spectrum.

3 A formal argument

The equation (2.2) can be viewed as a Newton's equation of motion on $H^0 = L^2(I, dx)$ and the function U in De nition 2.1 can be viewed as the potential function for the equation (2.2) in H^0. First, let us consider the Newton's equation of motion on $\mathbb{R}^d$:

$$x_t = \quad (\nabla U)(x_t), \quad x_t \in \mathbb{R}^d.$$

Then the classical Hamiltonian is $E = \frac{1}{2}|p|^2 + U(x)$ and the corresponding quantum Hamiltonian is the Schrodinger operator:

$$H = \quad \frac{\hbar^2}{2}\Delta + U \qquad \text{in } L^2(\mathbb{R}^d, dx).$$

Using the potential function U in De nition 2.1 we see that (2.2) reads

$$\frac{d^2}{dt^2} w_t = \quad \frac{1}{2}(\nabla U)(w_t), \qquad w_t \in H^0.$$

Hence the quantized formal Hamiltonian of (2.2) is

$$\frac{\hbar^2}{2}\Delta_{H^0} + \frac{1}{2}U \quad \text{in } L^2(H^0, dw), \tag{3.1}$$

where dw is the "Lebesgue measure" on H^0 and $\Delta_{H^0} = \mathrm{tr}\ {}^2 =$. Here is the adjoint operator of with respect to the "Lebesgue measure". $P(\ )_2$-Hamiltonian $\lambda\ {}^{1/2}(\ \ L + V_\lambda)$ $(\lambda = \hbar\ {}^2)$ is the rigorous version of this Schrodinger operator. To see this, we consider a formal unitary transformation $f(w) \to \hbar\ {}^{\dim H^0/2} f(\hbar\ {}^1 w)$. By this, (3.1) is unitarily equivalent to $\hbar H_\hbar$, where

$$\begin{aligned} H_\hbar &= \frac{1}{2}\left(\Delta_{H^0} \quad \hbar\ {}^2 U(\hbar w)\right) \\ &= \frac{1}{2} \quad \Delta_{H^0} \quad \lambda U \quad \frac{w}{\sqrt{\lambda}} \qquad\qquad \text{in } L^2(H^0, dw). \end{aligned} \tag{3.2}$$

Now we consider a unitary transformation of $\ L + V_\lambda$. Formally, we have

$$d\mu(w) = \quad (w)^2 dw, \tag{3.3}$$

where

$$(w) = \det\left(\frac{\tilde{A}}{\sqrt{2}}\right)^{1/2} \exp \quad \frac{1}{4}\|\tilde{A}w\|_{H^0}^2 \quad . \tag{3.4}$$

Also by (2.6)

$$\mathcal{E}(f, f) \quad = \int_{H^0} \| \ \ f(w)\|_{H^0}^2 \quad (w)^2 dw. \tag{3.5}$$

By the formal unitary transformation $f \to f \quad ^{1}$ from $L^2(H^0, dw)$ to $L^2(W, d\mu)$, $\quad L + V_\lambda$ is unitarily equivalent to

$$\Delta_{H^0} + \frac{1}{4}\|\tilde{A}^2 w\|^2_{H^0} + \lambda : V \quad \frac{w}{\sqrt{\lambda}} \quad : \quad \mathrm{tr}\tilde{A}^2. \tag{3.6}$$

Note that

$$\begin{aligned} &\frac{1}{4}\|\tilde{A}^2 w\|^2_{H^0} + \lambda : V \quad \frac{w}{\sqrt{\lambda}} \quad : \\ &= \lambda \left\{ \frac{1}{4} \int_I \left(\quad \frac{w'(x)}{\sqrt{\lambda}} \quad ^2 + m^2 \quad \frac{w(x)}{\sqrt{\lambda}} \quad ^2 \right) dx + \int_I : P \quad \frac{w(x)}{\sqrt{\lambda}} \quad : g(x)dx \right\} \end{aligned}$$

In this way, we can see that the $P(\;)_2$-Hamiltonian $\quad L + V_\lambda$ is an in nite dimensional Schrodinger operator. In quantum mechanics, there are many researches on the semi-classical limit of Schrodinger operators. Now we recall the results concerning the asymptotic behavior of the lowest eigenvalue of the Schrodinger operator. Our main result is an in nite dimensional analogue of this.

4 Results in nite dimensions

Let

$$H_{\lambda,U} := \quad \Delta + \lambda U \quad \frac{x}{\sqrt{\lambda}} \qquad \text{on} \quad L^2(\mathbb{R}^d, dx) \tag{4.1}$$

Note that $\lambda = \frac{1}{\hbar^2}$.

The following result can be found in [24].

Theorem 4.1. *Assume that*
(H1) $\min U(x) = 0$ *and* $\{x \in \mathbb{R}^d \mid U(x) = 0\} = \{h_1, \ldots, h_n\}$.
(H2)

$$U_i = \frac{1}{2} \quad \frac{\partial^2 U}{\partial x_k \partial x_l}(h_i) \quad > 0 \quad \textit{for all} \quad i.$$

(H3) $\liminf_{|x|\to\infty} U(x) > 0$.

Set $E_0(\lambda) = \inf \;(\quad H_{\lambda,U})$. *Then*

$$\lim_{\lambda\to\infty} E_0(\lambda) = \min_{1\le i\le n} \mathrm{tr}\sqrt{U_i}.$$

Also note that

$$\mathrm{tr}\sqrt{U_i} = \inf \;(\quad \Delta + (U_i x, x)).$$

5 Main result

Assumption 5.1. **(A1)** *$U(h)$ ($h \in H^1$) is a non-negative function and the zero point set of U is the finite set: $N = \{h_1, \ldots, h_n\}$.*
(A2) *Suppose* **(A1)**. *The Hessian $\frac{1}{2}D^2U(h_i) \in L(H^1, H^1)$ is a strictly positive operator for all $1 \le i \le n$.*

Lemma 5.2. *Assume that U is a nonnegative function and there exists $h \in H^1$ such that $U(h) = 0$. Then the following hold.*
(1) *h is a periodic C^∞ function.*
(2) *The following* (i), (ii) *are equivalent.*

(i) *$D^2U(h)$ is strictly positive.*

(ii) *Let $v(x) = \frac{1}{2}P''(h(x))g(x)$. Then* $\inf(m^2 - \Delta + 4v) > 0$.

Example 5.3. (1) *Assume that $g(x) \equiv 1$ and set $Q(x) = \frac{m^2}{4}x^2 + P(x)$.*
Suppose that $Q(x) \ge 0$ for all x and let $\{Q = 0\} = \{q_1, \ldots, q_n\}$.
Then the constant functions $\{q_1, \ldots, q_n\}$ are minimizers of U and $U(q_i) = 0$ for all i.
We have $m^2 - \Delta + 4v_i(x) = -\Delta + 2Q''(q_i)$.
Thus, (A1) *and* (A2) *is equivalent to that $Q''(q_i) > 0$ for all zero point q_i ($1 \le i \le n$).*
(2) *Let $P_a(u) = a(u^2 - 1)^2$ and $Q_a(u) = \frac{m^2}{4}u^2 + P_a(u)$.*
Let $a > \frac{m^2}{8}$, $x_a = \sqrt{1 - \frac{m^2}{8a}}$, $g \equiv 1$.
Then $P(u) = P_a(u) - Q_a(x_a)$ satisfies the assumption of the main theorem and $N = \{\pm x_a\}$. When g is a non-constant function, then we can prove that U has two minimizers $\{\pm h_a\}$ and satisfies the assumptions in main theorem for sufficiently large a. In this case, h_i is not a constant function.

Lemma 5.4. *Let v be a C^2-function on $\mathbb{R}$ with $v(x) = v(x + l)$ for all x. Assume* $\inf(m^2 - \Delta + 4v) > 0$ *and set $\tilde{A}_v = (m^2 - \Delta + 4v)^{1/4}$. Let $Q_v(w) = \int_I : w(x)^2 : g(x)dx$.*
(1) *It holds that* $\inf(-L + Q_v) > -\infty$ *and* $\inf(-L + Q_v)$ *is an eigenvalue with multiplicity* 1.
(2) *Let M_v be the multiplication operator by v in $L^2(I, dx)$. Then*

$$\begin{aligned}\inf(-L + Q_v) &= \frac{1}{2}\mathrm{tr}\left(\tilde{A}_v^2 - \tilde{A}^2 - 2\tilde{A}^{-1}M_v\tilde{A}^{-1}\right) \\ &= -\frac{1}{4}\left\|\left(\tilde{A}_v^2 - \tilde{A}^2\right)\tilde{A}^{-1}\right\|^2_{L_{(2)}(H^0)}.\end{aligned}$$

tr *denotes the trace in $H^0 = L^2(I, dx)$.*
(3) *Let Ω_v be the ground state function of $-L + Q_v$. $\Omega_v(w)^2 d\mu$ is the Gaussian measure whose covariance operator is $(m^2 - \Delta + 4v)^{-1/2}$ on $L^2(I, dx)$.*

Our main theorem is as follows.

Theorem 5.5. *Assume that* (A1) *and* (A2) *hold. Let* $E_0(\lambda) = \inf\ (\ \ L + V_\lambda)$*. Then*

$$\lim_{\lambda\to\infty} E_0(\lambda) = \min_{1\le i\le n} E_i, \tag{5.1}$$

where E_i *is the lowest eigenvalue of* $\ \ L + Q_{v_i}(w)$*, where* $Q_{v_i}(w) = \int_I : w(x)^2 : v_i(x)dx$ *and* $v_i(x) = \frac{1}{2}P''(h_i(x))g(x)$*.*

Remark 5.6. (1) The expression

$$\inf\ (\ \ L + Q_v) = \ \ \frac{1}{4}\|\ \ \tilde{A}_v^2\ \ \ \tilde{A}^2\ \ \tilde{A}\ \ ^1\|^2_{L_{(2)}(H^0)}.$$

can be found in the case of spatially cut-off $P(\ \ )_2$ Hamiltonians in *e.g.* [21, 10].

(2) Since $\int_W Q_v(w)d\mu(w) = 0$, it is trivial that $\inf\ (\ \ L + Q_v)\ \ \ 0$.

(3). Let $E_v = \inf\ (\ \ L + Q_v)$. Then $\ \ L + Q_v\ \ \ E_v$ is unitarily equivalent to the second quantization operator $d\ \ (\sqrt{m^2\ \ \ \Delta + 4v})$.

(4) We use the following result to prove the trace class property of the operator in the lemma:

Let $K(x,y)$ $(x, y \in I)$ be a Hilbert-Schmidt kernel on $L^2(I, dx)$. Assume that for almost all y, $x \to K(x,y)$ is absolutely continuous function and $\partial_x K(\ ,\ ) \in L^2(I\ \ \ I, dxdy)$. De ne $Tf(x) = \int_I K(x,y)f(y)dy$. Then T is a trace class operator on $L^2(I, dx)$.

6 Proof

The proof of the inequality LHS RHS in (5.1) is easy.

Let $\Omega_i(w)$ be the ground state of $\ \ L + Q_{v_i}$ and set

$$\tilde{\Omega}_i(w) \quad = \quad \Omega_i(w\ \ \ \sqrt{\lambda}h_i)\exp\left(\frac{\sqrt{\lambda}}{2}(h_i, w)_H\ \ \ \ \frac{\lambda}{4}\|h_i\|_H^2\right).$$

Then

$$(\ \ L + V_\lambda)\tilde{\Omega}_i, \tilde{\Omega}_i\ \ = E_i + O(\frac{1}{\lambda}),$$

which proves the estimate.

Next we consider the converse inequality. Let $\ \ \in C^\infty(\mathbb{R})$ be a non-negative function such that $\{\ \ = 1\} = [\ \ 1, 1]$, $\{\ \ = 0\} = (\ \ \infty,\ \ 2] \cup [2, \infty)$ and $0\ \ \ \ \ \ \ 1$. Let $\varepsilon > 0$ and set $\ \ _i(w) = \ \ \frac{\|(w\ \ \sqrt{\lambda}h_i)\|_W^2}{\varepsilon^2\lambda}$, $(1\ \ \ i\ \ \ n)$. $\ \ _\infty(w) = \sqrt{1\ \ \ \sum_{i=1}^n\ \ _i(w)^2}$. Let $f\ \ (w) = f(w)\ \ \ \ (w)$, where $* = i, \infty$ $(1\ \ \ i\ \ \ n)$. Then

$$\begin{aligned}((\ \ L + V_\lambda)f, f) \quad &= \sum_{\{\ \ =1,\ldots,n,\infty\}} ((\ \ L + V_\lambda)f\ \ , f\ \ ) \\ &\ \ \ \sum_{\{\ \ =1,\ldots,n,\infty\}} \int_W \|AD\ \ \ \ \|_H^2 f(w)^2 d\mu(w).\end{aligned} \tag{6.1}$$

Simple calculation shows that there exists a positive constant C such that $\|AD \quad (w)\|_H^2$ $\frac{C}{\varepsilon^2\lambda}$ μ-$a.s.$ w for all $*$. It is su cient to prove that there exists $C_1 > 0$ and $C_2 \in \mathbb{R}$,

$$\liminf_{\lambda\to\infty} ((\quad L + V_\lambda)f_i, f_i) \qquad E_i\|f_i\|_{L^2(\mu)}^2 \qquad \text{for all } i, \tag{6.2}$$

$$\liminf_{\lambda\to\infty} ((\quad L + V_\lambda)f_\infty, f_\infty) \qquad (C_1\lambda + C_2)\|f_\infty\|_{L^2(\mu)}^2, \tag{6.3}$$

since $\sum_{\ =1,\ldots,n,\infty} \|f\ \|_{L^2(\mu)}^2 = 1$.

(6.2) is the estimate near h_i. In nite dimensional cases, this is easy to prove by using the Taylor expansion of the potential function at h_i because the remainder term of the expansion is of small order. However this does not hold in the present case because the remainder term $R_{\lambda,i}$ in (6.12) is not a continuous function of w. Moreover, in nite dimensional case, the estimate (6.3) is proved by using the assumption (H3) in Theorem 4.1. However we cannot use such an estimate in the present case. Instead, we use GNS bound and large deviation estimates.

6.1 Sketch of the proof of (6.3)

We use

(i) A lower bound estimate for U outside $N = \{U = 0\}$,

(ii) Large deviation estimates for Wiener chaos ([5, 18, 19, 2]),

(iii) GNS bound.

We have the following lower bound estimate. When all $h_i \in N$ are constant functions, we do not need this lower bound. In fact, the rough lower bound $\|Ah\|_H^2 \quad m\|h\|_H^2$ is su cient for the following argument. However, minimizers of U are not constant functions in the case of spatially cut-off $P(\ )_2$-Hamiltonian and the following type lower bound estimate is necessary.

Lemma 6.1. *Let P_N be the projection onto the linear span of $\{e_k\}_{k=\ \ N}^N$ on H. We de ne a trace class operator $T_N = P_N \quad \frac{A}{\sqrt{m}} \quad I_H$.*
(1) *For any $h \in \mathrm{D}(A)$ and $N \in \mathbb{N}$,*

$$\|Ah\|_H^2 \quad m\|(I_H + T_N)h\|_H^2.$$

(2) *For any $\varepsilon > 0$, there exists $\ (\varepsilon) > 0$ and $N_0 \in \mathbb{N}$ such that for all $N \quad N_0$,*

$$\inf\left\{\frac{m}{4}\|(I_H + T_N)h\|_H^2 + V(h) \;\middle|\; h \in (\cup_{i=1}^n B_\varepsilon(h_i))^c \cap H\right\} \qquad (\varepsilon). \tag{6.4}$$

Using GNS estimate, we can prove

Lemma 6.2. *Let $\tilde{V}$ be a bounded measurable function and T be a trace class self-adjoint operator on H with* $\inf \quad (I_H + T) > 0$*. Then*

$$
\begin{aligned}
m \int_W \|(I_H + T)Df(w)\|_H^2 d\mu + \int_W \tilde{V}(w) f(w)^2 d\mu \\
\frac{m}{2} \log I + C(m,T)\|f\|_{L^2(\mu)}^2,
\end{aligned}
\tag{6.5}
$$

where $I = \int_W \exp\left(\quad \frac{2}{m}\tilde{V}(w) \quad (Tw, w)_H \quad \frac{1}{2}\|Tw\|_H^2 \right) d\mu(w)$ *and* $C(m,T)$ *is a constant.*

The proof of the following theorem can be found in [2].

Theorem 6.3 (Large deviation for Wiener chaos)**.** *Let be a non-negative bounded continuous function on W. For $w \in W$, set*

$$
F_\lambda(w) \quad = \qquad \frac{w}{\sqrt{\lambda}} \quad : V \quad \frac{w}{\sqrt{\lambda}} \quad :,
$$

and $F(h) = \quad (h)V(h)$ *for* $h \in H$*. The image measure of* μ *by the measurable map* F_λ *satis es the large deviation principle with the good rate function:*

$$
I_F(x) = \begin{cases} \inf \left\{ \frac{1}{2}\|h\|_H^2 \;\middle|\; \exists h \in H \text{ such that } F(h) = x \right\}, \\ +\infty \qquad \nexists h \in H \text{ such that } F(h) = x. \end{cases}
$$

Proof of (6.3)*.* Let be the function which was de ned before.
Let $\quad_i(w) = \quad \frac{3\|w \quad \sqrt{\lambda}h_i\|_W^2}{\varepsilon^2\lambda}$ and $\quad_\infty(w) = \sqrt{1 \quad \sum_{i=1}^n \quad_i(w)^2}$. $\quad_\infty$ satis es that $\quad_\infty(w) = 1$ for w with $\quad_\infty(w) \neq 0$ and

$$
\{w \in W \mid \quad_\infty(w) \neq 0\} \qquad \cup_{i=1}^n B_{\varepsilon\sqrt{\frac{\lambda}{3}}} \quad \sqrt{\lambda}h_i \quad {}^c .
$$

Let $\varepsilon' < \frac{\varepsilon}{\sqrt{3}}$. For this ε', we choose a natural number N_0 as in Lemma 6.1 and de ne a trace class operator $T = \quad \frac{A}{\sqrt{m}} \quad I_H \quad P_{N_0}$. We have

$$
\begin{aligned}
&((\quad L + V_\lambda)f_\infty, f_\infty) \\
&\quad m \int_W \|(I_H + T)Df_\infty(w)\|_H^2 d\mu(w) + \int_W \quad V_\lambda(w) \quad \frac{1}{2}\lambda \quad (\varepsilon') \qquad _\infty(w) f_\infty(w)^2 d\mu(w) \\
&\quad + \int_W \frac{1}{2}\lambda \quad (\varepsilon') \quad _\infty(w) f_\infty(w)^2 d\mu(w).
\end{aligned}
\tag{6.6}
$$

Note that

$$
\int_W \frac{1}{2}\lambda \quad (\varepsilon') \quad _\infty(w) f_\infty(w)^2 d\mu(w) = \frac{1}{2}\lambda \quad (\varepsilon')\|f_\infty\|_{L^2(\mu)}^2.
$$

Let $\tilde{V}_\lambda(w) = \big(V_\lambda(w) \quad \frac{1}{2}\lambda \ \ (\varepsilon')\big) \quad {}_\infty(w)$. Applying Lemma 6.2,

$$\begin{aligned} J_2(\lambda) = m \int_W \|(I_H + T)Df_\infty(w)\|_H^2 d\mu(w) + \int_W \tilde{V}_\lambda(w) f_\infty(w)^2 d\mu(w) \\ \frac{m}{2} \log \int_W \exp \quad \frac{2}{m}\tilde{V}_\lambda(w) \quad (Tw, w)_H \quad \frac{1}{2}\|Tw\|_H^2 \quad d\mu(w) \quad \|f_\infty\|_{L^2(\mu)}^2 \\ +C(m,T)\|f_\infty\|_{L^2(\mu)}^2. \end{aligned} \tag{6.7}$$

Let

$$\hat{\ }_\infty(h) = \sqrt{1 \quad \sum_{i=1}^n \quad (3\varepsilon \ \ ^2\|h \quad h_i\|_W^2)}. \tag{6.8}$$

If $\hat{\ }_\infty(h) \neq 0$, $h \in \quad \cup_{i=1}^n B_{\varepsilon'/\sqrt{3}}(h_i) \quad {}^c \cap H$ holds. Hence for all $h \in H$

$$\begin{aligned} &\frac{1}{2}\|(I+T)h\|_H^2 + \frac{2}{m} \quad V(h) \quad \frac{1}{2} \ (\varepsilon') \quad \hat{\ }_\infty(h) \\ &= \frac{1}{2}\|(I+T)h\|_H^2(1 \quad \hat{\ }_\infty(h)) + \quad \frac{1}{2}\|(I+T)h\|_H^2 + \frac{2}{m} \quad V(h) \quad \frac{1}{2} \ (\varepsilon') \quad \hat{\ }_\infty(h) \\ &\quad \frac{1}{2}\|(I+T)h\|_H^2 + \frac{2}{m} \quad V(h) \quad \frac{1}{2} \ (\varepsilon') \quad \hat{\ }_\infty(h) \quad 0. \end{aligned} \tag{6.9}$$

Using the large deviation estimate, for large λ, we have for any $\varepsilon'' > 0$,

$$J_2(\lambda) \quad (\quad \varepsilon''\lambda + C_m)\|f_\infty\|_{L^2(\mu)}^2$$

which proves (6.3).

6.2 Proof of (6.2)

We use

(i) Taylor expansion of Wick polynomials

(ii) Ground state transformation

(iii) Laplace method for Wick polynomials

It holds that

$$\begin{aligned} V_\lambda \quad w + \sqrt{\lambda}h_i \quad &= \quad \lambda \int_I P(h_i(x))dx + \sqrt{\lambda}\int_I P'(h_i(x))w(x)g(x)dx + \int_I : w(x)^2 : v_i(x)dx \\ &+ \sum_{k=3}^{2N} \lambda^{1 \quad \frac{k}{2}} \int_I : w(x)^k : \frac{P^{(k)}(h_i(x))}{k!} g(x)dx. \end{aligned} \tag{6.10}$$

Set $f_i(w) = f_i(w + \sqrt{\lambda} h_i) \exp \quad \frac{\sqrt{\lambda}}{2}(h_i, w)_H \quad \frac{\lambda}{4}\|h_i\|_H^2 \quad .$

By using the assumption that $U(h_i) = 0$ and $DU(h_i) = 0$, we have

$$
\begin{aligned}
((\quad L + V_\lambda) f_i, f_i) &= \int_W \|ADf_i(w)\|^2 d\mu + \int_W Q_{v_i}(w) f_i(w)^2 d\mu \\
&\quad + \int_W R_{\lambda,i}(w) f_i(w)^2 d\mu,
\end{aligned} \tag{6.11}
$$

where

$$
R_{\lambda,i}(w) = \sum_{k=3}^{2N} \lambda^{1 \quad \frac{k}{2}} \int_I : w(x)^k : g_{k,i}(x) dx \tag{6.12}
$$

and $g_{k,i}(x) = \frac{P^{(k)}(h_i(x))}{k!} g(x)$.

In order to get a lower bound estimate, we use the following lemmas. The lemma below is also proved by using GNS bound.

Lemma 6.4. *Let $\tilde{V}$ be a bounded measurable function on W. Let v be a C^2 function on $\mathbb{R}$ with period l. We assume that $m^2 \quad \Delta + 4v$ is a strictly positive operator on $L^2(I, dx)$. Let $c_v = \inf \quad \left(\sqrt{m^2 \quad \Delta + 4v}\right)$ and $E_v = \inf \quad (\quad L + Q_v)$. Then it holds that for any $f \in \mathfrak{F}C^\infty$,*

$$
\begin{aligned}
&(\quad L + Q_v + \tilde{V} \quad E_v) f, f \quad {}_{L^2(\mu)} \\
&\qquad \frac{c_v}{2} \log \int_W \exp \quad \frac{2}{c_v} \tilde{V}(w) \quad \Omega_v(w)^2 d\mu(w) \quad \|f\|_{L^2(\mu)}^2.
\end{aligned}
$$

Lemma 6.5 (Laplace asymptotics)**.** *Let be a smooth non-negative function such that $\{ \quad = 1\} = [\quad 1, 1]$ and $\{ \quad = 0\} = (\quad \infty, \quad 2] \cup [2, \infty)$. Set $\rho_{\lambda,\varepsilon}(w) = \quad \frac{\|w\|_W^2}{\lambda\varepsilon} \quad$. Let $f_k(x)$ $(3 \quad k \quad 2M)$ be continuous functions on I such that $\inf_x f_{2M}(x) > 0$. Let*

$$
\varphi_\lambda(w) = \sum_{k=3}^{2M} \int_I : \quad \frac{w(x)}{\sqrt{\lambda}} \quad {}^k : f_k(x) dx. \tag{6.13}
$$

Then for sufficiently small ε, we have

$$
\lim_{\lambda \to \infty} \int_W e^{\quad \lambda \varphi_\lambda(w) \rho_{\lambda,\varepsilon}(w)} d\mu(w) = 1. \tag{6.14}
$$

Remark 6.6. Let $\varphi(h) = \sum_{k=3}^{2M} h(x)^k f_k(x) dx$ and $G(h) = \frac{1}{2}\|h\|_H^2 + \varphi(h)$ for $h \in H$. Then $h = 0$ is a zero point of G and is a local minimizer of G. Hence, (6.14) is nothing but the Laplace asymptotic formula. The reader may think that the above asymptotics is trivial since

$$
\lim_{\lambda \to \infty} \lambda \varphi_\lambda(w) = 0. \quad \mu \quad a.s.
$$

However, if we do not put the cut-off function $\rho_{\lambda,\varepsilon}$ on the exponent, the limit may be not 1 if G has other zero points.

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Have Fun Exploring Circuit QED with Non-Commutative Oscillators
— From Mathematics to Experimental Physics —

In Honor of K. R. Ito and I. Ojima on their 60th birthdays

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1 Introduction

The quantum electrodynamics (QED) says that the interaction between an atom and the light in nature is governed by the fine structure constant, and thus, the perturbation theory is valid over the standard QED world. On the other hand, *cavity QED* supplies much stronger interaction than the standard QED does [10, 20, 47]. That is, we can obtain the *strong coupling regime* for the atom and the light in cavity QED. To get such a strong interaction, we usually handle a two-level atom and a 1-mode laser which are put into a mirror cavity (i.e., a mirror resonator). This is a physical set-up for the standard cavity QED. Many physicists have foresaw that several solidstate analogues of strong coupling regime of cavity QED in superconducting systems [1, 8, 9, 11, 36, 37, 45, 53, 54]. Namely, we can respectively replace the two-level atom, the laser, and the mirror resonator by an artificial atom, a microwave, and a microwave resonator. Here the artificial atom is a superconducting qubit implemented by using the Josephson junction. We note that the Josephson junction makes very strong anharmonicity (i.e., nonlinearity concerning the quantum number) compared with actual atoms. Thus, we can obtain a stable two-level system in this replaced cavity QED. Moreover, being free from several micro-parameters, we can control the physical set-up with some macro-parameters because we realize the system of cavity QED by a superconducting circuit. This replaced cavity QED is called the *circuit QED*, and it has been demonstrated experimentally by many experimenters [4, 12, 27, 28, 33, 35, 49, 51]. It is remarkable that the interaction between the atom and the light is much stronger in circuit QED than it is in the standard cavity QED [14, 15, 19, 40]. Thus, the interaction is said to be in the *ultra-strong coupling regime*. The circuit QED comes of age and it is among the hottest subjects of experimental physics now [34]. In this paper, we briefly report our attempt in circuit QED. The effective Hamiltonian in cavity QED or in circuit QED is among Hamiltonians of non-commutative oscillators. The ‘non-commutative oscillator’ is a mathematical concept introduced in [43] by A. Parmeggiani and M. Wakayama[1]. We determine our Hamiltonian so that it meets the real, physical situation. We analyze the Hamiltonian for the large coupling constant.

2 Non-Commutative Oscillators

Let $Q(p,q,r)$ be a non-commutative quadratic form given by

$$Q(p,q,r) = A_{11}p^2 + A_{12}pq + A_{13}pr + A_{21}qp + A_{22}q^2 + A_{23}qr + A_{31}rp + A_{32}rq + A_{33}r^2$$

for variables p, q, r into which some operators are inserted, where $A_{k\ell} \in M_2(\mathbb{C})$, the set of all 2×2 matrices with complex coefficients ($k, \ell = 1, 2, 3$). The variable r plays a role of the alternative to insert the null operator 0 or the identity operator Id into it. Then the Hamiltonian of non-commutative oscillators is the canonical quantization of $Q(p,q,r)$. Namely, variables p and q are replaced by the differential operator $-id/dx$ and the multiplication operator $x\times$ respectively in $Q(p,q,r)$. Thus, $Q(-id/dx, x\times, r)$ is a kind of matrix-valued Schrödinger operator as $A_{11} \neq 0$. For example, the differential operator $Q(-id/dx, x\times, 0)$ becomes the differential operator which has been studied by A. Parmeggiani, M. Wakayama, and T. Ichinose [30, 43, 44] with setting $A_{k\ell}$ $(k,\ell = 1,2)$ as

$$A_{11} = A_{22} = \frac{\alpha+\beta}{4}\sigma_0 + \frac{\alpha-\beta}{4}\sigma_3, \qquad \alpha, \beta \in \mathbb{R},$$

[1] In fact, originally, they named their concept ‘non-commutative harmonic oscillator’ in [43] because the use of ‘harmonic’ comes from mathematics. But we omitted ‘harmonic’ from their original naming because harmonicity or anharmonicity plays an important role in cavity QED.

and $A_{12} = A_{21} = \frac{1}{2}\sigma_2$. Here we denote the Pauli matrices by σ_j $(j = 1, 2, 3)$ and σ_0 is the identity matrix.

As described in Sec.1 we handle the Hamiltonian $H = Q(-id/dx, x\times, r) + W$ with an extra term W to study cavity QED and circuit QED. In particular, we analyze H for the large coupling constant. Thus, we have to determine the coefficients $A_{k\ell}$ of $Q(p, q, r)$ so that our H meets the real, physical situation. The coupling constant appears in $A_{k\ell}$ $(k \neq \ell)$. The operator W describes, for instance, the energy of the pumping field or the so-called counter-rotating terms. There are typical two examples of the non-commutative oscillators. One of them is the Jaynes-Cummings model and other the fully coupled model. The Jaynes-Cummings Hamiltonian is the 1-photon version of the Wigner-Weisskopf Hamiltonian, and the fully coupled Hamiltonian (or the Rabi Hamiltonian) the 1-photon version of the spin-boson Hamiltonian.

At the tail end of this section, we note the following. H. G. Reik and M. Doucha conjectured that the solution of the eigenvalue problem for the fully coupled Hamiltonian in Bargmann's Hilbert space would be exactly represented [5, 6, 48]. This conjecture, however, has not been settled yet.

3 Dicke-Preparata Superradiance

G. Preparata had found that a non-perturbative ground state appears as the coupling strength gets large, by using the path-integral method [46]. C. P. Enz also showed that fact analyzing the equation of motion [17]. We call the non-perturbative ground state (Preparata's) *superradiant ground state*. The author gave a mathematical proof for the existence of the superradiant ground state and its some mathematical properties in the Hamiltonian formalism [22, 23]. The Hamiltonian is for the *Wigner-Weisskopf model* (or *Dicke model*) describing a two-level atom coupled to the light.

We employ the so-called N*-level approximation* when we take a great interest in the transition between the N states of the atom. To do that, we need a physical condition. It is anharmonicity, that is, nonlinearity concerning the level (i.e., quantum number) as in Fig.1. The Josephson junction on

(a) (b)

Figure 1: (a) harmonicity and anharmonicity among energy levels; (b) N states isolated from others.

a superconducting circuit makes the isolated N states, the 2 states in particular. Under the strong anharmonicity, we can take only the N states and ignore the other states by projecting the whole physical system into N state space. This approximated system with N sates is called N*-level atom*.

Let us consider a 2-level atom coupled with the light-field now. The Hamiltonian H_{WW} is given by that of the Wigner-Weisskopf model:

$$H_{\mathrm{WW}} := \frac{\hbar\omega_{\mathrm{a}}}{2}\sigma_z + \int d^3k\hbar\omega_{\mathrm{c}}(k)a^\dagger(k)a(k) + \hbar\mathrm{g}\int d^3k\left(\lambda(k)^*\sigma_+ a(k) + \lambda(k)a^\dagger(k)\sigma_-\right)$$

with $\sigma_+ := (\sigma_x + i\sigma_y)/2$ and $\sigma_- := (\sigma_x - i\sigma_y)/2$, where $\omega_{\mathrm{c}}(k)$ is the dispersion relation of photon, ω_{a} the atom transition frequency, g the atom-photon coupling constant with $\mathrm{g} \geq 0$, $a^\dagger(k)$ (resp. $a(k)$) the creation (resp. annihilation) operator, $\sigma_\sharp$ the Pauli matrices, $\sharp = x, y, z$. The Wigner-Weisskopf Hamiltonian is derived from the spin-boson Hamiltonian H_{SB} through the so-called *rotating wave approximation* (RWA):

$$H_{\mathrm{SB}} := \frac{\hbar\omega_{\mathrm{a}}}{2}\sigma_z + \int d^3k\hbar\omega_{\mathrm{c}}(k)a^\dagger(k)a(k) + \hbar\mathrm{g}\int d^3k\sigma_x\left(\lambda(k)^* a(k) + \lambda(k)a^\dagger(k)\right).$$

Namely, we can neglect the *counter-rotating terms* $W := \sigma_- a + a^\dagger\sigma_+$ in H_{SB} as g makes the weak or the standard strong coupling regimes because of a physical reason.

Here we recall a fact concerning the ground state energy of H_{SB}. In [21] we made the argument similar to the functional-integral computations for the transition amplitude of the so-called *instanton* *gas* [2]. Then, the constant $\hbar G(\mathrm{g})$ below plays a role similar to the action in the expression of the functional integral. Following the result in [21], the ground state energy E_{SB} of H_{SB} is expressed as:

$$E_{\mathrm{SB}} = -\hbar\mathrm{g}^2\int d^3k\frac{|\lambda(k)|^2}{\omega_{\mathrm{c}}(k)} - \lim_{\beta\to\infty}\frac{\hbar}{\beta}\ln\{I_{even}(\beta)+I_{odd}(\beta)\} \tag{3.1}$$

for arbitrary coupling constant g under some conditions, where

$$\begin{aligned}I_{even}(\beta) :=& 1+\sum_{\ell=1}^{\infty}\left(\frac{\omega_{\mathrm{a}}}{2}\right)^{2\ell}\int_0^{\beta}d\beta_1\int_0^{\beta_1}d\beta_2\cdots\int_0^{\beta_{2\ell-1}}d\beta_{2\ell}\\ &\times\exp\left[-2\mathrm{g}^2\int d^3k\frac{|\lambda(k)|^2}{\omega_{\mathrm{c}}(k)^2}\left\{2G_{\beta_1,\cdots,\beta_{2\ell}}(k)+2\ell\right\}\right],\end{aligned}$$

$$\begin{aligned}I_{odd}(\beta) :=& \beta\frac{\omega_{\mathrm{a}}}{2}\exp\left[-2\mathrm{g}^2\int d^3k\frac{|\lambda(k)|^2}{\omega_{\mathrm{c}}(k)^2}\right]+\sum_{\ell=1}^{\infty}\left(\frac{\omega_{\mathrm{a}}}{2}\right)^{2\ell+1}\int_0^{\beta}d\beta_1\int_0^{\beta_1}d\beta_2\cdots\int_0^{\beta_{2\ell}}d\beta_{2\ell+1}\\ &\times\exp\left[-2\mathrm{g}^2\int d^3k\frac{|\lambda(k)|^2}{\omega_{\mathrm{c}}(k)^2}\left\{2G_{\beta_1,\cdots,\beta_{2\ell}}(k)+2F_{\beta_1,\cdots,\beta_{2\ell+1}}(k)+(2\ell+1)\right\}\right],\end{aligned}$$

$$\begin{aligned}G_{\beta_1,\cdots,\beta_{2\ell}}(k) =& -\sum_{p=1}^{\ell}e^{-(\beta_{2p-1}-\beta_{2p})\omega_{\mathrm{c}}(k)}\\ &+\sum_{p,q=1;p<q}^{\ell}\left(e^{-\beta_{2p-1}\omega_{\mathrm{c}}(k)}-e^{-\beta_{2p}\omega_{\mathrm{c}}(k)}\right)\times\left(e^{\beta_{2q-1}\omega_{\mathrm{c}}(k)}-e^{\beta_{2q}\omega_{\mathrm{c}}(k)}\right)\le 0,\end{aligned}$$

and

$$F_{\beta_1,\cdots,\beta_{2\ell+1}}(k) = e^{\beta_{2\ell+1}\omega_{\mathrm{c}}(k)}\sum_{p=1}^{\ell}\left(e^{-\beta_{2p-1}\omega_{\mathrm{c}}(k)}-e^{-\beta_{2p}\omega_{\mathrm{c}}(k)}\right)\le 0,$$

This expression is strict, but it is very complicated because $I_{even}(\beta)$'s and $I_{odd}(\beta)$'s are so. We can make it simpler with a constant: First, we modify $I_{even}(\beta)$ and $I_{odd}(\beta)$ replacing $G_{\beta_1,\cdots,\beta_{2\ell}}(k)$ and $F_{\beta_1,\cdots,\beta_{2\ell+1}}(k)$ in them by ℓG and $G/2$, respectively, with any constant G. Namely,

$$I_{even}^{G}(\beta) := \cosh\left(\beta\frac{\omega_{\mathrm{a}}}{2}\exp\left[-2\mathrm{g}^2\int d^3k\frac{|\lambda(k)|^2}{\omega_{\mathrm{c}}(k)^2}(G+1)\right]\right),$$

and

$$I_{odd}^{G}(\beta) := \sinh\left(\beta\frac{\omega_{\mathrm{a}}}{2}\exp\left[-2\mathrm{g}^2\int d^3k\frac{|\lambda(k)|^2}{\omega_{\mathrm{c}}(k)^2}(G+1)\right]\right).$$

Then, Theorem 1.5 of [21] says that for every coupling constant g there is a unique constant $G(\mathrm{g})\in[-1,0]$ so that

$$\begin{aligned}E_{\mathrm{SB}} =& -\hbar\mathrm{g}^2\int d^3k\frac{|\lambda(k)|^2}{\omega_{\mathrm{c}}(k)} - \lim_{\beta\to\infty}\frac{\hbar}{\beta}\ln\left\{I_{even}^{G(\mathrm{g})}(\beta)+I_{odd}^{G(\mathrm{g})}(\beta)\right\}\\ =& -\hbar\mathrm{g}^2\int d^3k\frac{|\lambda(k)|^2}{\omega_{\mathrm{c}}(k)} - \frac{\hbar\omega_{\mathrm{a}}}{2}\exp\left[-2\mathrm{g}^2\int d^3k\frac{|\lambda(k)|^2}{\omega_{\mathrm{c}}(k)^2}\left(G(\mathrm{g})+1\right)\right].\end{aligned} \tag{3.2}$$

We use this exact expression later.

In [22, 23] the following facts were mathematically proved for H_{WW}: We denote the two energies of the decoupled 2-level atom by E_0 and E_1. Following the standard QED, E_1 gets unstable since E_1 is embedded in the continuous energy spectrum when there is no perturbation, and thus, it disappears after a perturbative interaction works as in Fig.2(a). If we make the coupling strength g strong, then an extra state with energy E_* appears as in Fig.2(b). It is dressed with 1 photon. Namely, there is a g_0 so that the state with E_* is unstable for $|\mathrm{g}|<\exists\mathrm{g}_0$, but it gets stable whenever $|\mathrm{g}|>\mathrm{g}_0$. C. Billionnet has investigated how such an extra energy state appears [7]. As we make g strong more, E_* goes down and overtakes E_0, and thus, E_* becomes a new ground state energy as in Fig.2(c). This new ground state energy is the *superradiant ground state energy.* It is clear that there is an energy level crossing to appear a superradiant ground state energy, which is called the *Dicke-type energy level crossing* [24, 25] because caused by Dicke's superradiance (see the remark at the tail end of Sec.4.3 of [41]). That is, a non-perturbative ground state appears as a phenomenon of phase transition [22, 23]. As g gets strong more and more, much to our surprise, another new superradiant ground state with energy E_{**} appears as in Fig.2(d). The state with E_{**} is dressed with more than 1 photon. According to Preparata, such a superradiant ground state appears as: (coupling strength) $\propto \sqrt{(\text{numbers of photons})}$.

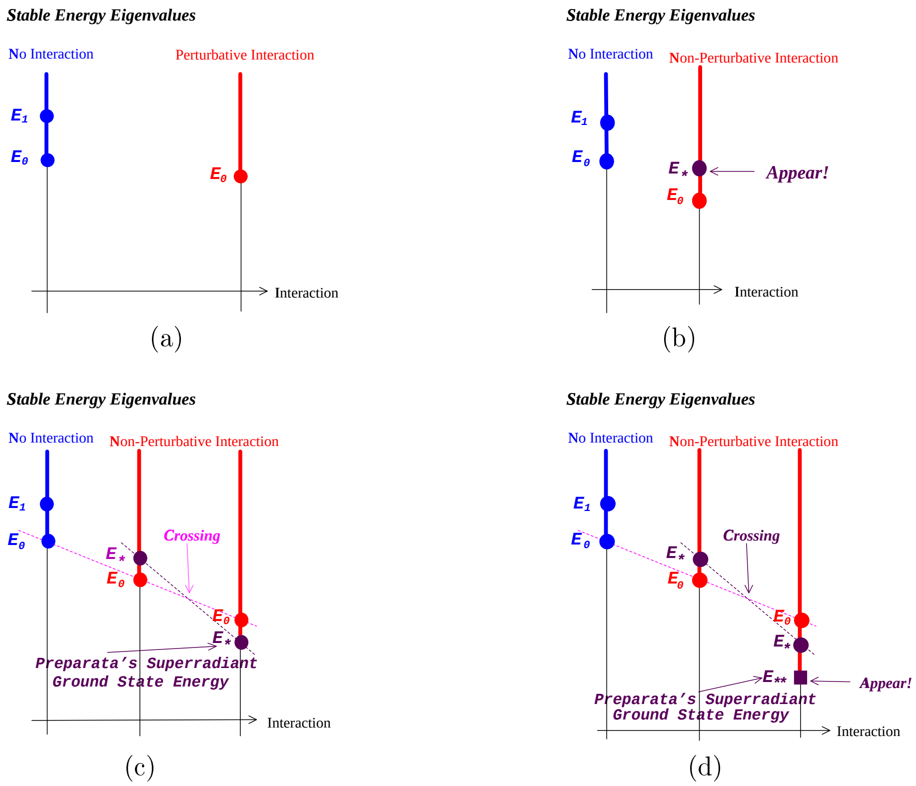


Figure 2: (a) disappearance of embedded eigenstate energy; (b) appearance of an extra energy state; (c) appearance of superradiant ground state energy; (d) another superradiant ground state appears.

We can find such superradiant ground states and energy level crossings for the Jaynes-Cummings model by the numerical result in Fig.4 below or Fig.1 of [32].

4 Dicke-Type Energy Level Crossings and Superradiant Ground State Energy

In this section we consider the Jaynes-Cummings model. Its Hamiltonian $H_{\mathrm{JC}}^{\mathrm{g}}$ is given as the 1-photon version of H_{WW}:

$$H_{\mathrm{JC}}^{\mathrm{g}} := \frac{\hbar\omega_{\mathrm{a}}}{2}\sigma_z + \hbar\omega_{\mathrm{c}}\left(a^\dagger a + \frac{1}{2}\right) + \hbar\mathrm{g}\left(\sigma_+ a + \sigma_- a^\dagger\right).$$

This is a solvable model:

$$\mathrm{Spec}(H_{\mathrm{JC}}^{\mathrm{g}}) = \left\{E_{-,-1}^{\mathrm{g}}, E_{+,m}^{\mathrm{g}}, E_{-,n}^{\mathrm{g}} \,\middle|\, m, n = 0, 1, 2, \cdots\right\},$$

where

$$\begin{cases} E_{-,-1}^{\mathrm{g}} = -\dfrac{\hbar(\omega_{\mathrm{a}} - \omega_{\mathrm{c}})}{2}, \\ E_{\pm,n}^{\mathrm{g}} = \hbar\omega_{\mathrm{c}}(n+1) \pm \dfrac{\hbar}{2}\sqrt{\Delta_0^2 + 4\mathrm{g}^2(n+1)}\,. \end{cases} \tag{4.1}$$

When g is so small that it makes a perturbative interaction, each eigenvalues sits near its original position as in Fig.3(a). If we employ so strong g that it is beyond perturbation theory, E_1 which is primarily the first excited state energy overtakes the original ground state energy E_0 as in Fig.3(b).

Then, a trivial crossing appears. Of course, there is a case where a non-trivial crossing appears as in Fig.3(c). The explanation of trivial and non-trivial crossings is in [24]. As g gets strong more and more, the Dicke-type energy level crossing takes place and E_1 becomes a superradiant ground state energy as in Fig.3(d).

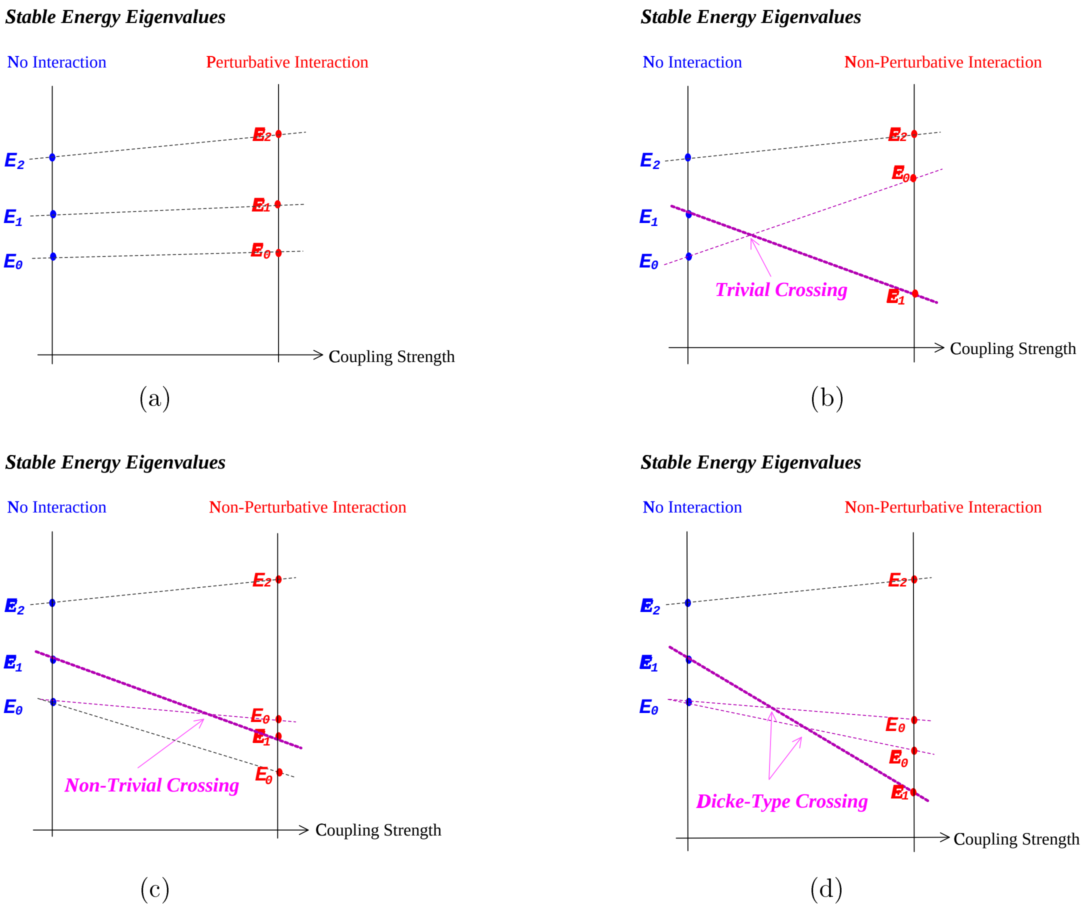


Figure 3: (a) no energy level crossing; (b) trivial crossing; (c) non-trivial crossing; (d) Dicke-type energy level crossing and superradiant ground state energy.

In [24, 25], it is precisely shown and mathematically proved that:

Theorem 4.1 *The Jaynes-Cummings model plus an extra term $\gamma\widetilde{W}$, $H_{\mathrm{JC}}^{\mathrm{g}} + \gamma\widetilde{W}$, has the Dicke-type energy level crossings, and then, the superradiant ground state appears as the coupling strength* g *grows large, provided that the extra term are sufficiently small compared with the coupling strength (i.e., $\gamma \ll \mathrm{g}$).*

An example in the case where $\omega_{\mathrm{a}} = \omega_{\mathrm{c}} = \omega$ and $\gamma = 0$ is in Fig.4.

In [25] the so-called *cavity-induced atom cooling* (CIAC), proposed by P. Horak *et al.* [29], was developed. The CIAC has been demonstrated by P. Maunz *et al.* [38]. The idea in [25] is to use the emission of the (multi) photon(s) for a superradiant cooling. In this process, because we use the energy level crossings, we do not have to excite the atom in the cavity with an extra laser. The energy level crossings and the emission of multi photons were theoretically and experimentally studied for a spin-half atom in a static magnetic field and a strong oscillating radio-frequency field by T. Yabuzaki *et al* [52]. They demonstrated the energy level crossings in the optical-pumping experiment with cesium vapor, and moreover, proved that Haroche-like resonance found by N. Tsukada *et al.* [50] is caused by the interference between the energy level crossings. We note that Y. Nakamura *et al.* had performed a similar experiment in circuit QED already [39].

5 A Chirality in the Ground State Energy

Cavity QED and circuit QED realize the weak, the strong, and the ultra-strong coupling regimes. Roughly speaking, the interaction order of each regime is $\mathrm{g}/\omega_{\mathrm{c}} \sim 0.01$ (the weak coupling regime); $\mathrm{g}/\omega_{\mathrm{c}} \sim 0.1$ (the strong coupling regime); and $\mathrm{g}/\omega_{\mathrm{c}} \sim 1$ (ultra-strong coupling regime).

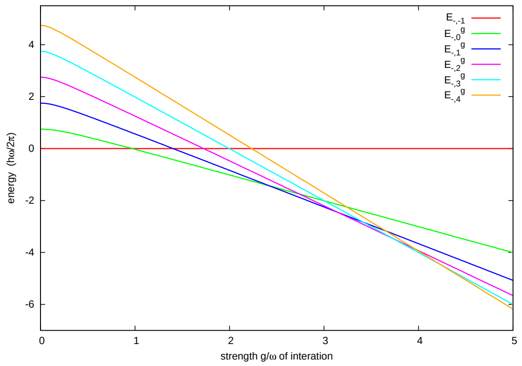


Figure 4: Dicke-type energy level crossings and superradiant ground state energy: $\omega_{\mathrm{a}} = \omega_{\mathrm{c}} = \omega$, $E^{g}_{-,n}$(in the figure) $= E^{\mathrm{g}}_{-,n}/\hbar\omega$.

The energy of the atom coupled with the 1-mode photon in a cavity is exactly described by the fully coupled Hamiltonian H_{FC} (i.e., the Rabi Hamiltonian):

$$H_{\mathrm{FC}} := H^{\mathrm{g}}_{\mathrm{JC}} + W = \frac{\hbar\omega_{\mathrm{a}}}{2}\sigma_z + \hbar\omega_{\mathrm{c}}\left(a^{\dagger}a + \frac{1}{2}\right) + \hbar \mathrm{g}\sigma_x\left(a^{\dagger} + a\right).$$

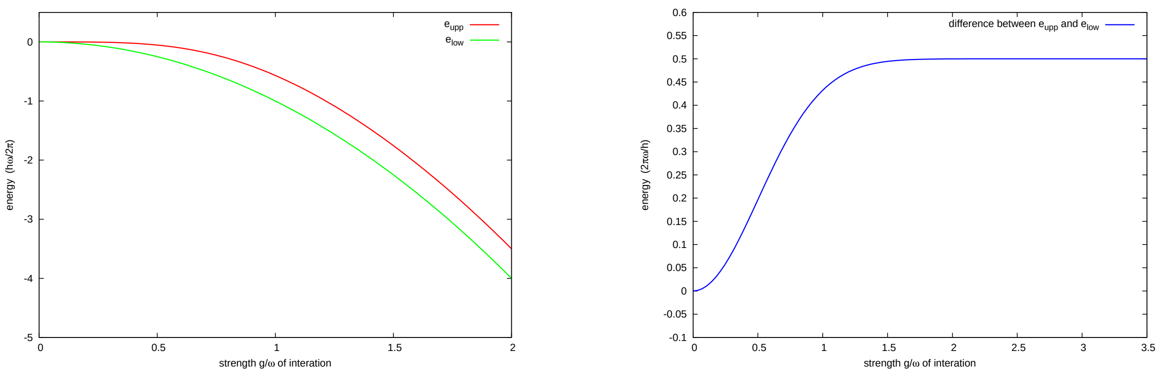
That is, H_{FC} is the 1-photon version of H_{SB}. Thus, applying the exact expression (3.2) of E_{SB} to the ground state energy E_{FC} of H_{FC}, we have

$$E_{\mathrm{FC}} = \frac{\hbar\omega_{\mathrm{c}}}{2} - \frac{\hbar \mathrm{g}^2}{\omega_{\mathrm{c}}} - \frac{\hbar\omega_{\mathrm{a}}}{2}\exp\left[-\frac{2\mathrm{g}^2}{\omega_{\mathrm{c}}^2}\left(G(\mathrm{g})+1\right)\right]$$

for a certain constant $G(\mathrm{g}) \in [-1, 0]$, which easily implies

$$e_{\mathrm{low}}(\mathrm{g}) := \frac{\hbar\omega_{\mathrm{c}}}{2} - \frac{\hbar \mathrm{g}^2}{\omega_{\mathrm{c}}} - \frac{\hbar\omega_{\mathrm{a}}}{2} \leq E_{\mathrm{FC}} \leq \frac{\hbar\omega_{\mathrm{c}}}{2} - \frac{\hbar \mathrm{g}^2}{\omega_{\mathrm{c}}} - \frac{\hbar\omega_{\mathrm{a}}}{2}\exp\left[-\frac{2\mathrm{g}^2}{\omega_{\mathrm{c}}^2}\right] =: e_{\mathrm{upp}}(\mathrm{g}). \tag{5.1}$$

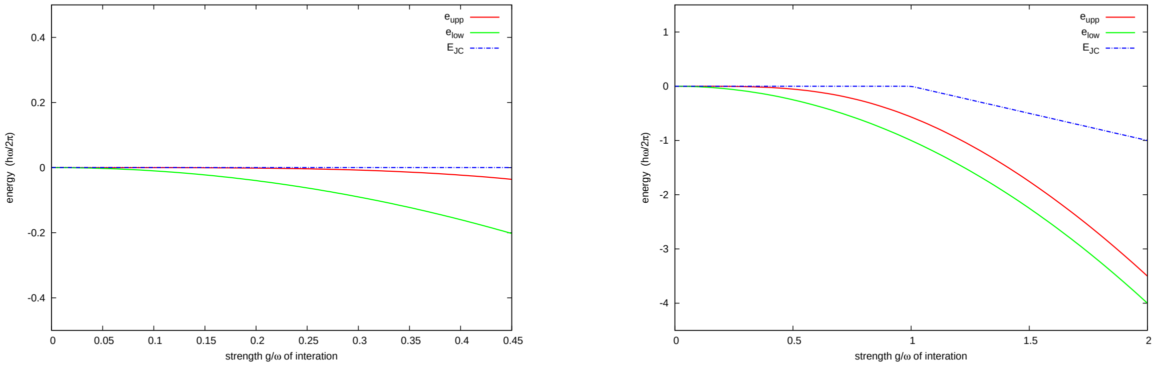
Then, we obtain $\lim_{\mathrm{g}\to\infty}(e_{\mathrm{upp}}(\mathrm{g}) - e_{\mathrm{low}}(\mathrm{g}))/\hbar\omega_{\mathrm{a}} = \dfrac{1}{2}$ as in Fig.5. Namely, although (5.1) is the most rough estimates at $G(\mathrm{g}) = -1$ and 0 respectively, the difference between $e_{\mathrm{upp}}(\mathrm{g})$ and $e_{\mathrm{low}}(\mathrm{g})$ is the zero-point energy $\hbar\omega_{\mathrm{a}}/2$ at most.



(a) (b)

Figure 5: $\omega_{\mathrm{a}} = \omega_{\mathrm{c}} = \omega$, $e_{\mathrm{upp}} = e_{\mathrm{upp}}(\mathrm{g})/\hbar\omega$, $e_{\mathrm{low}} = e_{\mathrm{low}}(\mathrm{g})/\hbar\omega$: (a) upper bound e_{upp} and lower bound e_{low}; (b) difference between e_{upp} and e_{low}.

The standard cavity QED realizes the weak and strong coupling regimes, but it hardly make the ultra-strong coupling regime. In the weak or the strong coupling regime, the fully coupled Hamiltonian H_{FC} is well approximated by the Jaynes-Cummings Hamiltonian $H^{\mathrm{g}}_{\mathrm{JC}}$. On the other hand, circuit QED can realize ultra-strong coupling regime as well as other regimes. In the ultra-strong coupling regime, $H^{\mathrm{g}}_{\mathrm{JC}}$ does not function well as a good approximation, which is pointed out theoretically and experimentally in physics [3, 13, 16, 40]. As far as the ground state energy goes, according to the numerical analysis as in Fig.6(a), the ground state energy of the standard Jaynes-Cummings model can supply a good approximation for the upper bound e_{upp} of H_{FC} until about $\mathrm{g}/\omega \approx 0.25$, but after that strength it works well no longer. The Jaynes-Cummings model has the phase transition at $\mathrm{g} = \omega$. Namely, the superradiant ground state appears: $E_{\mathrm{JC}} = 0$ $(0 \le \mathrm{g}/\omega \le 1)$; $\hbar\omega - \hbar\mathrm{g}$ $(1 \le \mathrm{g}/\omega \le 1+\sqrt{2})$ as in Fig.6(b).



(a) (b)

Figure 6: $\omega_{\mathrm{a}} = \omega_{\mathrm{c}} = \omega$, $e_{\mathrm{upp}} = e_{\mathrm{upp}}(\mathrm{g})/\hbar\omega$, $e_{\mathrm{low}} = e_{\mathrm{low}}(\mathrm{g})/\hbar\omega$, E_{JC}(in the figure)$= E_{\mathrm{JC}}/\hbar\omega$.

E. K. Irish gave a generalization of the RWA, which is called the *generalized rotating wave approximation* (GRWA) [32]. Following the GRWA we have an expression of the eigenstate energies which are same as given by I. D. Feranchuk *et al.* [18]. In numerical analysis, the GRWA is useful even for ultra-strong coupling regime. I. D. Feranchuk *et al.* used an operator-method algorithm in numerical analysis to approximate the eigenenergies of H_{FC}, using a parity conservation law. Recently, also using the parity conservation, F. Pan *et al.* proposed another approximation method, named progressive diagonalization scheme [42]. This parity conservation law is important to give a strict expression of the ground state energy of H_{FC} (see (3.1) and (3.2)). Though their method is to compute the approximate eigenvalues of H_{FC}, the final goal of our future studies is to establish the approximate Hamiltonian formalism so that we can approximately analyze the fully coupled Hamiltonian with it. Because we are interested in the qualitative properties as well as the quantitative. Thus, we had its first step with the attempt for the ground state energy in our joint work [26].

Define $H(\gamma)$ by

$$H(\gamma) := H_{\mathrm{JC}}^{\mathrm{g}} + \gamma W, \quad 0 \le \gamma \le 1.$$

We know $\gamma \to 0$ as $\mathrm{g} \ll 1$ and $\gamma \to 1$ as $1 \ll \mathrm{g}$. But, we have not obtained a mathematical theory which clarifies the physical relation between γ and g yet. Here we note that $H(0) = H_{\mathrm{JC}}^{\mathrm{g}}$ and that $H(1) = H_{\mathrm{FC}}$. Namely, $H(\gamma)$ describes almost the Jaynes-Cummings Hamiltonian as the coupling strength is so small $(\mathrm{g} \ll 1)$, and the fully coupled one as the coupling strength is so large $(\mathrm{g} \gg 1)$. Based on [21, 22], we can prove the following:

Theorem 5.1 *The standard Jaynes-Cummings Hamiltonian $H_{\mathrm{JC}}^{\mathrm{g}}$ can be parameterized so that a modified Jaynes-Cummings Hamiltonian $H_{\mathrm{JC}}^{\mathrm{g}}(\varepsilon)$ with a parameter $0 < \varepsilon < 1$ and its chiral-counter Hamiltonian $H_{\mathrm{CC}}^{\mathrm{g}}(\gamma, \varepsilon)$ restore $H(\gamma)$ as $H(\gamma) = H_{\mathrm{JC}}^{\mathrm{g}}(\varepsilon) + H_{\mathrm{CC}}^{\mathrm{g}}(\gamma, \varepsilon)$. Then, $H_{\mathrm{CC}}^{\mathrm{g}}(\gamma, \varepsilon)$ is also a modified Jaynes-Cummings Hamiltonian in a chiral space, and*

$$\sup_{0<\varepsilon<1} E_{\mathrm{low}}(\mathrm{g}, \varepsilon) \le \inf \mathrm{Spec}(H(\gamma)) \le \inf_{0<\varepsilon<1} E_{\mathrm{upp}}(\mathrm{g}, \varepsilon),$$

where

$$E_{\mathrm{low}}(\mathrm{g}, \varepsilon) := \inf \mathrm{Spec}(H_{\mathrm{JC}}^{\mathrm{g}}(\varepsilon)) + \inf \mathrm{Spec}(H_{\mathrm{CC}}^{\mathrm{g}}(\gamma, \varepsilon)),$$
$$E_{\mathrm{upp}}(\mathrm{g}, \varepsilon) := \inf \mathrm{Spec}(H_{\mathrm{JC}}^{\mathrm{g}}(\varepsilon)) + \langle \ell_*, -|H_{\mathrm{CC}}^{\mathrm{g}}(\gamma, \varepsilon)|-, \ell_* \rangle$$

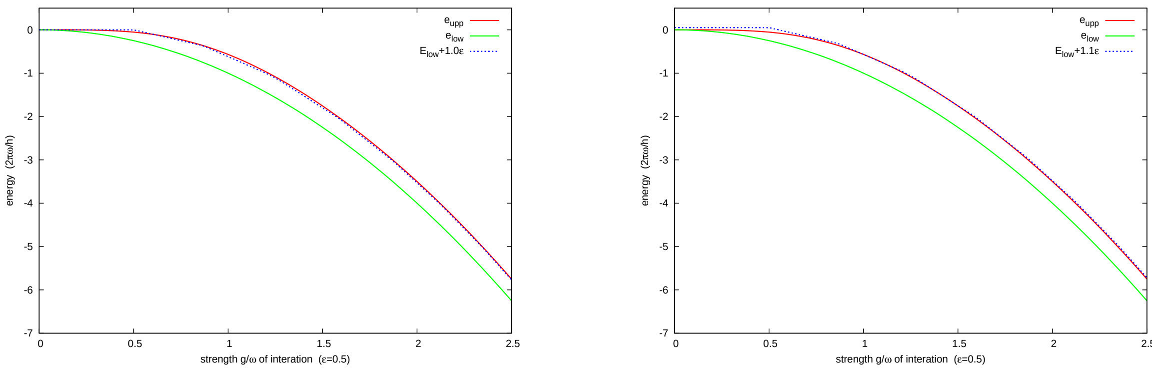
with $|-, \ell_*\rangle$ is a (superradiant) ground state of $H_{\mathrm{JC}}^{\mathrm{g}}(\varepsilon)$.

We note $E_{\mathrm{low}}(\mathrm{g}, \varepsilon)$ and $E_{\mathrm{upp}}(\mathrm{g}, \varepsilon)$ have the concrete expressions similar to those in (4.1).

In [26] we explain how to determine $H_{\mathrm{JC}}^{\mathrm{g}}(\varepsilon)$ and $H_{\mathrm{CC}}^{\mathrm{g}}(\gamma, \varepsilon)$, and argue how the ground state of $H_{\mathrm{FC}} = H(1)$ is dressed with photons by using $H_{\mathrm{JC}}^{\mathrm{g}}(\varepsilon)$ and $H_{\mathrm{CC}}^{\mathrm{g}}(\gamma, \varepsilon)$. Moreover, our mathematical argument and numerical experiments in [26] say that the energy curve of $E_{\mathrm{low}}(\mathrm{g}, \varepsilon)$ almost restores the ground-state-energy curve of H_{FC} as Fig.7. According to this result and others in [26], we conjecture that for $\gamma = 1$

$$E_{\mathrm{FC}} \le e_{\mathrm{upp}}(\mathrm{g}) \approx E_{\mathrm{low}}(\mathrm{g}, \varepsilon) + \alpha\varepsilon\hbar\omega$$

with $\varepsilon \approx 0.5$ and $\alpha \approx 1.0$, when $\omega = \omega_{\mathrm{a}} = \omega_{\mathrm{c}}$. Actually, the right hand side of (5.1) gives a good estimate as $\hbar^2\omega_{\mathrm{a}}^2/\mathrm{g} \ll 1$. Thus, since the upper bound $e_{\mathrm{upp}}(\mathrm{g})$ is a good approximation for E_{FC}, our numerical experiments as in Fig.7 say $E_{\mathrm{low}}(\mathrm{g}, \varepsilon) + \alpha\varepsilon\hbar\omega$ ($\varepsilon \approx 0.5$, $\alpha \approx 1.0$) may give a good approximation for E_{FC}.



(a) (b)

Figure 7: $\omega_{\mathrm{a}} = \omega_{\mathrm{c}} = \omega$, $e_{\mathrm{upp}} = e_{\mathrm{upp}}(\mathrm{g})/\hbar\omega$, $e_{\mathrm{low}} = e_{\mathrm{low}}(\mathrm{g})/\hbar\omega$, E_{JC}(in the figure)$= E_{\mathrm{JC}}/\hbar\omega$, $E_{\mathrm{low}} = E_{\mathrm{low}}(\mathrm{g}, \varepsilon)/\hbar\omega$: (a) $(E_{\mathrm{low}}(\mathrm{g}, 0.5) + 1.0 \times 0.5\hbar\omega)/\hbar\omega$; (b) $(E_{\mathrm{low}}(\mathrm{g}, 0.5) + 1.1 \times 0.5\hbar\omega)/\hbar\omega$.

At the tail end of this section, we point out the following. The energy properties as in Fig.3 c & d of [40] are never explained with the Jaynes-Cummings Hamiltonian, which are special to the ultra-strong coupling regime. We are trying to explain these properties using our chiral-counter Hamiltonian.

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Existence and absence of ground state of the Nelson model on a pseudo Riemannian manifold

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Abstract

The Nelson model describing a quantum particle interacting through a quantized scalar Bose field on the Minkowski spacetime is extended to one on a class of pseudo Riemannian spacetimes. In this note, we announce several results [3, 4] concerning the existence and absence of the ground state for such a model without infrared cutoff.

1 Introduction

We consider a confined quantum particle interacting through a scalar Bose field whose Hamiltonian is given by

$$H = K \otimes I + I \otimes d\Gamma(\omega) + \phi(\omega^{-1/2}\rho_X). \tag{1.1}$$

The particle, whose position is denoted by X, is described by a Schrödinger operator

$$K = -\frac{1}{2}\sum_{1\le j,k\le 3} \frac{\partial}{\partial X_j} A_{jk}(X) \frac{\partial}{\partial X_k} + V(X), \quad \text{acting in } \mathcal{K} = L^2(\mathbb{R}^3; dX)$$

with a confining potential $V : \mathbb{R}^3 \to \mathbb{R}$ such that K has a compact resolvent. Let

$$h = -\frac{1}{c(x)}\sum_{1\le j,k\le 3} \frac{\partial}{\partial x_j} a_{jk}(x) \frac{\partial}{\partial x_k} \frac{1}{c(x)} + m^2(x) \tag{1.2}$$

with a positive function $m^2(x)$. The *one-boson energy* is given by

$$\omega = h^{1/2}, \quad \text{acting in } \mathfrak{h} = L^2(\mathbb{R}^3; dx).$$

The second quantization $d\Gamma(\omega)$ of ω acting in the boson Fock space $\Gamma(\mathfrak{h})$ over $\mathfrak{h}$ is the free Hamiltonian of the Bose field. The Segal field operator $\phi(f)$ ($f \in \mathfrak{h}$) is given by $\phi(f) = (a(f) + a^*(f))/\sqrt{2}$, where the annihilation and creation operators $a(f)$ and $a^*(f)$ satisfy the canonical commutation relations (CCR): $[a(f), a^*(g)] = \langle f, g\rangle_{\mathfrak{h}}$ and $[a^\sharp(f), a^\sharp(g)] = 0$ ($a^\sharp = a$ or a^*). The charge distribution $\rho : \mathbb{R}^3 \to [0,\infty)$ is assumed to be an infinitely differentiable function with compact support and the constant

$$g := \int \rho(x)dx \ge 0$$

is called the charge which describes the strength of the interaction. We set $\rho_X(x) = \rho(x - X)$. Then ρ introduces an ultraviolet cutoff and satisfies

$$\sup_{X\in\mathbb{R}^3} \|(\omega^{-1/2} + \omega^{-1})\rho_X\|_{\mathfrak{h}} < \infty, \tag{1.3}$$

which yields that H is a (well-defined) bounded below self-adjoint operator.

The bottom of the spectrum of the one-boson energy

$$m_{\mathrm{b}} := \inf \sigma(\omega) \geq 0$$

is viewed as the (rest) *mass* of the boson. The Hamiltonian H is called *massless* (resp. *massive*) if $m_{\mathrm{b}} = 0$ (resp. $m_{\mathrm{b}} > 0$). We say that H is *infrared divergent* (resp. *infrared convergent*) if H has no ground state (resp. if H has a ground state). As is well known, the massive Hamiltonian H is infrared convergent since the strictly positive mass $m_{\mathrm{b}} > 0$ plays the role of an infrared cutoff. The infrared behavior is of interest only for massless case $m_{\mathrm{b}} = 0$.

1.1 Massless Nelson model on Minkowski spacetime

When $A_{jk}(X) = \mathbf{1}$, $a_{jk}(x) = \mathbf{1}$, $c(x) = 1$ and $m(x) \equiv 0$, then $m_{\mathrm{b}} = 0$ and H becomes the Hamiltonian of the so called *massless Nelson model* describing a quantum particle coupled to a massless scalar Bose field on Minkowski spacetime. Let $\mathscr{F}$ be the Fourier transform and $\hat{\rho}(k) = (\mathscr{F}\rho)(k)$. Then $w := \mathscr{F}\omega\mathscr{F}^{-1}$ is the multiplication operator by $w(k) = |k|$, which is the dispersion relation of the boson. We observe that H is unitarily transfomed by the second quantization $\Gamma(\mathscr{F})$ of $\mathscr{F}$ into

$$H_{\mathrm{Nelson}} = (-\Delta_X + V(X)) \otimes I + I \otimes d\Gamma(w) + \phi\left(\overline{\psi_0(\cdot, X)} w^{-1/2}\hat{\rho}\right),$$

where $\psi_0(k, X) = e^{ik\cdot X}$ is the plane wave. By assumption, we observe that

$$\int dk (w(k)^{-1} + w(k)^{-2})|\hat{\rho}(k)|^2 < \infty,$$

which implies (1.3) and yields that the Hamiltonian H_{Nelson} is a (well-defined) bounded below self-adjoint operator. Lőrinczi, Minlos and Spohn [5] study the infrared behavior of H_{Nelson} and show that H_{Nelson} is infrared divergent if the charge $g = (2\pi)^{3/2}\hat{\rho}(0)$ is strictly positive. Note that, if $g > 0$, then $\rho_X \notin D(w^{-3/2})$ and

$$\int dk \frac{|\hat{\rho}(k)|^2}{w(k)^3} = \infty. \tag{1.4}$$

As was shown in [1], if the left hand side of (1.4) is finite, then H_{Nelson} is infrared convergent. But this is not the case if $\rho \geq 0$ and $g > 0$.

1.2 Nelson model on a static spacetime

Let $g_{\mu\nu}(t, x)$ be a pseudo Riemannian metric on $\mathbb{R}^{1+3}$ and set $|g| := |\det[g_{\mu\nu}]|$ and $[g_{\mu\nu}]^{-1} = [g^{\mu\nu}]$. We consider the Klein-Gordon equation:

$$(\Box_g + \tilde{m}^2(t, x))\phi(x) = 0,$$

where

$$\Box_g = |g|^{-1/2}\partial_\mu |g|^{1/2} g^{\mu\nu}\partial_\nu$$

and

$$\tilde{m}^2(t, x) = m_0^2 + \theta R(t, x)$$

with a constant θ, the mass $m_0 \geq 0$ and the scalar curvature R. In this note, we always assume that $\tilde{m}^2(x)$ is a positive function. If $m \equiv 0$ and $\theta = 1/6$, then one obtains the so-called conformal wave equation. We say that the metric $g_{\mu\nu}$ is static if the line element is of the form

$$g_{\mu\nu}(t,x)dx^\mu dx^\nu = \lambda(x)dt^2 - \lambda(x)^{-1}\gamma_{jk}(x)dx^j dx^k,$$

where $\lambda(x) > 0$ is a smooth function independent of t and $\gamma_{jk}(x)$ is a Riemannian metric on $\mathbb{R}^3$. If $g_{\mu\nu}$ is static, then $\tilde{m}(t,x) = \tilde{m}(x)$ is independent of t and $\tilde{\phi} = \lambda^{-1}|\gamma|^{1/4}\phi$ satisfies

$$\partial_t^2\tilde{\phi} + h_g\tilde{\phi} = 0,$$

where

$$h_g = -\lambda|\gamma|^{-1/4}\partial_j|\gamma|^{1/2}\gamma^{jk}\partial_k|\gamma|^{-1/4}\lambda\tilde{\phi} + \tilde{m}^2\lambda$$

with $|\gamma| = |\det[\gamma_{jk}]|$ and $[\gamma^{jk}] = [\gamma_{jk}]^{-1}$. h_g is a typical example of h defined by (1.2) setting $c = \lambda^{-1}|\gamma|^{1/4}$, $a_{jk} = |\gamma|^{1/2}\gamma^{jk}$ and $m^2 = \lambda\tilde{m}^2$. Thus the Nelson model on the Minkowski spacetime is naturally extended to one on the static spacetime, which is a typical example of the abstract Hamiltonian H defined by (1.1).

1.3 Infrared behavior

We are interested in the infrared behavior of the abstract Hamiltonian H and concentrate on the massless case. To this end, we always assume that

$$\lim_{|x|\to\infty} m(x) = 0. \tag{1.5}$$

As is shown in Lemma 2.2, we know that our Hamiltonian H is massless, i.e., $m_{\mathrm{b}} = 0$, under some reasonable conditions for a_{jk}, c and m. The main result is the infrared behavior depends on the decay rate of the function $m(x)$. Indeed, we show in Theorem 3.5 that H is infrared convergent if the function $m(x)$ satisfies

$$m(x) \geq a\langle x\rangle^{-1} \quad \text{for some } a > 0.$$

This result is sharp with respect to the decay rate of $m(x)$ since we show in Theorem 3.4 H is infrared divergent if $m(x)$ satisfies

$$0 \leq m(x) \leq a\langle x\rangle^{-1-\epsilon} \quad \text{for some } a > 0 \text{ and } \epsilon > 0.$$

2 Definition of the Hamiltonian H

2.1 Particle Hamiltonian

For the particle Hamiltonian, we assume the following:

($\mathbf{K}_1$) $A_{jk}(X) \in W^{1,\infty}(\mathbb{R}^3)$ satisfies

$$A_0\mathbf{1} \leq [A_{jk}(X)] \leq A_1\mathbf{1}, \quad X \in \mathbb{R}^3$$

with some constants $A_0, A_1 > 0$.

Let K_0 be the self-adjoint operator associated with

$$\mathcal{E}_0(f,g) = \frac{1}{2} \sum_{1 \le j,k \le 3} \int dX A_{jk}(X) \frac{\partial \bar{f}}{\partial X_j} \frac{\partial g}{\partial X_k}, \quad f, g \in H^1(\mathbb{R}^3).$$

It follows from (K_1) that K_0 is self-adjoint on $H^2(\mathbb{R}^3)$. We further require the following:

($\mathbf{K}_2$) The potential function $V \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ satisfies

$$V(X) \ge C_0 |X|^{2q} - C_1, \quad X \in \mathbb{R}^3$$

with some constants $C_0 > 0$, $C_1 \ge 0$ and $q > 0$.

The following is standard:

Lemma 2.1. *Suppose that (K_1) and (K_2) hold. Then:*

(i) $K := K_0 \dot{+} V$ *is a self-adjoint operator associated with the closure of*

$$\mathcal{E}(f,g) = \mathcal{E}_0(f,g) + (\tilde{V}^{1/2} f, \tilde{V}^{1/2} g) + V_0(f,g),$$

where $0 \le \tilde{V}(x) := V(x) - V_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ *and* $V_0 := \inf_x V(x) \ge -C_1$*.*

(ii) K *has a compact resolvent.*

(iii) $|X|^q (K+b)^{-1/2}$ *is bounded for* $b > C_1$*.*

2.2 One-boson Hamiltonian

For the one-boson Hamiltonian, we assume that a_{jk}, c and m are real function safisfying:

($\mathbf{b}_1$) $a_{j,k} \in W^{1,\infty}(\mathbb{R}^3)$ satisfies

$$a_0 \mathbf{1} \le [a_{jk}(x)] \le a_1 \mathbf{1}, \quad x \in \mathbb{R}^3$$

with some constants $a_0, a_1 > 0$.

($\mathbf{b}_2$) $0 \le m \in L^\infty(\mathbb{R}^3)$ and

$$\lim_{x \to \infty} m(x) = 0. \tag{2.1}$$

($\mathbf{b}_3$) $c \in W^{2,\infty}(\mathbb{R}^3)$ satisfies

$$c_0 \le c(x) \le c_1, \quad x \in \mathbb{R}^3$$

with some constants $c_0, c_1 > 0$.

Let

$$h_0 := -\frac{1}{c(x)} \sum_{1 \le j,k \le 3} \frac{\partial}{\partial x_j} a_{jk}(x) \frac{\partial}{\partial x_k} \frac{1}{c(x)}.$$

Then h_0 is self-adjoint on $H^2(\mathbb{R}^3)$. For an operator A on $\mathfrak{h}$, the integral kernel is denoted by $A(x,y)$ (if it exists): $(Af)(x) = \int dy A(x,y) f(y)$.

Lemma 2.2. *Assume (b_1)-(b_3). Then:*

(i) $h := h_0 + m^2$ *is self-adjoint on* $H^2(\mathbb{R}^3)$*.*

(ii) $m_{\rm b} = 0$ *and* $\ker h = \{0\}$.

(iii) *For any* $t \geq 0$ *and* $x, y \in \mathbb{R}^3$,

$$0 \leq (e^{-th})(x,y) \leq C(e^{ct\Delta})(x,y),$$

where

$$(e^{T\Delta})(x,y) := (4\pi T)^{-3/2} \exp\left[\frac{-|x-y|^2}{4T}\right]$$

is the three dimensional Gaussian heat kernel.

2.3 Total Hamiltonian

Let

$$H = H_0 + \phi(\omega^{-3/2}\rho_X),$$

where

$$H_0 = K \otimes I + I \otimes d\Gamma(\omega).$$

The following holds:

Proposition 2.3. *Suppose that* (K_1) *-* (K_2) *and* (b_1) *-* (b_3) *hold. Then* H *is a (well-defined) bounded below self-adjoint operator on* $D(H_0)$.

Proof. Note that, for $j = 1, 2 \cdots$ and $\nu_j := (j-4)/4$,

$$\omega^{-j/2} = \gamma_j \int_0^\infty dt e^{-th} t^{\nu_j} \tag{2.2}$$

with some $\gamma_j > 0$. By (iii) of Lemma 2.2 and (2.2), we obtain:

$$\sup_X \|\omega^{-j/2}\rho_X\| \leq C\|w^{-j/2}\hat{\rho}\| < \infty, \quad j = 1, 2,$$

which yields $\phi(\omega^{-1/2}\rho_X)$ is well-defined and infinitesimally small w.r.t. H_0. Hence the proposition follows from the Kato-Rellich Theorem. □

3 Ground state of H

3.1 Gaussian heat kernel bounds

As was shown in (iii) of Lemma 2.2, due to the positivity of $m^2(x)$, the integral kernel $(e^{-th})(x,y)$ of the semigroup generated by $h = h_0 + m^2$ is bounded by the Gaussian heat kernel, from which one obtains the bound for $\|\omega^{-j/2}\rho_X\|$ for $j = 1, 2$. This is, however, not the case for $j = 3$ since $\|w^{-3/2}\hat{\rho}\|$ diverges (see (1.4)). To estimate $\|\omega^{-3/2}\rho_X\|$, we improve the bound on $(e^{-th})(x,y)$ assuming the following decay conditions for $m(x)$. We set $\langle x \rangle = \sqrt{1+|x|^2}$.

(QD) $m \in L^\infty(\mathbb{R}^3)$ satisfies

$$0 \leq m(x) \leq a\langle x \rangle^{-1-\epsilon}.$$

with some $a > 0$ and $\epsilon > 0$.

(SD) $m \in L^\infty(\mathbb{R}^3)$ and

$$v(x) \geq a\langle x\rangle^{-1}$$

with some $a > 0$.

Indeed, the following holds:

Proposition 3.1. *Assume* (b_1) *-* (b_2)*. Then:*

(i) Suppose that (QD) holds. Then

$$(e^{-th})(x, y) \geq D_1(e^{\delta_1 t\Delta})(x, y)$$

with some $D_1, \delta_1 > 0$*.*

(ii) Suppose that (SD) holds and that:

$$\sum_{jk} |\nabla a_{jk}(x)| \leq C\langle x\rangle^{-1} \tag{3.1}$$

with some $C > 0$*. Then*

$$(e^{-th})(x, y) \leq D_2\Phi_\alpha(x, t)\Phi_\alpha(y, t)(e^{\delta_2 t\Delta})(x, y)$$

with some $D_2, \delta_2 > 0$*. Here the function* Φ_α *is given by*

$$\Phi_\alpha(x, t) = \left[\frac{\langle x\rangle^2}{t + \langle x\rangle^2}\right]^\alpha$$

with some $\alpha > 0$*.*

Proof. See [4] for (i) and [3] for (ii). □

Remark 3.2. *If* $c(x) \equiv 1$*, then Proposition 3.1 is due to Semenov [6] and Zhang [7].*

Using the proposition above and the Laplace transform (2.2) we obtain the following, which exhibits the main difference between (QD) and (SD):

Proposition 3.3. *Let* $\rho \geq 0$ *belong to* $C_0^\infty(\mathbb{R}^3)$*.*

(i) Suppose that (QD) holds. Then

$$\rho_X \notin D(\omega^{-3/2}).$$

(ii) Suppose that (SD) holds. Then $\rho_X \in D(\omega^{-3/2})$ *and*

$$\|\omega^{-3/2}\rho_X\|_{\mathfrak{h}}^2 \leq C\langle X\rangle^\mu,$$

for any $\mu > 3/2$ *with some* $C \geq 0$ *depending only on* μ*.*

3.2 Absence of ground state

In the paper [4], we show the absence of ground state employing the functional integral methods and requiring the following:

(K$_3$) $A_{jk}(x)$ belongs to $C^1(\mathbb{R}^3)$ $(j,k=1,2,3)$ and the vector-valued function $b(x) := (b_1(x), b_2(x), b_3(x))$ with $b_j(x) := \frac{1}{2}\sum_{j=1}^{3} \partial_j A_{jk}(x)$ and the matrix-valued function $\sigma(x) := A(x)^{1/2}$ obey the Lipschitz condition:

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq D|x - y|$$

for arbitrary $x, y \in \mathbb{R}^3$ with some constant D, where for a matrix σ, we denote

$$|\sigma| = \sqrt{\sum_{1\leq j,k\leq 3} |\sigma_{jk}|^2}.$$

In this subsection, we assume that (K$_1$) - (K$_3$) hold. Then there is a unique solution $(X_t^x)_{t\geq 0}$ to the stochastic differential equation:

$$dX_t^x = \sigma(X_t^x) \cdot dB_t + b(X_t^x)dt,$$

where $(B_t)_{t\geq 0}$ is the three dimensional Brownian motion. Furthermore we have the Feynman-Kac type formula:

$$(e^{-tK} f)(x) = \mathbb{E}_{\mathcal{W}}\left[e^{-\int_0^t V(X_s^x)ds} f(X_t^x)\right],$$

where $\mathcal{W}$ denotes the Wiener measure. Using the above equation, one can show that K has a unique strictly positive ground state φ which decays as

$$0 < \varphi(x) \leq C e^{-c|x|^{q+1}}. \tag{3.2}$$

We define a unitary transformation U from $\mathfrak{h}$ to the probability space $L^2(\mathbb{R}^3; \varphi(x)^2 dx)$ by

$$U : f \longmapsto \varphi^{-1} f.$$

One can also construct a stochastic process $(X_t)_{t\in\mathbb{R}}$ associated with the generator

$$U(K - \inf \sigma(K))U^{-1}.$$

Then, by a similar argument as in [5] where the Nelson model on the Minkowski spacetime is considered, one can show that H has no ground state if

$$\lim_{T\to+\infty} \mathbb{E}_{\mu_T}\left[\exp\left[-\frac{1}{2}\int_{-T}^{0} ds \int_0^T dt W(X_t, X_s, |t-s|)\right]\right] = 0, \tag{3.3}$$

where μ_T is a probability measure and W is a double potential given by

$$W(x,y,t) = \frac{1}{2}\left(\rho_x, \omega^{-1} e^{-t\omega} \rho_y\right)_{\mathfrak{h}}.$$

It follows from the Gaussian heat kernel bounds (iii) of Lemma 2.2 and (i) of Propositon 3.1 that:

$$d_1 W_\infty(x,y,d_2 t) \leq W(x,y,t) \leq d_3 W_\infty(x,y,d_4 t) \tag{3.4}$$

with some d_1, d_2, d_3 and $d_4 > 0$. Here

$$W_\infty(x, y, t) = \frac{1}{2}\left(\hat{\rho_x}, w^{-1}e^{-tw}\hat{\rho}_y\right)_{\mathfrak{h}}$$

is the double potential for the Nelson model on the Minkowski spacetime that is infrared divergent. The estimates (3.2) and (3.4) yield the equation (3.3) by making use of a modification of [5] (see also [2]). Thus we obtain:

Theorem 3.4. *Suppose that* (K_1) *-* (K_3), (b_1) *-* (b_3) *and (QD) hold. Then* H *has no ground state.*

3.3 Existence of ground state

In this subsection, we assume that (K_1), (K_2) and (b_1) - (b_3) hold. In order to show the existence of ground state, we introduce massive Hamiltonians H_μ by replacing ω in (1.1) with $\omega_\mu := (h + \mu)^{1/2}$. Then, by a standard argument, one can show that H_μ has a normalized ground state Ψ_μ. Since the unit ball in a Hilbert space is compact for the weak topology, there exist a sequence $\mu_j \to 0$ ($j \to \infty$) and a vector Ψ_0 such that Ψ_{μ_j} tends weakly to Ψ_0. To prove that H has a ground state, it suffices to show that $\Psi_0 \neq 0$, which comes from the so-called boson number bound:

$$\sup_\mu \langle \Psi_\mu, N\Psi_\mu\rangle < \infty.$$

Indeed, the left hand side above is controlled by (iii) of Lemma 2.1 and (ii) of Proposition 3.3 if (K_2) is satisfied with $q > 3/2$. Thus we have:

Theorem 3.5. *Suppose that* (K_1) *and* (b_1) *-* (b_3) *hold. Assume in addition that* (3.1), *(SD) and* (K_2) *with* $q > 3/2$. *Then* H *has a ground state.*

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The Relativistic Pauli-Fierz model

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Dedicated to Izumi Ojima and Kei-ichi Ito
on the occasion of their sixtieth birthdays

Abstract

The relativistic Pauli-Fierz model is discussed. The Feynman-Kac type formula with cádlág path is shown and its applications are given.

In Sections 1 and 2 we review the results on the Pauli-Fierz model and in Section 3 we are concerned with the relativistic Pauli-Fierz model.

1 The Pauli-Fierz model

1.1 Definition

We begin with reviewing results on the Pauli-Fierz model. The Pauli-Fierz model describes the minimal interaction between electrons and a quantized radiation field, but electrons are assumed to be low energy and to be governed by a Schrödinger equation.

Let $L^2(\mathbb{R}^3)\otimes\mathcal{F}$ be the total Hilbert space describing the joint electron-photon state vectors. Here $\mathcal{F}=\mathcal{F}(\mathcal{H})$, $\mathcal{H}=L^2(\mathbb{R}^3\times\{\pm\})$, denotes the Boson Fock space over the one-photon Hilbert space $\mathcal{H}$. The elements of the set $\{\pm\}$ account for the fact that a photon is a transversal wave perpendicular to the direction of its propagation, thus it has two components. The Fock vacuum in $\mathcal{F}$ is denoted by Ω. Let $a(f)$ and $a^*(f)$, $f\in\mathcal{H}$, be the annihilation operator and the creation operator, respectively. We also use the identification: $\mathcal{H}\cong L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$ and set $a^\sharp(f,+)=a^\sharp(f\oplus 0)$ and $a^\sharp(f,-)=a^\sharp(0\oplus f)$ for $f\in L^2(\mathbb{R}^3)$. The annihilation operator and the creation operator satisfy the canonical commutation relations:

$$[a(f,i),a^*(g,j)]=\delta_{ij}(\bar{f},g),\quad [a^\sharp(f,i),a^\sharp(g,j)]=0.$$

Let T be a contraction operator on $\mathcal{H}$. Then

$$\Gamma(T) = \bigoplus_{n=0}^{\infty} \underbrace{T \otimes \cdots \otimes T}_{n}$$

is also contraction on $\mathcal{F}$. The second quantization of a self-adjoint operator h on $\mathcal{H}$ is defined by

$$\mathrm{d}\Gamma(h) = \bigoplus_{n=0}^{\infty} \sum_{j=1}^{n} \underbrace{1\!\!1 \otimes \cdots \overset{j}{h} \cdots \otimes 1\!\!1}_{n}.$$

The quantized radiation field with a given cutoff function $\hat{\varphi}$ is defined by

$$A_\mu(x) = \frac{1}{\sqrt{2}} \sum_{j=\pm} \int dk \frac{e_\mu^j(k)}{\sqrt{\omega(k)}} \left(a^*(k,j)\hat{\varphi}(-k)e^{-ik\cdot x} + a(k,j)\hat{\varphi}(k)e^{ik\cdot x} \right) \quad (1.1)$$

for $x \in \mathbb{R}^3$, where $\omega(k) = |k|$. The vectors $e^+(k)$ and $e^-(k)$ are polarization vectors.

The Hamiltonian of one electron is given by the Schrödinger operators with external potential V: $-\frac{1}{2}\Delta + V$, where we assume that the mass of electron is one. On the other hand the free Hamiltonian of the field is defined by $\mathrm{d}\Gamma(\omega)$. Then the decoupled Hamiltonian is

$$(-\frac{1}{2}\Delta + V) \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega).$$

Let $\mathrm{D} = -i\nabla_x$. The Pauli-Fierz Hamiltonian is defined by the minimal coupling of the decoupled Hamiltonian with the quantized radiation field:

$$H = \frac{1}{2}(\mathrm{D} \otimes 1\!\!1 - eA)^2 + V \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega). \quad (1.2)$$

Here e denotes the coupling constant. Throughout we use the following assumptions (1) $\hat{\varphi}(-k) = \overline{\hat{\varphi}(k)}$, (2) $\sqrt{\omega}\hat{\varphi}, \hat{\varphi}/\omega \in L^2(\mathbb{R}^3)$, (3) there exists $0 \le a < 1$ and $0 \le b$ such that

$$\|Vf\| \le a\| - \frac{1}{2}\Delta f\| + b\|f\|$$

for $f \in D(-\frac{1}{2}\Delta)$. We put $D_{\mathrm{PF}} = D\left(-\frac{1}{2}\Delta \otimes 1\!\!1\right) \cap D(1\!\!1 \otimes \mathrm{d}\Gamma(\omega))$. Then H is self-adjoint on D_{PF} and essentially self-adjoint on any core of the decoupled Hamiltonian.

Remark 1.1 *We notice that Pauli-Fierz Hamiltonians with different polarization vectors are isomorphic with each other. Then we fix polarization vectors throughout.*

1.2 Function space

We introduce a function space $(\mathscr{Q},\mu)$ associated with the quantized radiation field and reformulate the Pauli-Fierz Hamiltonian on $L^2(\mathbb{R}^3)\otimes L^2(\mathscr{Q},d\mu)$ instead of $L^2(\mathbb{R}^3)\otimes\mathcal{F}$. In order to have a functional integral representation of $(F,e^{-tH}G)$, $F,G\in L^2(\mathbb{R}^3)\otimes\mathcal{F}$, we construct probability spaces $(\mathscr{Q}_\beta,\Sigma_\beta,\mu_\beta)$, $\beta=0,1$, and the Gaussian random variables $\mathscr{A}_\beta(\mathbf{f})$ indexed by $\mathbf{f}=(f_1,f_2,f_3)\in\oplus^3L^2_{\mathbb{R}}(\mathbb{R}^{3+\beta})$ of mean zero and covariance given by

$$\mathrm{q}_\beta(\mathbf{f},\mathbf{g})=\begin{cases}\frac{1}{2}(\hat{\mathbf{f}},\delta^\perp\hat{\mathbf{g}})_{L^2(\mathbb{R}^3)}, & \beta=0,\\ \frac{1}{2}(\hat{\mathbf{f}},\mathbb{1}\otimes\delta^\perp\hat{\mathbf{g}})_{L^2(\mathbb{R}^4)}, & \beta=1.\end{cases}$$

Note that transversal delta function $\delta^\perp(k)=(\delta_{\mu\nu}-k_\mu k_\nu/|k|^2)_{1\le\mu,\nu\le 1}$ depends only on $k\in\mathbb{R}^3$. In what follows we denote

$$\begin{array}{lll}\text{(Minkowskian)} & \mathscr{A}=\mathscr{A}_0, & \mathscr{Q}=\mathscr{Q}_0,\\ \text{(Euclidean)} & \mathscr{A}_{\mathrm{E}}=\mathscr{A}_1, & \mathscr{Q}_{\mathrm{E}}=\mathscr{Q}_1\end{array}\tag{1.3}$$

using the subscript E for Euclidean objects. Let $\mathscr{A}(x)=\mathscr{A}(\oplus^3\tilde{\varphi}(\cdot-x))$, where $\tilde{\varphi}$ is the inverse Fourier transformation of $\hat{\varphi}/\sqrt{\omega}$. The Pauli-Fierz Hamiltonian in function space is defined by

$$H=\frac{1}{2}(\mathrm{D}\otimes\mathbb{1}-e\mathscr{A}(x))^2+V\otimes\mathbb{1}+\mathbb{1}\otimes\mathrm{d}\Gamma(\omega(\mathrm{D})).\tag{1.4}$$

Let $\mathrm{j}_t:L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^4)$, $t\in\mathbb{R}$, be the family of isometries such that $j_t^*j_s=e^{-|t-s|\omega(\mathrm{D})}$ and we define $\mathrm{J}_t:L^2(\mathscr{Q})\to L^2(\mathscr{Q}_{\mathrm{E}})$ by $\mathrm{J}_t=\Gamma(\mathrm{j}_t)$, and $\mathrm{J}_t^*\mathrm{J}_s=e^{-|t-s|\mathrm{d}\Gamma(\mathrm{D})}$ follows.

Let $(B_t)_{t\ge0}$ denote the three dimensional Brownian motion on the probability space $(\mathscr{X},B(\mathscr{X}),\mathscr{W}^x)$, where $\mathscr{X}=C([0,\infty);\mathbb{R}^3)$ endowed with the locally uniform topology, $B(\mathscr{X})$ is the Borel σ-field on $\mathscr{X}$, and $\mathscr{W}^x$ the Wiener measure. Write $\mathbb{E}^x[\cdots]=\int_{\mathscr{X}}\cdots d\mathscr{W}^x$.

Theorem 1.2 *Let $F,G\in L^2(\mathbb{R}^3)\otimes L^2(\mathscr{Q}_{\mathrm{E}})$. Then*

$$(F,e^{-tH}G)=\int dx\mathbb{E}^x\left[e^{-\int_0^tV(B_s)ds}\left(\mathrm{J}_0F(B_0),e^{-ie\mathscr{A}_{\mathrm{E}}(K_t)}\mathrm{J}_tG(B_t)\right)_{L^2(\mathscr{Q}_{\mathrm{E}})}\right].\tag{1.5}$$

Here $K_t=\bigoplus_{\mu=1}^3\int_0^t\mathrm{j}_s\tilde{\varphi}(\cdot-B_s)dB_s^\mu$ denotes the $\bigoplus^3L^2(\mathbb{R}^4)$-valued stochastic integral.

From this functional integral representation a lot of properties of ground state of H can be derived in the non-perturbative way.

1.3 Translation invariant Pauli-Fierz model

We consider the translation invariant Pauli-Fierz Hamiltonian. This is obtained by setting the external potential V identically zero. Put $\mathrm{P_f}_\mu = \mathrm{d}\Gamma(k_\mu)$, which describes the field momentum. The total momentum operator P on $L^2(\mathbb{R}^3) \otimes \mathcal{F}$ is defined by

$$\mathrm{P}_\mu = \mathrm{D}_\mu \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{P_f}_\mu, \quad \mu = 1, 2, 3. \tag{1.6}$$

We can see that $[H, \mathrm{P}_\mu] = 0$. This leads us to decompose H on the spectrum of the total momentum operator P. The Pauli-Fierz Hamiltonian with a fixed total momentum $H(p)$ is defined by

$$H(p) = \frac{1}{2}(p - \mathrm{P_f} - eA(0))^2 + \mathrm{d}\Gamma(\omega), \quad p \in \mathbb{R}^3, \tag{1.7}$$

with domain $D(H(p)) = D(\mathrm{d}\Gamma(\omega)) \cap D(\mathrm{P_f^2})$. Here $p \in \mathbb{R}^3$ is called the total momentum. Define the unitary operator $\mathscr{T} : L^2(\mathbb{R}^3_x) \otimes \mathcal{F} \to L^2(\mathbb{R}^3_p) \otimes \mathcal{F}$ by

$$(\mathscr{T}\Psi)(p) = \frac{1}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3} e^{-ix\cdot p} e^{ix\cdot \mathrm{P_f}} \Psi(x) dx.$$

Then $H(p)$ is a nonnegative self-adjoint operator and

$$\mathscr{T} \left(\int_{\mathbb{R}^3}^{\oplus} H(p) dp \right) \mathscr{T}^{-1} = H \tag{1.8}$$

holds. As in the previous section, we move to Schrödinger representation from Fock representation to construct a functional integral representation. In that picture $H(p)$ becomes

$$H(p) = \frac{1}{2}(p - \mathrm{d}\Gamma(\mathrm{D}) - e\mathscr{A}(0))^2 + \mathrm{d}\Gamma(\omega(\mathrm{D})), \quad p \in \mathbb{R}^3, \tag{1.9}$$

on $L^2(\mathscr{Q})$. The functional integral representation of $e^{-tH(p)}$ can be also constructed as an application of that of e^{-tH}.

Theorem 1.3 *Let $\Psi, \Phi \in L^2(\mathscr{Q})$. Then*

$$(\Psi, e^{-tH(p)}\Phi) = \mathbb{E}^0 \left[e^{ip\cdot B_t} \left(\mathrm{J}_0 \Psi, e^{-ie\mathscr{A}_\mathrm{E}(K_t)} \mathrm{J}_t e^{-i\mathrm{d}\Gamma(\omega(\mathrm{D}))\cdot B_t} \Phi \right)_{L^2(\mathscr{Q}_\mathrm{E})} \right]. \tag{1.10}$$

1.4 Effective mass

Let $E(p) = \inf \mathrm{Spec}(H(p))$. Introducing a cutoff function with infrared cutoff $\kappa > 0$:

$$\hat{\varphi}(k) = \begin{cases} 0, & |k| < \kappa, \\ (2\pi)^{-3/2} & \kappa \le |k| \le \Lambda, \\ 0, & |k| > \Lambda, \end{cases}$$

we can see that $E(p)$ is analytic in p_μ for sufficiently small e. The effective mass m_{eff} is defined by

$$\frac{1}{m_{\text{eff}}} = \frac{1}{3}\Delta_p E(p)\Big\lceil_{p=0} \tag{1.11}$$

and we have expansion with respect to $\alpha = e^2/4\pi$:

$$m_{\text{eff}} = 1 + \frac{8}{3\pi}\left(\int_\kappa^\Lambda \frac{1}{r+2}dr\right)\alpha + a_2\alpha^2 + \cdots .$$

Then $a_1 \sim \log \Lambda$. The conventional claim is $a_n \sim (\log \Lambda)^n$ but our model does not satisfies this. In particular $a_2 \sim \sqrt{\Lambda}$ as $\Lambda \to \infty$ is shown in [HS05].

When the Hamiltonian includes spin 1/2, then

$$H(p) = \frac{1}{2}(p - \mathrm{P_f} - eA(0))^2 + \mathrm{d}\Gamma(\omega) - \frac{1}{2}\sigma \cdot B(0),$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the 2×2 Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $B(0) = \nabla \times A(0)$ denotes the quantized magnetic field. In this case the effective mass is also computed: $m_{\text{eff}} = 1 + a_1\alpha + a_2\alpha^2 + \cdots$, where

$$a_1 = \frac{8}{3\pi}\left(\int_\kappa^\Lambda \frac{1}{r+2}dr + \int_\kappa^\Lambda \frac{r^2}{(r+2)^3}dr\right) \tag{1.12}$$

and the behavior of a_2 is [HI05, HI07]

$$-C_1 < \lim_{\Lambda\to\infty} a_2/\Lambda^2 < C_2.$$

2 The dipole approximation

2.1 Symplectic structure

We first of all consider the perturbation of the annihilation operator and the creation operator by c-number. Then CCR leaves invariant.

Let $c(f) = a(f) + (g, f)$ and $c^*(f) = a^*(f) + (\bar{g}, f)$. Then $c(f)$ and $c^*(f)$ satisfy CCR and adjoint relation: $c(f)^* = c^*(\bar{f})$. Thus $c(f)$ and $c^*(f)$ satisfy the same CCR and adjoint relation as those of $a(f)$ and $a^*(f)$. Moreover the unitary operator $U = e^{-a^*(\bar{g})+a(g)}$ induces the unitary equivalence:

$$Ua^\sharp(f)U^{-1} = c^\sharp(f)$$

and also transforms the free Hamiltonian $\mathrm{d}\Gamma(\omega)$ to

$$U\mathrm{d}\Gamma(\omega)U^{-1} = \mathrm{d}\Gamma(\omega) + a^*(\omega\bar{g}) + a(\omega g) + (\omega g, g).$$

This can be extended to more complicated transformation $a(f) \mapsto b(f)$ and $a^*(f) \mapsto b^*(f)$ such that $b(f)$ and $b^*(f)$ satisfy the same CCR and adjoint relation as those of $a(f)$ and $a^*(f)$.

Let $B(\mathcal{H})$ denote the set of bounded operators on $\mathcal{H}$. Let

$$J = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix},$$

where $\mathbb{1}$ denotes the identity operator on $\mathcal{H}$. For $S \in B(\mathcal{H})$ we define $\bar{S}f = \overline{S\bar{f}}$. Define

$$\mathrm{Sp}_\infty = \left\{ A = \begin{pmatrix} S & \bar{T} \\ T & \bar{S} \end{pmatrix} \in B(\mathcal{H}) \oplus B(\mathcal{H}) \middle| AJA^* = A^*JA = J \right\}.$$

Sp_∞ is called the infinite dimensional symplectic group. Let $A = \begin{pmatrix} S & \bar{T} \\ T & \bar{S} \end{pmatrix} \in \mathrm{Sp}_\infty$ and we set

$$b(f) = a(Sf) + a(Tf),$$
$$b^*(f) = a^*(\bar{S}f) + a(\bar{T}f).$$

Since $A \in \mathrm{Sp}_\infty$, $\{b(f), b^*(g)\}$ satisfies CCR and $b(f)^* = b^*(\bar{f})$. We furthermore define the subgroup of Sp_∞ by

$$\Sigma_2 = \left\{ A = \begin{pmatrix} S & \bar{T} \\ T & \bar{S} \end{pmatrix} \in \mathrm{Sp}_\infty \middle| T \text{ is a Hilbert Schmidt class} \right\}.$$

It is known ([Ber66, HI03]) that there exists a projective unitary representation[1] $U : \Sigma_2 \mapsto \{\text{unitary on } \mathcal{F}\}$ such that[2]

$$U(A)a^\sharp(f)U(A)^{-1} = b^\sharp(f) \tag{2.1}$$

for all $f \in \mathcal{H}$. Conversely if a unitary operator U satisfies (2.1), then $A \in \Sigma_2$. Using this fact, one can diagonalize quadratic Hamiltonians as

$$U\left(\mathrm{d}\Gamma(\omega) + (a^*(\bar{f}) + a(f))^2\right)U^{-1} = \mathrm{d}\Gamma(\omega) + C$$

with some constant C under some conditions. Furthermore we can see that there exists a unitary operator $\mathscr{U}_p$ such that

$$\mathscr{U}_p\left(\mathrm{d}\Gamma(\omega) + (p + a^*(\bar{f}) + a(f))^2\right)\mathscr{U}_p^{-1} = \mathrm{d}\Gamma(\omega) + C_p,$$

where $p \in \mathbb{R}$ is a parameter. See [Ara90].

2.2 Dipole approximation

Let us now consider the Pauli-Fierz Hamiltonian. We replace $A(x)$ in H with $\mathbb{1} \otimes A(0)$, and the mass of electron is assumed to be m. Then H turns to be

$$H_{\mathrm{dip}} = \frac{1}{2m}(\mathrm{D} \otimes \mathbb{1} - e\mathbb{1} \otimes A(0))^2 + V \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega). \tag{2.2}$$

This is called the dipole approximation. Let $V = 0$. In the dipole approximation the Hamiltonian without external potential is *not* translation invariant but it commutes with the momentum operator of particle. Define $H_{\mathrm{dip}}(p)$ by

$$H_{\mathrm{dip}}(p) = \frac{1}{2m}(p - eA(0))^2 + \mathrm{d}\Gamma(\omega), \quad p \in \mathbb{R}^3,$$

acting on $\mathcal{F}$. Note that

$$\int_{\mathbb{R}^3}^{\oplus} H_{\mathrm{dip}}(p)dp = \frac{1}{2m}(\mathrm{D} \otimes \mathbb{1} - e\mathbb{1} \otimes A(0))^2 + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega).$$

---

[1] $U(A)U(B) = \omega(A,B)U(AB)$ with some phase $\omega(A,B)$.

[2] $U(A)$ is of the form

$$U(A) = \det(\mathbb{1} - K_1^* K_1)^{1/4} e^{-\frac{1}{2}\langle a^*|K_1|a^*\rangle} : e^{-\frac{1}{2}\langle a^*|K_2|a\rangle} : e^{-\frac{1}{2}\langle a|K_3|a\rangle},$$

where $K_1 = TS^{-1}$, $K_2 = 2(\mathbb{1} - (S^{-1})^T)$ and $K_3 = -S^{-1}\bar{T}$. See [HI07].

Taking the dipole approximation makes the model drastically simpler. It is a quadratic operator as mentioned in the previous section. For each $p \in \mathbb{R}^3$ it can be indeed constructed the family of operators

$$\{b^*(f,p), b(f,p), f \in \mathcal{H}\}$$

such that [Ara83]

(1) $b^*(f,p)$ and $b(g,p)$ satisfy CCR;

(2) $b(g,p)^* = b^*(\bar{g},p)$;

(3) $[H_{\rm dip}(p), b(f,p)] = -b(\omega f,p)$ and $[H_{\rm dip}(p), b^*(f,p)] = b^*(\omega f,p)$.

We can also see that there exists a bounded operator S, a Hilbert-Schmidt operator T and a function L_p such that

$$\begin{aligned} b(f,p) &= a(Sf) + a^*(Tf) + (L_p, f), \\ b^*(f,p) &= a(\bar{T}f) + a^*(\bar{S}f) + (\bar{L}_p, f). \end{aligned}$$

Then $A = \begin{pmatrix} S & \bar{T} \\ T & \bar{S} \end{pmatrix} \in \Sigma_2$. There exists a unitary operator $S_p = e^{iep\cdot\phi}$ such that

(1) ϕ is of the form
$$\phi = i\sum_{j=\pm}(a^*(\bar{F}_j, j) - a(F_j, j))$$
with some function F_j,

(2) $U_p = S_p U(A)$ satisfies that
$$U_p a^\sharp(f) U_p^{-1} = b^\sharp(f), \quad U_p H_{\rm dip}(p) U_p^{-1} = {\rm d}\Gamma(\omega) + \frac{1}{2m_{\rm eff}} p^2 + g,$$

(3) constants $m_{\rm eff}$ and g are given by
$$m_{\rm eff} = m + \frac{2}{3}e^2\|\hat{\varphi}/\omega\|^2, \quad g = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^2 t^2 \|\hat{\varphi}/(t^2+\omega^2)\|^2}{m + e^2\frac{2}{3}\|\hat{\varphi}/\sqrt{t^2+\omega^2}\|^2} dt. \quad (2.3)$$

Let $U = e^{ie{\rm D}\otimes\phi}(\mathbb{1}\otimes U(A))$. Then

$$UH_{\rm dip}U^{-1} = -\frac{1}{2m_{\rm eff}}\Delta\otimes\mathbb{1} + \mathbb{1}\otimes{\rm d}\Gamma(\omega) + g + V(\cdot - e\phi), \qquad (2.4)$$

In particular $\inf \mathrm{Spec}(H_{\mathrm{dip}}) = g$ follows when $V = 0$. Let us take a special cutoff function

$$\hat{\varphi}(k) = \begin{cases} (2\pi)^{-3/2} & |k| \le \Lambda, \\ 0, & |k| > \Lambda. \end{cases}$$

Then $g \to \infty$ as $\Lambda \to \infty$. Indeed we can directly see that g has the bound:

$$e^2 \frac{8}{3} \left(\frac{3}{8\pi} \frac{1}{m} \right)^{1/2} \frac{\pi}{2} \le \lim_{\Lambda \to \infty} \frac{g}{\Lambda^{3/2}} \le e^2 \frac{8}{3} \left(\frac{9}{8\pi} \frac{1}{m} \right)^{1/2} \frac{\pi}{2}.$$

From (2.4) it follows that H_{dip} is unitary equivalent to

$$\left(-\frac{1}{2m_{\mathrm{eff}}} \Delta + V \right) \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega) + g + e(V(\cdot - e\phi) - V). \tag{2.5}$$

It is seen that $m_{\mathrm{eff}} \sim e^2$ and $e(V(\cdot - e\phi) - V) \sim e\phi \cdot \nabla V \sim e$ when $\nabla \cdot V \in L^\infty$. Hence heuristically enhanced binding may occur under some conditions, i.e., the existence of ground state of H_{dip} can be shown for sufficiently large $|e|$ even when we do not assume the existence of ground state of $-\frac{1}{2m}\Delta + V$. The enhanced binding arising in H_{dip} is shown in [HS01].

2.3 Lorentz covariant Pauli-Fierz model

Quantization of the electromagnetic field does not cohere with normal postulates such as Lorentz covariance and existence of a positive definite metric. Then we chose to quantize in a manner sacrificing manifest Lorentz covariance; conversely if the electromagnetic field is quantized in a manifestly covariant fashion, the notion of a positive definite metric must be sacrificed and the existence of negative probability arising from the indefinite metric renders invalid a probabilistic interpretation of quantum field theory. One prescription for quantization of the electromagnetic field in a Lorentz covariant manner is the Gupta-Bleuler procedure ([Ble50, Gup50] and [KO79]).

Let us construct $A_\mu(f, x)$, $x \in \mathbb{R}^3$, $\mu = 0, 1, 2, 3$, with test function $f \in L^2(\mathbb{R}^3)$ such that $[A_\mu(f), A_\nu(g)] = -ig_{\mu\nu}(\bar{f}, g)$, where

$$g_{\mu\nu} = \begin{cases} 1 & \mu = \nu = 0, \\ -1, & \mu = \nu = 1, 2, 3, \\ 0, & \mu \ne \nu. \end{cases}$$

Let $\mathcal{F} = \mathcal{F}(\oplus^4 L^2(\mathbb{R}^3))$. The annihilation operator and the creation operator are denoted by $a(f, \mu)$ and $a^*(f, \mu)$, respectively. Define

$$a^\dagger(f, \mu) = \begin{cases} -a^*(f, 0), & \mu = 0, \\ a^*(f, \mu), & \mu = 1, 2, 3. \end{cases}$$

Then it follows that

$$[a(f,\mu),a^\dagger(g,\nu)]=-ig_{\mu\nu}(\bar{f},g).$$

Let $e^j(k)\in\mathbb{R}^3$, $k\in\mathbb{R}^3$, $j=1,2,3$, be unit vectors such that $e^3(k)=k/|k|$, and three vectors $e^1(k)$, $e^2(k)$ and $e^3(k)$ form a right-hand system for each $k\in\mathbb{R}^3$. We fix them. The quantized radiation field, smeared by the test function $f\in L^2(\mathbb{R}^3)$ at the time zero is defined by

$$A_\mu(f,x)=\frac{1}{\sqrt{2}}\sum_{j=1}^3\int dk\frac{e^j_\mu(k)}{\sqrt{\omega(k)}}\left(a^*(k,j)\hat{f}(k)e^{-ikx}+a(k,j)\hat{f}(-k)e^{ikx}\right),$$
$$A_0(f,x)=\frac{1}{\sqrt{2}}\int dk\frac{1}{\sqrt{\omega(k)}}\left(a^*(k,0)\hat{f}(k)e^{-ikx}+a(k,0)\hat{f}(-k)e^{ikx}\right)$$

and their conjugate momenta by

$$\dot{A}_\mu(g,x)=\frac{i}{\sqrt{2}}\sum_{j=1}^3\int dke^j_\mu(k)\sqrt{\omega(k)}\left(a^*(k,j)\hat{g}(k)e^{-ikx}-a(k,j)\hat{g}(-k)e^{ikx}\right),$$
$$\dot{A}_0(g,x)=\frac{i}{\sqrt{2}}\int dk\sqrt{\omega(k)}\left(a^*(k,0)\hat{g}(k)e^{-ikx}-a(k,0)\hat{g}(-k)e^{ikx}\right).$$

Set $A_\mu(f)=A_\mu(f,0)$. Note that $A_\mu(f)$, $\mu=1,2,3$, are symmetric but $A_0(f)$ skew symmetric. We then have commutation relations between A_μ and $\dot{A}_\nu$:

$$[A_\mu(f),\dot{A}_\nu(g)]=-ig_{\mu\nu}(\bar{f},g),\quad \mu,\nu=0,1,2,3,$$

and $[A_\mu(f),A_\nu(g)]=0$, $[\dot{A}_\mu(f),\dot{A}_\nu(g)]=0$. Then the Lorentz covariant Pauli-Fierz Hamiltonian with the dipole approximation is defined by

$$H=\frac{1}{2}(\mathrm{D}\otimes\mathbb{1}-e\mathbb{1}\otimes A(0))^2+\mathbb{1}\otimes\mathrm{d}\Gamma(\omega)+e\mathbb{1}\otimes A_0(0).$$

Take the fiber p. Then we define

$$H(p)=\frac{1}{2}(p-eA(0))^2+\mathrm{d}\Gamma(\omega)+eA_0(0).$$

This Hamiltonian is not self-adjoint on $\mathcal{F}$, since A_0 is skew symmetric. We introduce the indefinite scalar product on $\mathcal{F}$ by $(F|G)=(F,\Gamma[g]G)$, where $[g]=[g_{\mu\nu}]:\oplus^4L^2(\mathbb{R}^3)\to\oplus^4L^2(\mathbb{R}^3)$. Then $H(p)$ is symmetric with respect to $(\cdot|\cdot)$.

In [HS09] we prove the asymptotic completeness of $H(p)$ based on the LSZ method, and characterize the physical subspace of $H(p)$.

3 Relativistic Pauli-Fierz model

In quantum mechanics the relativistic Schrödinger operator is defined by

$$H_{\mathrm{R}}(a) = \sqrt{(p-a)^2 + m^2} - m + V.$$

In this section the analogue version of the Pauli-Fierz model is defined and its functional integral representation is given. We would like to study the spectral property, effective mass and enhanced binding of the relativistic Pauli-Fierz Hamiltonian as well as the standard Pauli-Fierz Hamiltonian mentioned in the previous section. Some spectral property of the relativistic Pauli-Fierz model is studied in e.g., [HS10, KMS09, MS09].

In this section we overview the relativistic Pauli-Fierz Hamiltonian and the detail [Hir10] will be published somewhere.

3.1 Definition

The so-called relativistic Pauli-Fierz Hamiltonian is defined by

$$H_{\mathrm{R}} = \sqrt{(\mathrm{D} \otimes \mathbb{1} - eA)^2 + m^2} - m + V \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega) \tag{3.1}$$

on $L^2(\mathbb{R}^3) \otimes \mathcal{F}$ as a self-adjoint operator.

First of all we have to define H_{R}. It is however not trivial to do it, since H_{R} has non-local operator $\sqrt{(\mathrm{D} \otimes \mathbb{1} - eA)^2 + m^2}$. Although one standard way to define $(\mathrm{D} \otimes \mathbb{1} - eA)^2 + m^2$ as a self-adjoint operator is to take the self-adjoint operator associated with the quadratic form:

$$F, G \mapsto \frac{1}{2} \sum_{\mu=1}^{3} ((\mathrm{D} \otimes \mathbb{1} - eA)_\mu F, (\mathrm{D} \otimes \mathbb{1} - eA)_\mu G) + m^2 (F, G),$$

we do not take it. Instead of this we will find a core of $(\mathrm{D} \otimes \mathbb{1} - eA)^2 + m^2$ by using a functional integration. Let

$$L_t = \oplus_{\mu=1}^{3} \int_0^t \tilde{\varphi}(\cdot - B_s) dB_s^\mu.$$

Then we can see that $\int dx \mathbb{E}^x \left[(F(B_0), e^{-ie\mathscr{A}(L_t)} G(B_t)) \right]$ defines the semigroup generated by a self-adjoint operator K such that

$$(F, e^{-tK} G) = \int dx \mathbb{E}^x \left[(F(B_0), e^{-ie\mathscr{A}(L_t)} G(B_t)) \right], \tag{3.2}$$

and see that

$$K \supset \frac{1}{2}(\mathrm{D} \otimes 1\!\!1 - e\mathscr{A})^2 \lceil_{D_{\mathrm{PF}}}. \tag{3.3}$$

Let $N = 1\!\!1 \otimes \mathrm{d}\Gamma(1\!\!1)$ be the number operator and $\mathscr{D} = D(\Delta) \cap \cap_{n=1}^{\infty} D(N^n)$.

Lemma 3.1 *Suppose that $\omega^{3/2}\hat{\varphi} \in L^2(\mathbb{R}^3)$. Then $\frac{1}{2}(\mathrm{D} \otimes 1\!\!1 - e\mathscr{A})^2 \lceil_{\mathscr{D}}$ is essentially self-adjoint.*

Proof: By using (3.2) we will show that e^{-tK} leaves $\mathscr{D}$ invariant. First of all it can be proven that $e^{-tK}\mathscr{D} \subset D(\Delta)$. Next let us see that $e^{-tK}\mathscr{D} \subset \cap_{n=1}^{\infty} D(N^n)$. Let $z \in \mathbb{N}$ and $F, G \in D(N^\alpha)$. We have

$$(N^\alpha F, e^{-tK}G) = \int dx \mathbb{E}^x \left[(N^\alpha F(B_0), e^{-ie\mathscr{A}(L_t)} G(B_t)) \right]. \tag{3.4}$$

Let $\Pi(f) = i[N, A(f)]$. Note that

$$e^{ie\mathscr{A}(L_t)} N e^{-ie\mathscr{A}(L_t)} = N - e\Pi(L_t) + \frac{e^2}{2}\|L_t\|^2 \tag{3.5}$$

and then

$$\begin{aligned} &(N^\alpha F, e^{-tK}G) \\ &= \int dx \mathbb{E}^x \left[(F(B_0), e^{-ie\mathscr{A}(L_t)} \left(N - e\Pi(L_t) + \frac{e^2}{2}\|L_t\|^2 \right)^\alpha G(B_t)) \right]. \end{aligned} \tag{3.6}$$

By the Burkholder-Davis-Gundy type inequality,

$$\mathbb{E}^x \left[\left\| \int_0^t \tilde{\varphi}(\cdot - B_s) dB_s^\mu \right\|^{2z} \right] \leq \frac{(2z)!}{2^\alpha} t^\alpha \|\hat{\varphi}\|^{2z}.$$

we can see that

$$\int dx \mathbb{E}^x \left[\left\| \left(N - e\Pi(L_t) + \frac{e^2}{2}\|L_t\|^2 \right)^\alpha F(B_t) \right\|^2 \right] \leq C_\alpha^2 \|(N+1)^\alpha F\|^2 \tag{3.7}$$

with some constant C_α. Combining (3.6) and (3.7) we have

$$|(N^\alpha F, e^{-tK}G)| \leq C_\alpha \|F\| \|(N+1)F\|. \tag{3.8}$$

This implies $e^{-tK} \cap_{n=1}^{\infty} D(N^n) \subset \cap_{n=1}^{\infty} D(N^n)$ and $e^{-tK}\mathscr{D} \subset \mathscr{D}$ follows. Hence K is essential self-adjoint on $\mathscr{D}$. $\square$

We denote the self-adjoint extension of $K\lceil_{\mathscr{D}}$ by the same symbol K for simplicity, and $\sqrt{2K+m^2}$ by the spectral resolution of K. Let $(T_t)_{t\geq 0}$ be the subordinator on a probability space $(\mathcal{T}, \mathcal{B}_{\mathcal{T}}, \nu)$ such that

$$\mathbb{E}_\nu^0[e^{-uT_t}] = \exp\left(-t(\sqrt{2u+m^2}-m)\right), \quad u \geq 0.$$

Since

$$(F, e^{-t(\sqrt{2K+m^2}-m)}G) = \mathbb{E}_\nu^0[(F, e^{-T_t K}G)],$$

we immediately have

$$(F, e^{-t(\sqrt{2K+m^2}-m)}G) = \int ds \mathbb{E}^{x,0}\left[(F(B_0), e^{-ie\mathscr{A}(L_{T_t})}G(B_{T_t}))\right]. \tag{3.9}$$

From (3.9) we directly see the diamagnetic inequality:

$$|(F, e^{-t(\sqrt{2K+m^2}-m)}G)| \leq (|F|, e^{-t(\sqrt{-\Delta+m^2}-m)}|G|). \tag{3.10}$$

From the diamagnetic inequality we have:

(1) Suppose that V is $\sqrt{-\Delta+m^2}-m$ -form bounded with a relative bound a. Then $|V|$ is also K-form bounded with a relative bound smaller than a.

(2) Suppose that V is relatively bounded with respect to $\sqrt{-\Delta+m^2}-m$ with a relative bound a, then V is also relatively bounded with respect to K with a relative bound a.

Let $\omega^{3/2}\hat{\varphi} \in L^2(\mathbb{R}^3)$. Suppose that $V = V_+ - V_-$ satisfies that V_- is relatively form bounded with respect to $\sqrt{-\Delta+m^2}-m$ and $D(V_+) \supset D(\Delta)$. Then H_{R} is defined by

$$H_{\mathrm{R}} = \sqrt{2K+m^2} - m \dotplus V_+ \otimes \mathbb{1} \dot{-} V_- \otimes \mathbb{1} \dotplus \mathbb{1} \otimes \mathrm{d}\Gamma(\omega). \tag{3.11}$$

3.2 Functional integration

Now we will construct the functional integral representation of $e^{-tH_{\mathrm{R}}}$ through the Trotter product formula. We fix $t > 0$. Let $t_j = tj/2^n$, $j = 0, ..., 2^n$. Define $L^2(\mathbb{R}^4)$-valued stochastic process S_n^μ on $\mathscr{X} \times \mathcal{T}$ by

$$S_n^\mu = \sum_{j=1}^{2^n} \int_{T_{t_{j-1}}}^{T_{t_j}} \mathrm{j}_{t_{j-1}} f(\cdot - B_s) dB_s^\mu, \tag{3.12}$$

where $f \in L^2(\mathbb{R}^3)$ and $\int_{T_{t_{j-1}}}^{T_{t_j}} \cdots dB_s^\mu = \int_T^S \cdots dB_s^\mu$ evaluated at $T = T_{t_{j-1}}$ and $S = T_{t_j}$.

Lemma 3.2 *$\{S_n^\mu\}_{n=1}^\infty$ is a Cauchy sequence in $L^2(\mathscr{X}\times\mathcal{T};\mathscr{W}^x\otimes\nu)\otimes L^2(\mathbb{R}^4)$.*

Proof: Set S_n for S_n^μ for simplicity. We can directly see that

$$\int dx\mathbb{E}^{x,0}[\|S_{n+1}-S_n\|^2]\le\sum_{j=1}^{2^n}\int_{(2j-1)t/2^n}^{2jt/2^n}2\mathbb{E}^{0,0}[\|f(\cdot-x)\|^2]\frac{t}{2^{n+1}}.$$

Hence we have

$$\left(\int dx\mathbb{E}^{x,0}[\|S_m-S_n\|^2]\|\right)^{1/2}\le\|f\|\sum_{j=n+1}^{m}\frac{t}{2^{(j+1)/2}}$$

and it follows that S_n is a Cauchy sequence. $\square$

We define the $L^2(\mathbb{R}^4)$-valued stochastic process $\int_0^{T_t}\mathrm{j}_{(T^{-1})_s}f(\cdot-B_s)dB_s^\mu$ on the probability space $(\mathscr{X}\times\mathcal{T},B(\mathscr{X})\times B_{\mathcal{T}},\mathscr{W}^x\otimes\nu)$ by the strong limit of S_n^μ:

$$\int_0^{T_t}\mathrm{j}_{(T^{-1})_s}f(\cdot-B_s)dB_s^\mu=\mathrm{s}-\lim_{n\to\infty}S_n^\mu. \tag{3.13}$$

Remark 3.3 *We give a remark with respect to (3.13). The subordinator $[0,\infty)\ni t\mapsto T_t\in[0,\infty)$ is monotonously increasing, but not injective. So the inverse T^{-1} can not be defined. (3.13) is a formal description of the limit of S_n^μ.*

Theorem 3.4 *Let $\omega^{3/2}\hat{\varphi}\in L^2(\mathbb{R}^3)$. Suppose that $V=V_+-V_-$ satisfies that V_- is relatively form bounded with respect to $\sqrt{-\Delta+m^2}-m$ and $D(V_+)\supset D(\Delta)$. Then*

$$(F,e^{-tH_{\mathrm{R}}}G)=\int dx\mathbb{E}^{x,0}\left[e^{-\int_0^t V(B_{T_s})ds}(\overline{\mathrm{J}_0F(B_0)},e^{-ie\mathscr{A}_{\mathrm{E}}(K_t^{\mathrm{rel}})}\mathrm{J}_tG(B_{T_t}))\right], \tag{3.14}$$

where $K_t^{\mathrm{rel}}=\oplus_{\mu=1}^3\int_0^{T_t}\mathrm{j}_{(T^{-1})_s}\tilde{\varphi}(\cdot-B_s)dB_s^\mu$.

Proof: We set $V=0$ for simplicity. By the Trotter product formula we have

$$(F,e^{-tH_{\mathrm{R}}}G)=\lim_{n\to\infty}\left(F,\left(e^{-t/2^nK}e^{-t/2^n\mathrm{d}\Gamma(\omega)}\right)^{2^n}G\right).$$

By the Markov property of $\mathrm{E}_t=\mathrm{J}_t^*\mathrm{J}_t$ the right hand side above is equal to

$$\lim_{n\to\infty}\left(\mathrm{J}_0F,\left(\prod_{j=0}^{2^n}e^{-t/2^n(\sqrt{(\mathrm{D}\otimes\mathbb{1}-e\mathscr{A}_{\mathrm{E}}(\mathrm{j}_{tj/2^n}\tilde{\varphi}))^2+m^2}-m)}\right)\mathrm{J}_tG\right).$$

Thus we have

$$(F, e^{-tH_{\rm R}}G) = \lim_{n\to\infty}\int dx\mathbb{E}^{x,0}\left[(\overline{{\rm J}_0F(B_0)}, e^{-ie\mathscr{A}_{\rm E}(K_t(n))}{\rm J}_tG(B_{T_t}))\right],$$

where

$$K_t(n) = \sum_{j=1}^{2^n}\int_{T_{t(j-1)/2^n}}^{T_{tj/2^n}} {\rm j}_{t(j-1)/2^n}\tilde{\varphi}(\cdot - B_s)dB_s^\mu.$$

By Lemma 3.2 and a limiting argument we can show the theorem for $V = 0$. In the case of $H_{\rm R}$ with a bounded continuous V, we can also prove the theorem by the Trotter product formula. It can be also extended to $V = V_+ - V_-$ such that V_- is relatively form bounded with respect to $\sqrt{-\Delta + m^2} - m$ and $D(V_+) \supset D(\Delta)$ by a limiting argument. $\square$

By using this functional integral representation we can see similar results to those of H.

Corollary 3.5 *Suppose the same assumptions as Theorem 3.4.*

(1) Let $E(e) = \inf {\rm Spec}(H_{\rm R})$. Then

$$|(F, e^{-tH_{\rm R}}G)| \leq (|F|, e^{-t(\sqrt{-\Delta+m^2}-m+{\rm d}\Gamma(\omega))}|G|). \tag{3.15}$$

In particular $E(0) \leq E(e)$.

(2) Let $\mathfrak{S} = e^{-i(\pi/2)N}$. Then $\mathfrak{S}e^{-tH_{\rm R}}\mathfrak{S}^{-1}$ is positivity improving. In particular the ground state of $H_{\rm R}$ is unique.

3.3 Translation invariant relativistic Pauli-Fierz Hamiltonian

In the case of the relativistic Pauli-Fierz Hamiltonian with $V = 0$, we can also show similar results to those of H by using the functional integral representation of $e^{-tH_{\rm R}}$, but we omit the detail. We give only the results. The relativistic Pauli-Fierz Hamiltonian with a fixed total momentum p, $H_{\rm R}(p)$, is defined by

$$H_{\rm R}(p) = \sqrt{(p - {\rm P_f} - eA(0))^2 + m^2} - m + {\rm d}\Gamma(\omega), \quad p \in \mathbb{R}^3, \tag{3.16}$$

with domain $D(H_{\rm R}(p)) = D({\rm d}\Gamma(\omega)) \cap D(|{\rm P_f}|)$.

Theorem 3.6 *Suppose $\omega^{3/2}\hat{\varphi} \in L^2(\mathbb{R}^3)$.*

(1) $H_{\rm R}(p)$ is essentially self-adjoint and $H_{\rm R} \cong \int_{\mathbb{R}^3}^{\oplus} H_{\rm R}(p)dp$.

(2) Let $\Psi, \Phi \in \mathscr{Q}$. Then

$$(\Psi, e^{-tH_{\mathrm{R}}(p)}\Phi) = \mathbb{E}^{0,0}\left[e^{ip\cdot B_{T_t}}\left(\mathrm{J}_0\Psi, e^{-ie\mathscr{A}_{\mathrm{E}}(K_t^{\mathrm{rel}})}\mathrm{J}_t e^{-i\mathrm{P}_{\mathrm{f}}\cdot B_{T_t}}\Phi\right)\right]. \quad (3.17)$$

From this functional integral representation we immediately have corollaries. Let $E(p) = \inf \mathrm{Spec}(H_{\mathrm{R}}(p))$.

Corollary 3.7 *(1) It follows that*

$$|(\Psi, e^{-tH_{\mathrm{R}}(p)}\Phi)| \leq (|\Psi|, e^{-t(\sqrt{(p-\mathrm{P}_{\mathrm{f}})^2+m^2}-m+\mathrm{d}\Gamma(\omega))}|\Phi|). \quad (3.18)$$

(2) $\mathfrak{S}^{-1}e^{-tH_{\mathrm{R}}(0)}\mathfrak{S}$ is positivity improving. In particular

(a) $E(0) \leq E(p)$,

(b) the ground state of $H_{\mathrm{R}}(0)$ is unique if it exists.

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Critical Behavior of Stochastic Geometric Models and the Lace Expansion

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Abstract

Rigorous results on critical behavior of stochastic geometric models (self-avoiding walk, lattice tress and animals, and percolation) in high dimensions are presented. I try to explain why different models exhibit different upper critical dimensions. I also explain the main tool of the analysis, *lace expansion*.

Contents:

1 Models and Results: overview of critical phenomena of stochastic geometric models

We begin by defining models we consider, and by reviewing known facts about their critical behaviour. The models we consider exhibit critial behaviour which is similar to those in classical spin systems; but we can observe model-specific properties (such as different values for critical dimensions). We rstrict ourselves to models in high dimensions[1].

1.1 About the lattice

In this article, we always consider models on the d-dimensional hypercubic lattice:
$\mathbb{Z}^d = \{(x_1, x_2, \ldots, x_d) \ \ x_j \in \mathbb{Z}\}$. An element $x \in \mathbb{Z}^d$ is called a *site*; a pair of distinct sites are called a *bond*. We denote the set of all bonds by Ω.

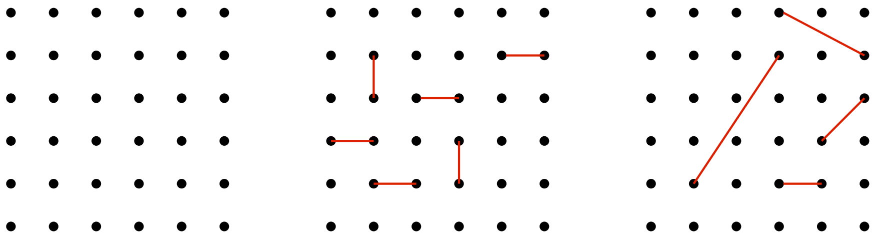
[*] On the occasion of Professor Ito and Professor Ojima's 60^{th} birthdays

[1] Recently, our understanding of critical phenomena in *two-dimensional* systems has improved quite a bit, thanks to the development of theory of SLE (stochastic Loewner evolution, or, Schramm-Loewner evolution). Although these developments are quite important, I restrict myself to critical behaviours in high dimensional systems.

As sets of bonds, we mainly consider two cases[2] :

- Nearest-neighbour (n.n.) model: $\Omega_{\rm nn} := \big\{\{x,y\}\,|\,x,y\in\mathbb{Z}^d,\ |x-y|=1\big\}$
- Spread-out model: $\Omega_L := \big\{\{x,y\}\,|\,x,y\in\mathbb{Z}^d,\ 0<|x-y|\le L\big\}$, where L is a (large) positive integer.

Examples are illustrated in the following figure:



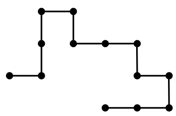
2-dim lattice   n.n. bonds   spread-out bonds

1.2 Self-Avoiding Walk (SAW)

An n-step *self-avoiding walk* (SAW) is (see [14] for details)

- a set of ordered $(n+1)$ sites: $\omega=(\omega(0),\omega(1),\dots,\omega(n))$, with $\omega(j)\in\mathbb{Z}^d$,
- where pairs of consecutive sites are bonds: $\{\omega(j),\omega(j+1)\}\in\Omega$ $(0\le j<n)$,
- and which satisfies *self-avoiding constraint*: $\omega(i)\ne\omega(j)$ for $i\ne j$.

We denote by $|\omega|$ the number of steps of the walk ω. An example of 12-step SAW is here:



n=12

Note that the self-avoiding constraint is essential (otherwise, the problem would be trivial).

Quantities of interest. We are interested in the following quantities:

- $c_n(x,y) := \#\{\omega : x\to y, |\omega|=n, \text{SAW}\}$: the number of n-step SAW's from x to y.
- $c_n := \#\{\omega : 0\to\bullet, |\omega|=n, \text{SAW}\}$: the number of n-step SAW's from 0 to anywhere. Here and in the following, I use $\bullet$ to denote arbitrary point on the lattice.
- $\ell_n := \big\langle|\omega(n)|^2\big\rangle_n^{1/2}$: *mean-square displacement.*
  Here, $\langle\cdots\rangle_n$ is the expectation with respect to the uniform measure on all the n-step SAW's starting from the origin. So in the above, $\omega(n)$ denotes the endpoint of ω, which starts from the origin. Therefore, the quantity is roughly the average distance between two endpoints of n-step SAW's.

We also consider following quantities (first two are generating functions of the above quantities). p is a parameter, and these are defined for[3] $|p|<1/\mu$ —— μ itself is defined in the following.

[2]The reason why we consider these two types of bond models is the following: To our great regret, the current technology to analyze stochastic geometic models are limited. In particular, we can obtain almost complete information on the spread-out models, but not for the nearest-neighbour models. However, the common wisdom strongly suggests that these two types of models belong to the same universality class. Therefore, rigorous results on spread-out models can be considered as strog indications that similar critical behaviour would be observed for the nearest-neighbour models as well. Of course it is quite important to keep in mind that the above indications have not been proved rigorously.

[3]With a suitable care, we can extend these definitions to $p=1/\mu$, but we will not go into details

- $G_p(x,y) := \sum_n p^n\, c_n(x,y) = \sum_{\omega: x\to y} p^{|\omega|}$: *two-point function* from x to y

- $\chi_p := \sum_x G_p(0,x) = \sum_n p^n\, c_n = \sum_{\omega:0\to\bullet} p^{|\omega|}$: *susceptibility*

- $\xi_p := -\lim_{n\to\infty} \dfrac{n}{\log G_p(0, ne_1)}$: *correlation length*
  (e_1 is the unit vector in the first coordinate direction)

Connective Constant. The limit $\mu := \lim_{n\to\infty} (c_n)^{1/n}$ exists and is called a *connective constant.* We write $p_c := 1/\mu$.

(Why the above limit exists?) If we cut an $(m+n)$-step SAW ω after its n^{th}-step, we get an n-step SAW ω_1 and an m-step SAW ω_2. Moreover, different ω's yield different pairs (ω_1, ω_2). Therefore $c_{n+m} \le c_n\, c_m$ holds, and taking the log of both sides, we get $\log c_{m+n} \le \log c_n + \log c_m$. The limit $\lim_{n\to\infty} \dfrac{\log c_n}{n}$ is now seen to exist by a standard *subadditivity argument.*

Critical phenomena. For $d > 1$, it has been proven:

- The connective constant μ is positive (in fact, it is easy to see that $\mu \ge d$). So c_n diverges (as $n \uparrow \infty$) like $c_n \approx \mu^n$.

- For $p < p_c$, the *mass* $m_p = 1/\xi_p$ is positive so that $G_p(0,x) \approx \exp(-m_p|x|)$. That is, the two-point function decays exponentially in $|x|$.

- χ_p and $\xi_p = 1/m_p$ diverge as $p \uparrow p_c$.

Singular behavior around $p \approx p_c$ is called *critical behavior.*

Expected Details of Critical Behaviour. Like spin models in statistical mechanics, it has been expected:

- *Critical exponents* $\gamma, \nu, \eta, \ldots$ exist and satisfy

$$c_n \sim A\mu^n\, n^{\ -1}, \qquad (\ell_n)^2 \sim D\, n^{2\nu}, \qquad (n \uparrow \infty) \tag{1.1}$$
$$\chi_p \sim A'(p_c - p)^{-\ } , \qquad \xi_p \sim D'(p_c - p)^{-\nu}, \qquad (p \uparrow p_c) \tag{1.2}$$
$$G_{p_c}(0,x) \sim C|x|^{-(d-2+\eta)} \qquad (|x| \uparrow \infty) \tag{1.3}$$

  for some constants A, D, A', D', C.

- Critical exponents are *universal*, in the sense that they do not depend on details of the model considered (e.g. their values are the same for the nearest-neighbour model and for the spread-out model). μ, A, D, A', D', C are not expected to be universal.

- *Scaling relations*, such as $(2-\eta)\nu = \gamma$ hold.

- There exists an *upper critical dimension*, d_c, above which critical exponents take on "simple" values (called *mean-field values*). It is expected that $d_c = 4$ for SAW.

Expected values for critical exponents for SAW are summarized in the following table.

d	γ	ν	η
2	$\frac{43}{32}$	$\frac{3}{4}$	$\frac{5}{24}$
3	$1.162\dots$	$0.59\dots$	0.03 (?)
4	1 (log)	$\frac{1}{2}$ (log)	0
> 4	1	$\frac{1}{2}$	0

Now the important question is how much of the above *expectations* have been rigorously proven.

Rigorous results in high dimensions. Rigorous results obtained so far (in high dimensions) can be summarized as follows. First, on its critical behaviour,

Theorem 1.1 ([3, 16, 12, 8, 7]) *For n.n. SAW in $d \geq 5$, and for spread-out SAW in $d > 4$ and $L \gg 1$, we have $\gamma = 1, \nu = 1/2, \eta = 0$. That is, there are constants μ, A, D, A', D', C such that*

$$c_n \sim A\mu^n, \qquad \ell_n \sim \sqrt{D}n^{1/2} \qquad (n \uparrow \infty) \tag{1.4}$$

$$\chi_p \sim A'(p_c - p)^{-1}, \qquad \xi_p \sim D'(p_c - p)^{-1/2} \qquad (p \uparrow p_c) \tag{1.5}$$

$$G_{p_c}(x) \sim \frac{C}{|x|^{d-2}} \qquad (|x| \uparrow \infty) \tag{1.6}$$

Next, on its scaling (continuum) limit, we have

Theorem 1.2 ([17, 12]) *For n.n. SAW in $d \geq 5$, and for spread-out SAW in $d > 4$ and $L \gg 1$, their scaling limits are* Brownian Motion. *More precisely, for an n-step SAW ω which starts from the origin, define a piecewise linear curve $X_n(t)$ as the linear interpolation of*

$$X_n\left(\frac{j}{n}\right) := \frac{1}{\sqrt{n}}\omega(j) \qquad (j = 0, 1, 2, \dots, n). \tag{1.7}$$

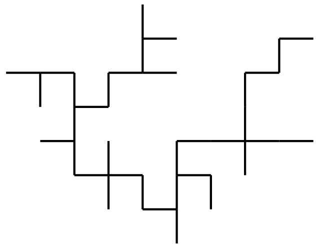
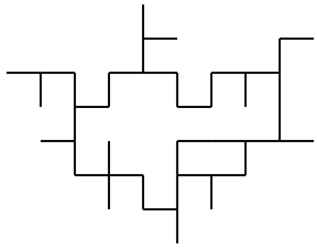
(Note that we scale the space by $n^{1/2}$*.) Then, X_n converges in distribution to a Brownian Motion with diffusion constant D. [D appears in Theorem 1.1].*

1.3 Lattice Trees and Animals

Definitions of lattice trees and animals are quite simple.

- Lattice animal (LA) is just a connected set of bonds.
- Lattice tree (LT) is a lattice animal, without cycles (i.e. LT is a tree-like object).

Examples of LT and LA are shown below:

a lattice tree   a lattice animal

Quantities of Interest.

- a_n, t_n: number of LA (LT)'s with n bonds, containing the origin
- ℓ_n: *radius of gyration* of LA or LT of size n.
- $G_p(x,y) := \sum_{T \ni x,y} p^{|T|}$: *two-point function*, where $|T|$ denotes the number of bonds in T.

As for SAW, by subadditivity argument, there exists a constant λ such that $\lim_{n\to\infty} (t_n)^{1/n} = \lambda$. Expected critical behaviour is:

$$t_n \sim A'\lambda^n n^{\;\;-2}, \qquad \ell_n \sim D' n^{\nu} \qquad (n \uparrow \infty)$$

with critical exponents γ, ν.

Rigorous results can be summarized as follows. First, on its critical behaviour, we have

Theorem 1.3 ([10, 11]) *For n.n. LTLA in $d \gg 1$, and for spread-out LTLA in $d > 8$ and $L \gg 1$, we have $\gamma = 1/2, \nu = 1/4$*

Next, on its scaling (continuum) limit,

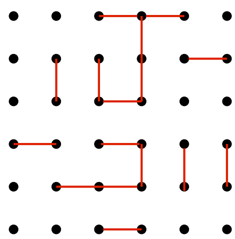
Theorem 1.4 ([6, 5]) *Distribution of scaled LT converges to that of* Integrated Super-Brownian Excursion (ISE). *Here scaling means shrinking a LT by (size)$^{1/4}$ in space.*

1.4 Percolation

Assign independent identically distributed (i.i.d.) random variable n_b on each bond b, according to ($p \in [0,1]$ is a parameter)

$$n_b = \begin{cases} 1 & \text{with probability } \; p \qquad \text{(bond } \textit{occupied}\text{)} \\ 0 & \text{with probability } \; 1-p \quad \text{(bond } \textit{vacant}\text{)} \end{cases}$$

Given a bond configuration, we say $x \longleftrightarrow y$ (*x and y are connected*), if there exists a path of occupied bonds connecting x and y. We denote by $C(x)$ the *connected cluster* of x (= the set of sites connected to x).



red = occupied bonds

Quantities of interest:

- $G_p(x,y) = \mathbb{P}[x \longleftrightarrow y]$: *two-point function*
- $\chi_p := \sum_y G_p(0,y) = \langle |C(0)| \rangle_p$: *susceptibility*
- $\xi_p := -\lim_{n\to\infty} \dfrac{n}{\log G_p(0, ne_1)}$: *correlation length.* $G_p(0,x) \approx const\, e^{-|x|/\xi_p}$
- $\theta_p := \mathbb{P}[|C(0)| = \infty]$: *percolation density*

Critical Phenomena. It has been proven that there exists $p_c > 0$, the *critical point*, such that[4]

- For $p < p_c$, the model is in its *subcritical phase.* $\chi_p < \infty$ and $\xi_p < \infty$, and $\theta_p = 0$.
- For $p > p_c$, the model is in its *supercritical phase.* $\chi_p = \xi_p = \infty$, and $\theta_p > 0$.

Crossover of these two occur at the critical point, p_c. In particular, χ_p and ξ_p diverge as $p \uparrow p_c$. Moreover, we expect *power laws* (as for SAW):

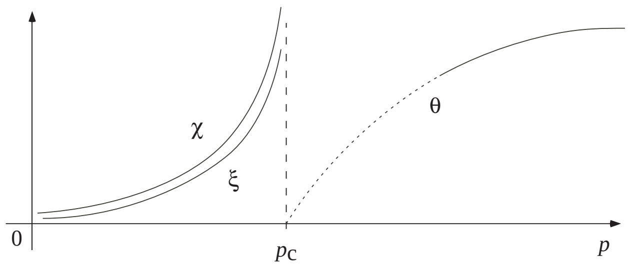
$$\chi_p \approx (p_c - p)^{-}\ , \qquad \xi_p \approx (p_c - p)^{-\nu}, \qquad \frac{\langle |C(0)|^2 \rangle_p}{\langle |C(0)| \rangle_p} \approx (p_c - p)^{-\Delta} \qquad (p \uparrow p_c)$$

$$\theta_p \approx (p - p_c)^{\beta} \qquad (p \downarrow p_c)$$

$$G_{p_c}(x) \approx |x|^{-(d-2+\eta)} \qquad (p = p_c,\ |x| \uparrow \infty)$$

$$\mathbb{P}\left[\, |C(0)| = n \,\right] \approx n^{-1-1/\delta} \qquad (p = p_c,\ n \uparrow \infty)$$

And, we expect similar properties (universality, existence of upper critical dimension, scaling, etc) for percolation as we do for SAW. Expected critical behaviour is summarized schematically in the following figure[5].



Rigorous results in high dimensions. Rigorous results in high dimensions can be summarized as follows. First, as for its critical behaviour,

Theorem 1.5 ([9, 2, 8, 7]) *For n.n. percolation in $d \gg 1$, and for spread-out percolation in $d > 6$ and $L \gg 1$, we have $\gamma = 1, \nu = 1/2, \beta = 1, \eta = 0, \delta = \Delta = 2$.*

The above result should be contrasted with the following result.

---

[4] The existence of two phases can be proved easily. More precisely, it is not difficult to prove that $\chi_p, \xi_p < \infty$ for $0 \le p \ll 1$, and that $\theta_p > 0$ for $p \approx 1$. However, proving the absence of an intermediate phase (that is, proving that θ_p becomes positive as soon as χ_p and ξ_p becomes infinite) is quite nontrivial, and has been first proven in [1] and [15]

[5] Whether θ_p is continuous in p at $p = p_c$ or not (in low dimensions) is a long-standing open problem

Theorem 1.6 ([4, 18]) *For percolations in $d < 6$ dimensions, it is* impossible *for all the critical exponents to take on their* mean-field *values. So, $d_c \geq 6$.*

Theorem 1.5 and theorem 1.6 strongly suggest that the upper critical dimension for percolation is six. (These theorems estabilish $d_c = 6$ for spread-out percolation. For nearest-neighbour percolation, we do not have a complete proof yet.)

To our regret, scaling limit of critical percolation clusters has not been identified rigorously. However, we expect:

Conjecture 1.7 Consider the connected cluster of the origin $C(0)$ at $p = p_c$. Scale it by $|C(0)|^{1/4}$. Then, the scaling limit of $C(0)$ *will be* ISE, as for lattice trees.

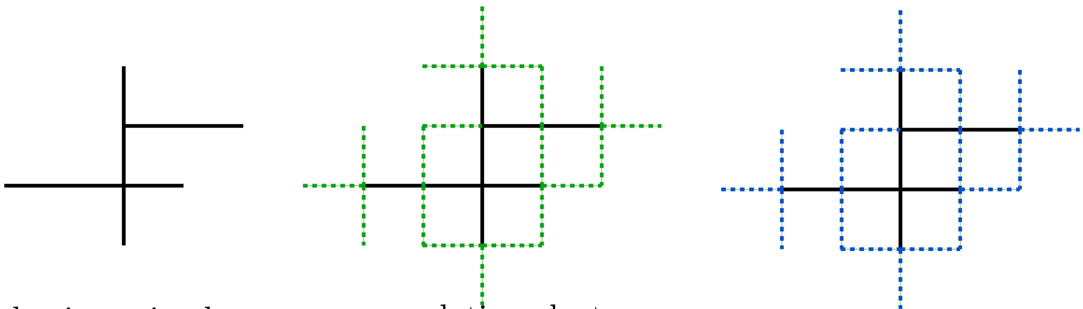
Currently, we have only partial results towards the above conjecture.

Proposition 1.8 ([13]) *First and second moments of scaled connected cluster at $p = p_c$ is the same as those of ISE.*

An important open problem. *Currently, there are almost NO rigorous results on two-point functions in the super-critical phase, even in high dimensions.*

1.5 q-Lattice Animals

Connected clusters of percolation are lattice animals, but the weight is different.



a lattice animal	a percolation cluster	a q-lattice animal
each bond $= p$	each bond $= p$	each bond $= p$
vacant bond $= 1$	vacant bond $= 1 - p$	vacant bond $= q$

It would be natural to try to interpolate these two models, so assign the weight q on each vacant bond ($1 - p < q < 1$). Fix q and change p, and investigate the critical behavior.

Theorem 1.9 (H-Tamenaga '10) *N.n. q-lattice animals in $d \gg 1$, and spread-out q-lattice animals in $d > 8$ and $L \gg 1$, exhibit the* same *critical behavior as that for high dimensional* LTLA.

The above theorem, and its proof, strongly suggests that the q-lattice animals and LTLA belong to the same universality class.

Remark: Very important (and often earlier) analysis of Ising models by many people are abundant. Also, oriented percolation and contact processes have been well analyzed. In this article, I do not go into these subjects further.

2 Intermezzo: bubble, triangle, square

We have seen that different models exhibit different upper critical dimensions. Part of the reason can be understood from the following theorem.

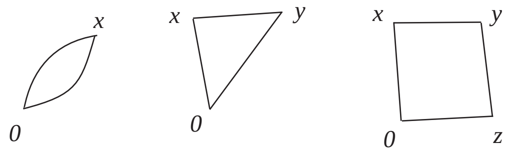
Theorem 2.1 (various people) *Sufficient conditions for mean-field values of* γ *are:*

SAW: $\displaystyle\sum_x G_{p_c}(0,x)\, G_{p_c}(x,0) < \infty$

Percolation: $\displaystyle\sum_{x,y} G_{p_c}(0,x)\, G_{p_c}(x,y)\, G_{p_c}(y,0) < \infty$

LTLA: $\displaystyle\sum_{x,y,z} G_{p_c}(0,x)\, G_{p_c}(x,y)\, G_{p_c}(y,z)\, G_{p_c}(z,0) < \infty$

Note that the above theorem refers to quantities expressed as sums of *critical* two-point functions. If we represent the two-point function $G(x,y)$ by a line x —— y, the above summands are represented as



So they are called (resp.) *bubble, triangle, square conditions.*

Above conditions provide hints on the values of upper critical dimensions. Suppose $d > d_c$. Above the upper critical dimension, two-point function would behave as ($\eta = 0$)

$$G_{p_c}(0,x) \approx |x|^{2-d} \qquad (|x| \uparrow \infty) \tag{2.1}$$

If we assume the above, it is easy to see that

- the bubble condition is satisfied if $d > 4$.
- the triangle condition is satisfied if $d > 6$.
- the square condition is satisfied if $d > 8$.

Let's move on to understand why bubble, triangle, and square conditions arise. Let me begin with the bubble condition.

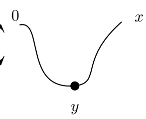
2.1 How to understand the bubble condition for SAW

We first explain how to derive $\gamma \geq 1$, which holds in all dimensions. We begin by differentiating the susceptibility for SAW:

$$p\frac{\partial}{\partial p}\chi_p = p\frac{\partial}{\partial p}\sum_x G_p(0,x) = p\frac{\partial}{\partial p}\sum_x \sum_{\omega:0\to x} p^{|\omega|} = \sum_x \sum_{\omega:0\to x} p^{|\omega|}\,|\omega|.$$

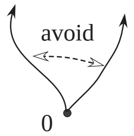
Because the number of sites on ω is $|\omega|+1$, and because we're interested in the limit of very large $|\omega|$, we can approximate $|\omega| \approx \sum_{y\in\omega} 1$ to get

$$\approx \sum_{x,y} \sum_{\omega:0\to x} p^{|\omega|}\, I[y \in \omega] = \sum_{x,y}$$



Now split the walk ω at y into ω_1 and ω_2, and rewrite the sum over ω as sums over ω_1 and ω_2. In so doing, don't forget the fact that ω_1 and ω_2 were originally a single SAW —— that is, ω_1 and ω_2 should avoid each other

$$= \sum_{x,y} \sum_{\substack{\omega_1:0\to x \\ \omega_2:0\to y}} p^{|\omega_1|} p^{|\omega_2|} I[\omega_1 \cap \omega_2 = \{0\}] = \tag{2.2}$$



Now bound the indicator (which indicates that ω_1 and ω_2 should avoid each other) by 1 to get:

$$p\frac{\partial}{\partial p}\chi_p \le \sum_{\substack{\omega_1:0\to\bullet \\ \omega_2:0\to\bullet\bullet}} p^{|\omega_1|} p^{|\omega_2|} = (\chi_p)^2 \tag{2.3}$$

where in the above, a dot $\bullet$ and double dots $\bullet\bullet$ means that walks can end anywhere. This can be rewritten as

$$p\frac{\partial}{\partial p}(\chi_p)^{-1} \le 1 \tag{2.4}$$

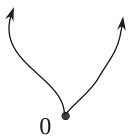
which can be integrated to yield

$$(\chi_p)^{-1} \le (p_c - p) \tag{2.5}$$

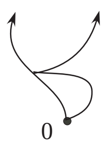
which would imply $\gamma \ge 1$, if the exponent exists.

To prove $\gamma \ge 1$ under the bubble condition, we need an inequality in the opposite direction. I first prove $\gamma \le 1$ under the condition that the bubble is less than one. For this purpose, let's use inclusion-exclusion to get

$$p\frac{\partial}{\partial p}\chi_p = (\chi_p)^2 - \sum_{\substack{\omega_1:0\to\bullet \\ \omega_2:0\to\bullet\bullet}} p^{|\omega_1|} p^{|\omega_2|} I[\omega_1 \cap \omega_2 \supsetneqq \{0\}] = \tag{2.6}$$



$-$



$$\ge (\chi_p)^2 - \sum_{z\neq 0} G(0,z)^2\,(\chi_p)^2 = (\chi_p)^2\Big[1 - \sum_{z\neq 0} G(0,z)^2\Big] \tag{2.7}$$

$\gamma = 1$ now follows if $\sum_{x\neq 0} G_{p_c}(0,x)^2 < 1$.

To prove $\gamma \le 1$ under the condition $\sum_{x\neq 0} G_{p_c}(0,x)^2 < \infty$, we have to try a bit more. Consider the second term of (2.6). As indicated in the above figure, ω_1 and ω_2 do intersect (at least) at one point. Let's call the intersection point z (if there are more than one intersection points, pick up the last intersection along ω_1 in the direction from 0 to $\bullet$). Cut ω_1 into ω_{11} and ω_{12} at z, and cut ω_2 into ω_{21} and ω_{22} at z, and rewrite the second term of (2.6). The result is

$$\sum_{\substack{\omega_1:0\to\bullet \\ \omega_2:0\to\bullet\bullet}} p^{|\omega_1|} p^{|\omega_2|} I[\omega_1 \cap \omega_2 \supsetneqq \{0\}] = \sum_z \sum_{\substack{\omega_{11}:0\to z \\ \omega_{12}:z\to\bullet}} p^{|\omega_{11}|} p^{|\omega_{12}|} I[\omega_{11} \cap \omega_{12} = \{z\}]$$
$$\times \sum_{\substack{\omega_{21}:0\to z \\ \omega_{22}:z\to\bullet\bullet}} p^{|\omega_{21}|} p^{|\omega_{22}|} I[\omega_{21} \cap \omega_{22} = \{z\}] \times I[\omega_{12} \cap \omega_{21} = \{0,z\}, \omega_{12} \cap \omega_{22} = \{z\}] \tag{2.8}$$

(The indicator $I[\omega_{11} \cap \omega_{12} = \{z\}]$ takes care of the fact that ω_{11} and ω_{12} were originally a single SAW, and the indicator $I[\omega_{21} \cap \omega_{22} = \{z\}]$ takes care of the fact that ω_{21} and ω_{22} were originally a single SAW. The indicator $I[\omega_{12} \cap \omega_{21} = \{0,z\}, \omega_{12} \cap \omega_{22} = \{z\}]$ takes care of the fact that z is the *last* (seen from ω_1, in the direction of $0 \to \bullet$) intersection point between ω_1 and ω_2.) Now discard most of the indicators to get an upper bound as

$$
\begin{aligned}
&\sum_{\substack{\omega_1:0\to\bullet\\ \omega_2:0\to\bullet\bullet}} p^{|\omega_1|}\,p^{|\omega_2|}\,I[\omega_1\cap\omega_2 \supsetneqq \{0\}]\\
&\le \sum_z \sum_{\substack{\omega_{11}:0\to z\\ \omega_{12}:z\to\bullet}} p^{|\omega_{11}|}\,p^{|\omega_{12}|} \sum_{\substack{\omega_{21}:0\to z\\ \omega_{22}:z\to\bullet\bullet}} p^{|\omega_{21}|}\,p^{|\omega_{22}|}\,I[\omega_{12}\cap\omega_{22}=\{z\}]\\
&= \sum_z \sum_{\substack{\omega_{11}:0\to z\\ \omega_{21}:0\to z}} p^{|\omega_{11}|+|\omega_{21}|} \sum_{\substack{\omega_{12}:z\to\bullet\\ \omega_{22}:z\to\bullet\bullet}} p^{|\omega_{12}|+|\omega_{22}|} I[\omega_{12}\cap\omega_{22}=\{z\}]
\end{aligned}
\tag{2.9}
$$

Now if we recall (2.2), we see that the last sum over ω_{12} and ω_{22} is nothing but[6] $p\dfrac{\partial\chi_p}{\partial p}$. Therefore, we have

$$
\sum_{\substack{\omega_1:0\to\bullet\\ \omega_2:0\to\bullet\bullet}} p^{|\omega_1|}\,p^{|\omega_2|}\,I[\omega_1\cap\omega_2 \supsetneqq \{0\}] \le \sum_z \sum_{\substack{\omega_{11}:0\to z\\ \omega_{21}:0\to z}} p^{|\omega_{11}|+|\omega_{21}|}\times p\frac{\partial\chi_p}{\partial p}
\tag{2.10}
$$

Or, recalling (2.6), we get

$$
p\frac{\partial\chi_p}{\partial p} \ge (\chi_p)^2 - \sum_{z\neq 0} G(0,z)^2 \times p\frac{\partial\chi_p}{\partial p} \qquad \Longrightarrow \qquad p\frac{\partial\chi_p}{\partial p} \ge \frac{(\chi_p)^2}{1+\sum_{z\neq 0} G(0,z)^2}
\tag{2.11}
$$

This immediately implies $\gamma \le 1$ if $\displaystyle\sum_{z\neq 0} G(0,z)^2$ is uniformly finite for $p < p_c$. $\square$

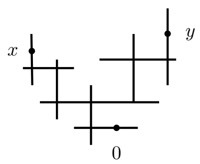
The intuitive reason for $d_c = 4$*:* Two SAW's, whose Hausdorff dimension will be 2, must avoid each other in $d > d_c$. For this, we need $d > 2+2 = 4$. So we can interpret 4 as $\mathbf{4 = 2 + 2}$.

2.2 How about LTLA?

Again, we start from the expression of its susceptibility.

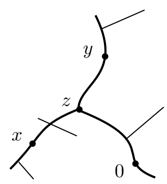
$$
p\frac{\partial\chi_p}{\partial p} = p\frac{\partial}{\partial p}\sum_x \sum_{T\ni 0,x} p^{|T|} = \sum_x \sum_{T\ni 0,x} |T|p^{|T|}
\tag{2.12}
$$

$$
= \sum_{x,y}\sum_{T\ni 0,x,y} p^{|T|} = \sum_{x,y}
\tag{2.13}
$$



Now, if three points 0, x, and y are on the same lattice tree, we can always find a point z on that tree, and three distinct branches which connect z and 0, z and x, and z and y. These three branches must not intersect, but if we relax this condition, we can get an upper bound as

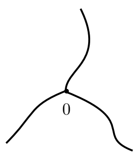
$$
\le \sum_{x,y,z} \qquad \le (\chi_p)^3
\tag{2.14}
$$

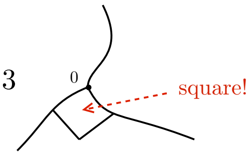


This implies $\gamma \ge 1/2$ if the exponent exists.

[6]Rigorously sepaking, we are here ignoring a small term which can be neglected in the limit of $\chi_p \uparrow \infty$

In the opposite direction, we again use inclusion-exclusion to get

$\geq$ 

$-\,3$ 

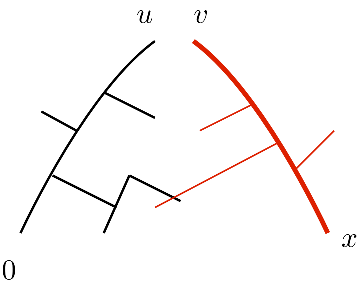
(2.15)

This leads to $\gamma = 1$ if the square is less than $1/3$.

The intuitive reason for $d_c = 8$: Two lattice trees, whose Haussdorf dimension will be 4, must avoid each other in $d > d_c$. For this, we need $d > 4 + 4 = 8$. So we can interpret $d_c = 8$ as $\mathbf{8} = \mathbf{4} + \mathbf{4}$.

2.3 How about Percolation?

Percolation is more subtle. Although connected clusters of percolation are just lattice animals, their weights are different. (Because of this difference, percolation allows for probabilistic interpretation.) Due to this, the avoidance condition now is: One connected cluster (4-dimensional, left) should avoid the backbone (2-dimensional, *thick right*) of another connected cluster. Therefore, $\mathbf{6} = \mathbf{4}+\mathbf{2}$.

$$\frac{\partial \chi_p}{\partial p} = \sum_x \sum_{(u,v)}$$ 

connect v and x without using black bonds

3 The lace expansion

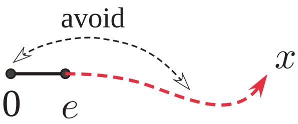
Almost all the results concerning high-dimensional critical behavior have been obtained by *lace expansion*, first introduced by Brydges and Spencer [3]. The lace expansion have the following characteristics:

- The lace expansion gives a *self-consistent equation* for two-point functions (even if we do not know explicit values of p_c).
- The number of n^{th} order terms of the expansion grows as $(const)^n$. This is in a striking contrast with usual expansions, which typically yield $n!$ terms. Because the number of terms grows only exponentially, *there is a hope that the expansion is absolutely convergent.*

In this section, we briefly explain basic ideas of the lace expansion (mainly for SAW).

3.1 Derivation of the expansion (for n.n. SAW)

Start from $G(x)$ $(x \neq 0)$, and cut at its first step, e. Call the second part as ω'.

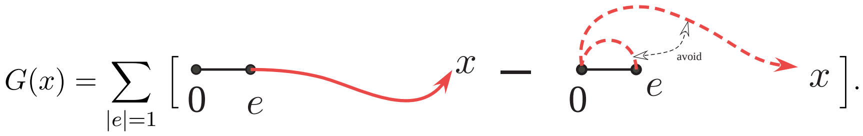
$$G(x) = \sum_{\omega:0\to x} p^{|\omega|} = \sum_{|e|=1}$$ 

(3.1)

The second part ω' should avoid the first part. I.e. it should not come back to the origin. Now use inclusion-exclusion to write this avoidance condition as:

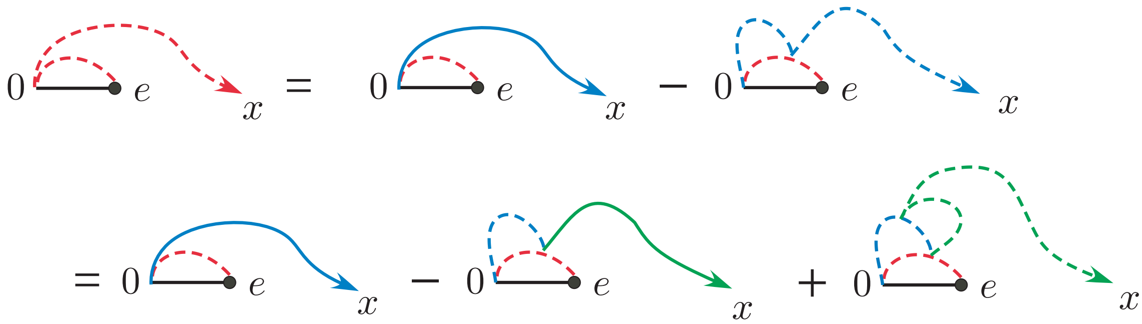
$$I[(0,e)\circ\omega' \text{ is SAW}] = I[0\notin\omega'] = 1 - I[0\in\omega'], \tag{3.2}$$

and plug it into (3.1). The result looks as:



(3.3)

Then cut ω' at its first visit at 0, and use inclusion-exclusion again and again:

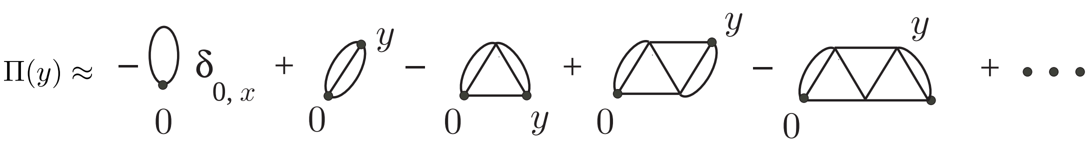


In the above figures, solid lines represent a SAW which has no constraint, while dashed lines of the same color must avoid each other.

Continuing this way, we get

$$G(x) = \delta_{0,x} + p\sum_{|e|=1} G(x-e) + \sum_y \Pi(y)\, G(x-y), \tag{3.4}$$

where (each line $\approx G$)



(3.5)

Taking the Fourier transform of the above, we get

$$\hat{G}(k) = \frac{1}{1-\{2dp\hat{D}(k)+\hat{\Pi}(k)\}} \tag{3.6}$$

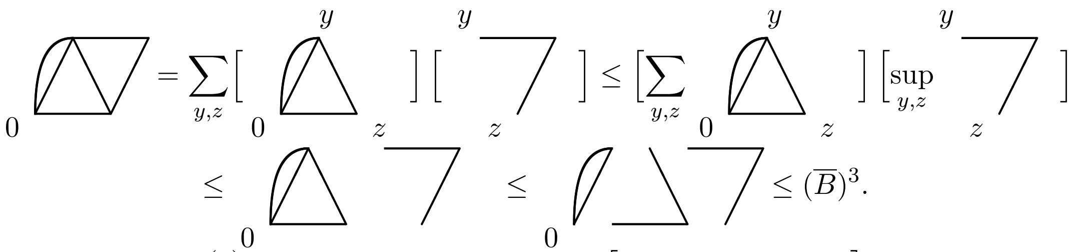
— A closed from equation for the two-point function, although $\Pi_p(x)$ itself is quite complicated. We now proceed to get info on Π_p in terms of G_p.

3.2 How to bound these diagrams?

Use (for $f, g \geq 0$) the following elementary inequalities

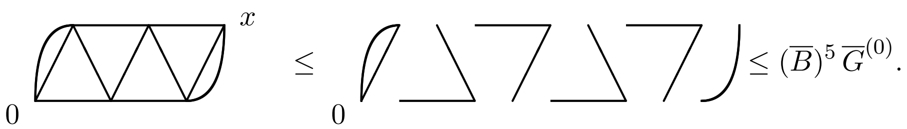
$$\sum_x f(x)g(x) \leq \Big[\sup_x f(x)\Big]\Big[\sum_x g(x)\Big] = \|f\|_\infty\,\|g\|_1$$

repeatedly, to break diagrams into basic units. E.g., (all vertices with degree ≥ 2 are summed over)



where
$$\overline{G}^{(\alpha)} := \sup_{x\neq 0} |x|^\alpha G(x), \quad \overline{B} := \sup_a \Big[\sum_{x\neq 0} G(x)\, G(x-a)\Big]$$

Similarly,



So we can bound various terms of Π_p in terms of two quantities $\overline{G}^{(\alpha)}$ and $\overline{B}$.

3.3 How to prove convergence?

To control the expansion and prove its convergence, we proceed as follows.

- Start from good bounds on $\overline{G}^{(2)}, \overline{B}$
- As explained in Section 3.2, derive good bounds on $\sum_x \Pi(x)$ and $\sum_x |x|^2\Pi(x)$
- From the expression (3.6), derive good bounds on $\hat{G}(k)$ and its second derivative
- Use Fourier analysis to get good bounds on $\overline{G}^{(2)}, \overline{B}$; in fact, we derive bounds which are *better* than what we had started.

Schematically,
$$\overline{G}^{(2)}, \overline{B} \;\underset{\text{Fourier analysis}}{\overset{\text{Diagrammatic estimates}}{\rightleftarrows}}\; \sum_x |x|^2 |\Pi(x)|$$

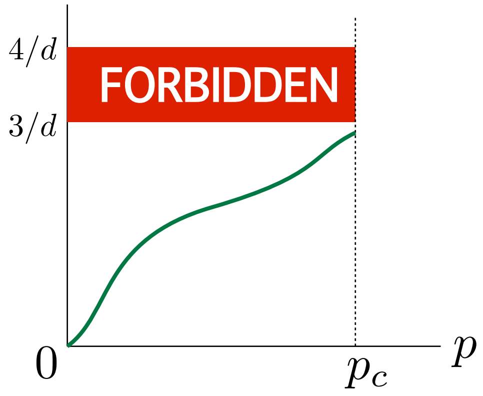
In high dimensions, the above procedure makes a nice closed loop — the following is true.

$$\boxed{\text{If } \overline{G}^{(2)}, \overline{B} \leq 4/d, \text{ then in fact } \overline{G}^{(2)}, \overline{B} < 3/d.}$$

Now, note that $\overline{G}$ and $\overline{B}$ are *continuous functions of* p for $p < p_c$. This is because for $p < p_c$, (1) the two-point function $G_p(0,x)$ is an absolutely convergent power series in p, and this is continuous, (2) $G_p(0,x)$ decays exponentially in $|x|$, and thus essentially finite number of x's are relevant to $\overline{G}$ and $\overline{B}$.

Also, $\overline{G}$ and $\overline{B}$ are 0 *at* $p = 0$.

So increasing p from 0 to p_c, $\overline{G}$ and $\overline{B}$ can *never exceed* $4/d$, as shown in the figure. Dominated convergence guarantees they are so, even at $p = p_c$.



These bounds on $\overline{G}$ and $\overline{B}$ now guarantee the convergence of the lace expansion, and lead to good bounds on $\hat{G}(k)$ and $\hat{\Pi}(k)$. $\square$

3.4 Summary for SAW lace expansion

Results of k-space analysis of the lace expansion up to '92 can be summarized as:

Lemma 3.1 *For n.n. SAW in $d \geq 5$,*

$$\hat{G}(k) = \frac{1}{1 - 2dp_c\hat{D}(k) - \hat{\Pi}(k)} \tag{3.7}$$

with

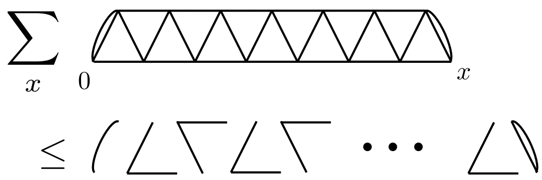
$$\sum_x |\Pi(x)| = O(d^{-1}), \quad \sum_x |x|^2|\Pi(x)| = O(d^{-1}). \tag{3.8}$$

$$\overline{G}^{(2)} = O(d^{-1}), \quad \overline{B} = O(d^{-1}). \tag{3.9}$$

3.5 Other models

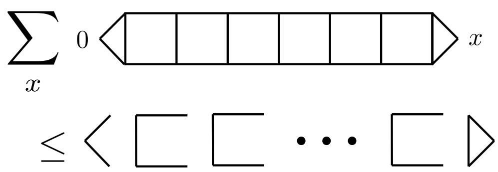
Lace expansion for other models are derived in a similar manner. Their diagrams are schematically as follows:

(SAW)



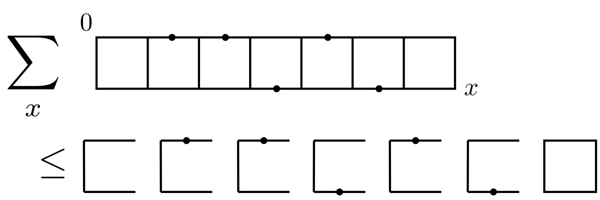
bubbles

(perc)



triangles

(LTLA)



squares

Note that bubble, triangles, squares (respectively) appear for SAW, percolation, LTLA (respectively); a further suggestion for the upper critical dimension $d_c = 4, 6, 8$.

4 Summary and ...

- Rigorous results on the critical behavior of stochastic geometric models (SAW, percolation, lattice trees and animals) have been reviewed.
- Some attempts were made to explain why these models exhibit different upper critical dimensions.
- The lace expansion, one of the main tools of the analysis, was explained.

We now have some understanding of critical behavior of these models in high dimensions. HOWEVER, even for high dimensions, our understanding is far from complete. In particular:

- Current proof of mean-field critical behavior for SAW in $d \geq 5$ is *unnatural*, in the sense that it requires the bubble to be "small." A natural proof would only require the bubble condition. Also, the proof of mean-field critical behavior for nearest-neighbour percolation in $d > 6$ and LTLA in $d > 8$ are missing, for the same reason.
- For percolation, there are almost no rigorous quantitative results on its two-point function in the *supercritical phase*.

And, needless to say, lower dimensions are wide open. Several important open problems:

- In the first place, do critical exponents exist?
- How about *universality*?
- How about *scaling and hyperscaling*?
- What happens if there is randomness involved?
- ...

I hope we will see substantial progress in (one or more of) these open problems in the near future.

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Stochastic ranking process and web ranking numbers

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1 Introduction.

This is a summary of a series of our studies [13, 14, 15, 16, 12] on stochastic ranking process, with its applications on ranking numbers found on the web, such as sales ranks at an online bookstore amazon.co.jp, and thread title listings of an online collected bulletin board 2ch.net. This is a joint work with K. Hattori, Y. Hariya, Y. Nagahata, Y. Takeshima, and T. Kobayashi.

In Section 2, we consider mathematical aspects of stochastic ranking process. We define the stochastic ranking process with the jump times of the particles determined by Poisson random measures, and state that the joint empirical distribution of scaled position and intensity measure converges almost surely in the infinite particle limit. We give an explicit formula for the limit distribution, which can be characterized as a unique global classical solution to an initial value problem for the inviscid Burgers system of non-linear partial differential equations with time dependent coefficients and with evaporation terms. This characterization is in accord with the hydrodynamic limit theories, where a macroscopic time development of collective microscopic random motion of particles is smooth, so that it satisfies a system of partial differential equations.

In Section 3, we show ranking data collected from actual websites at the Amazon online bookstore and at an online collected bulletin board 2ch.net, and show how they are explained by the properties of stochastic ranking process given in Section 2. It is a new social phenomena to have a large number of items aligned dynamically in an order of popularity, and real time values of ranks of thousands can be observed. By performing a statistical fit of the data to the formulas from the stochastic ranking process, one can analyze a 'long tail' structure of social activities at these websites. We conclude that the best hit or top sales items dominate the activities both at Amazon.co.jp and 2ch.net, so that, in particular, Amazon.co.jp, perhaps in contrast to its fame, is not an example of a long tail business.

2 Stochastic ranking process.

The latest version of stochastic ranking process, which extends the original model [13] to the case of time dependent intensities, is defined as follows [12]. Let $\mathcal{M}(\mathbb{R}_+)$ be the space of Radon measures ρ on the Borel σ-algebra $\mathcal{B}(\mathbb{R}_+)$ of non-negative reals $\mathbb{R}_+$. Let N be a positive integer, and let $\nu_i^{(N)}$, $i = 1, 2, \cdots, N$, be independent Poisson random measures (Poisson point processes) on $\mathbb{R}_+$, defined on a probability space $(\mathrm{P}, \mathcal{F}, \Omega)$. For each i, denote the intensity measure of $\nu_i^{(N)}$ by $\rho_i^{(N)}$;

$$\mathrm{E}[\ \nu_i^{(N)}(A)\ ] = \rho_i^{(N)}(A),\ A \in \mathcal{B}(\mathbb{R}_+). \tag{1}$$

We assume that $\rho_i^{(N)} \in \mathcal{M}(\mathbb{R}_+)$ and that $\rho_i^{(N)}$ is continuous (i.e., $\rho_i^{(N)}(\{t\}) = 0$, $t \geqq 0$) for all N and i. Let $x_1^{(N)}, x_2^{(N)}, \cdots, x_N^{(N)}$ be a permutation of $1, 2, \cdots, N$, and

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define a process $X^{(N)} = (X_1^{(N)}, \cdots, X_N^{(N)})$ by

$$\begin{aligned} &X_i^{(N)}(t) \\ &= x_i^{(N)} + \sum_{k=1}^{N} \int_0^t \mathbf{1}_{X_k^{(N)}(s-0) > X_i^{(N)}(s-0)}\ \nu_k^{(N)}(ds) + \int_0^t (1 - X_i^{(N)}(s-0))\, \nu_i^{(N)}(ds), \\ &i = 1, 2, \cdots, N,\ t \geqq 0\,, \end{aligned} \tag{2}$$

where, $\mathbf{1}_A$ is the indicator function of an event A. We call the process $X^{(N)}$ defined by (2), a stochastic ranking process, after [13, 14, 15].

Denote the unit measure concentrated on c by δ_c. With probability 1 we can write

$$\nu_i^{(N)} = \sum_{j=1}^{\infty} \delta_{\tau_{i,j}^{(N)}},\ i = 1, 2, \cdots, N, \tag{3}$$

where, with probability 1, $\tau_{i,j}^{(N)}$'s are random variables satisfying $0 < \tau_{i,1}^{(N)} < \tau_{i,2}^{(N)} < \cdots$, $i = 1, 2, \cdots, N$, and $\tau_{i,j}^{(N)} \neq \tau_{i',j'}^{(N)}$ if $(i, j) \neq (i', j')$. In the following, we work on the event that these inequalities hold. $X_i^{(N)}(t)$ has an explicit expression using $\tau_{i,j}^{(N)}$'s:

$$X_i^{(N)}(t) = \begin{cases} x_i^{(N)} + \displaystyle\sum_{i';\, x_{i'}^{(N)} > x_i^{(N)}} \mathbf{1}_{\tau_{i',1}^{(N)} \leqq t} & 0 \leqq t < \tau_{i,1}^{(N)}\,, \\ 1 + \displaystyle\sum_{i'=1}^{N} \mathbf{1}_{\exists j' \in \mathbb{N};\ \tau_{i,j}^{(N)} < \tau_{i',j'}^{(N)} \leqq t} & \tau_{i,j}^{(N)} \leqq t < \tau_{i,j+1}^{(N)},\ j = 1, 2, 3, \cdots, \end{cases} \tag{4}$$

for $i = 1, \cdots, N$.

In the time homogeneous case, namely, the case where there exists positive constants $w_i^{(N)}$ such that $\rho_i^{(N)}((0, t]) = w_i^{(N)} t$ for $t \geqq 0$, a discrete time version of the process (4) has been known for a long time [26, 23, 17, 6, 22, 21] and is called move-to-front (MTF) rules. The process has, in particular, been extensively studied as a model of least-recently-used (LRU) caching in the field of information theory [24, 8, 4, 7, 5, 25, 9, 11, 10, 18, 19, 20], and also is noted as a time-reversed process of top-to-random shuffling.

Put

$$X_C^{(N)}(t) = \sum_{i=1}^{N} \mathbf{1}_{\tau_{i,1}^{(N)} \leqq t}, \quad t \geqq 0. \tag{5}$$

$X_C^{(N)}(t)$ is a random variable which denotes the position of the boundary between the top side $x \leqq X_C^{(N)}(t)$ and the tail side $x > X_C^{(N)}(t)$, where each particle in the top side (i.e., i which satisfies $X_i^{(N)}(t) \leqq X_C^{(N)}(t)$) has experienced jump to the top by time t (i.e., $\tau_{i,1}^{(N)} \leqq t$), and the particles in the tail side are those particles which have not jumped to the top by time t.

Proposition 1 ([13, Prop. 2],[12, Prop. 1.1, Cor. 1.2]) *Let $t \geqq 0$. Assume that a sequence of distributions $\{\lambda_t^{(N)} \mid N \in \mathbb{N}\}$ on $\mathbb{R}_+$ defined by*

$$\lambda_t^{(N)} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\rho_i^{(N)}((0,t])} \tag{6}$$

converges weakly as $N \to \infty$ *to a probability distribution* λ_t *. Then the scaled position of the boundary*

$$Y_C^{(N)}(t) = \frac{1}{N} X_C^{(N)}(t) = \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{\tau_{i,1}^{(N)} \leqq t} \tag{7}$$

converges almost surely as $N \to \infty$ *to*

$$y_C(t) = 1 - \int_0^\infty e^{-s} \lambda_t(ds). \tag{8}$$

Assume furthermore that λ_t *is continuous in* t *with respect to the topology of weak convergence. Then for almost all sample* $\omega \in \Omega$, $Y_C^{(N)}(\cdot, \omega) : \mathbb{R}_+ \to [0,1)$ *defined by (7) converges pointwise in* t *as* $N \to \infty$ *to a deterministic function* $y_C : \mathbb{R}_+ \to [0,1)$ *defined by (8).* $\diamond$

Consider a joint empirical distribution $\mu^{(N)}$ of intensity measure $\rho_i^{(N)}$ and scaled position

$$Y_i^{(N)}(t) = \frac{1}{N}(X_i^{(N)}(t) - 1), \tag{9}$$

defined by

$$\mu_t^{(N)} = \frac{1}{N} \sum_{i=1}^{N} \delta_{(\rho_i^{(N)}, Y_i^{(N)}(t))}, \quad t \geqq 0. \tag{10}$$

$\mu_t^{(N)}$, $N \in \mathbb{N}$, are random variables whose samples are distributions on the product space $\mathcal{M}(\mathbb{R}_+) \times [0,1)$ of space of Radon measures $\mathcal{M}(\mathbb{R}_+)$ and an interval $[0,1) \subset \mathbb{R}_+$. We consider the standard vague topology on $\mathcal{M}(\mathbb{R}_+)$. Since $\mathbb{R}_+$ is a Polish space, i.e., complete and separable metric space, so is $\mathcal{M}(\mathbb{R}_+)$ [2, Theorem 31.5], and consequently, $\mathcal{M}(\mathbb{R}_+) \times [0,1)$ is also a Polish space [2, Example 26.2].

Assume that a sequence of initial configurations

$$\mu_0^{(N)} = \frac{1}{N} \sum_{i=1}^{N} \delta_{(\rho_i^{(N)}, N^{-1}(x_i^{(N)} - 1))}, \quad N = 1, 2, \cdots,$$

converges weakly as $N \to \infty$ to a probability distribution μ_0 on $\mathcal{M}(\mathbb{R}_+) \times [0,1)$. Then, in particular,

$$\begin{aligned} &\Lambda^{(N)}(d\rho) := \mu_0^{(N)}(d\rho \times [0,1)) = \frac{1}{N} \sum_{i=1}^{N} \delta_{\rho_i^{(N)}}(d\rho) \;\to\; \Lambda(d\rho) := \mu_0(d\rho \times [0,1)), \\ &\text{weakly, as } N \to \infty. \end{aligned} \tag{11}$$

Define, for $0 \leqq s \leqq t$,

$$\lambda_{s,t}^{(N)} = \int_{\mathcal{M}(\mathbb{R}_+)} \delta_{\rho((s,t])} \Lambda^{(N)}(d\rho). \tag{12}$$

Note that $\lambda_t^{(N)} = \lambda_{0,t}^{(N)}$ in (6).

Theorem 2 ([13, Thm. 1.5], [12, Thm. 1.3]) *Assume that* $\mu_0^{(N)} \to \mu_0$ *weakly as* $N \to \infty$ *for a probability distribution* μ_0 *on* $\mathcal{M}(\mathbb{R}_+) \times [0,1)$. *Assume that for each* (s,t) *satisfying* $t \geqq s \geqq 0$,

$$\lambda_{s,t}^{(N)} \;\to\; \lambda_{s,t} := \int_{\mathcal{M}(\mathbb{R}_+)} \delta_{\rho((s,t])} \Lambda(d\rho), \quad \textit{weakly as} \;\; N \to \infty, \tag{13}$$

where Λ is as in (11). Then for any $t > 0$, and for almost all sample $\omega \in \Omega$, the distribution $\mu_t^{(N)}(\omega)$ converges weakly to a non-random probability distribution μ_t on $\mathcal{M}(\mathbb{R}_+) \times [0,1)$.

μ_t has a following expression in terms of $U(d\rho, y, t) := \mu_t(d\rho \times [y,1))$.

$$U(d\rho, y, t) := \mu_t(d\rho \times [y,1)) = \begin{cases} e^{-\rho((t-t_0(y,t),t])}\,\Lambda(d\rho) & 0 \leqq y \leqq y_C(t), \\ e^{-\rho((0,t])}\,U(d\rho, \hat{y}(y,t), 0) & y_C(t) \leqq y < 1. \end{cases} \tag{14}$$

Here, $t_0(y,t)$ is the inverse function with respect to t_0 of

$$y_A(t_0,t) = 1 - \int_{\mathcal{M}(\mathbb{R}_+)} e^{-\rho((t-t_0,t])}\,\Lambda(d\rho), \quad 0 \leqq t_0 \leqq t, \tag{15}$$

and $\hat{y}(y,t)$ is the inverse function with respect to y of

$$y_B(y,t) = 1 - \int_{\mathcal{M}(\mathbb{R}_+)} e^{-\rho((0,t])}\,\mu_0(d\rho \times [y,1)), \quad t \geqq 0,\ 0 \leqq y < 1. \tag{16}$$

$\diamond$

Note that $y_C(t) = y_A(t,t) = y_B(0,t)$.

If we impose additional conditions, we may go further for Theorem 2 and prove almost sure convergence as a sequences of processes $\mu^{(N)} \to \mu$, on a finite time interval $[0,T]$. See [12, §4].

The structure of the explicit limit formula (14), in particular, the appearance of the inverse functions t_0 of y_A and $\hat{y}$ of y_B , can mathematically be understood through a system of partial differential equations. Consider the case that the limit distribution Λ is supported on a discrete set: $\Lambda = \sum_\alpha r_\alpha \delta_{\rho_\alpha}$. Then (14) implies, for $U_\alpha(y,t) := \mu_t(\{\rho_\alpha\} \times [y,1))$,

$$U_\alpha(y,t) = \begin{cases} r_\alpha\, e^{-\rho_\alpha((t-t_0(y,t),t])} & 0 \leqq y \leqq y_C(t), \\ U_\alpha(\hat{y}(y,t),0)\, e^{-\rho((0,t])} & y_C(t) \leqq y < 1, \end{cases} \tag{17}$$

where t_0 and $\hat{y}$ are inverse functions, respectively, of $y_A(t_0,t) = 1 - \sum_\alpha r_\alpha e^{-\rho_\alpha((t-t_0,t])}$, and $y_B(y,t) = 1 - \sum_\alpha U_\alpha(y,0) e^{-\rho_\alpha((0,t])}$.

Theorem 3 ([14, Thm. 1], [12, Thm. 1.4]) *Let k be a positive integer, and for each $\alpha = 1, 2, \cdots, k$, let r_α be a positive constant, $w_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ a measurable function satisfying $w_\alpha(t) > 0$, $t \geqq 0$, and $u_\alpha : [0,1) \to \mathbb{R}_+$ a non-negative smooth strictly decreasing function, satisfying*

$$\sum_{\beta=1}^k r_\beta = 1, \quad \sum_{\beta=1}^k r_\beta w_\beta(t) < \infty, \ t \geqq 0, \quad \textit{and} \quad \sum_{\beta=1}^k u_\beta(y) = 1 - y, \ 0 \leqq y < 1. \tag{18}$$

Then an initial value problem for a system of first order non-linear partial differential equations (inviscid Burgers equations with a term representing evaporation)

$$\begin{gathered} \frac{\partial U_\alpha}{\partial t}(y,t) + \sum_{\beta=1}^k w_\beta(t)\, U_\beta(y,t)\, \frac{\partial U_\alpha}{\partial y}(y,t) = -w_\alpha(t) U_\alpha(y,t), \\ (y,t) \in [0,1) \times \mathbb{R}_+, \ \alpha = 1, 2, \cdots, k, \end{gathered} \tag{19}$$

with a boundary condition

$$U_\alpha(0,t) = r_\alpha,\ t \geqq 0,\ \alpha = 1,2,\cdots,k, \tag{20}$$

and initial data

$$U_\alpha(\cdot,0) = u_\alpha,\ \alpha = 1,2,\cdots,k, \tag{21}$$

has a unique time global classical solution, whose formula is given by (17) with

$$\rho_\alpha((s,t]) = \int_s^t w_\alpha(u)\,du \quad \textit{and} \quad U_\alpha(y,0) = u_\alpha(y). \tag{22}$$

$\diamond$

The system (19) of partial differential equations is solved by a method of characteristic curves, and y_A, y_B, and y_C turn out to be the characteristic curves for (19), which mathematically explains how the inverse functions of these functions appear in the solutions. Theorem 3 indicates that the limit in Theorem 2 has an interpretation that a collective random motion of particles is macroscopically observed as a smooth time development explained by a system of partial differential equations, as in the theory of hydrodynamic limit.

3 Web rankings.

With great advance in the internet technologies, a new application of the process appeared [14, 15, 12]. The mathematical results on the stochastic ranking process have successfully been applied to statistical explanation of practical ranking data, such as the ranking numbers of books found in the web pages of an online bookstore Amazon.co.jp [15, 14], or the order of the subject titles in the title listing pages of a collected web bulletin board 2ch.net [14, 12]. A ranking of a book at Amazon.co.jp jumps close to top of the ranking whenever the book is sold at Amazon.co.jp [15], and a subject title in the web page for the list of 2ch.net jumps to the top whenever a comment (a 'response') concerning the subject is submitted [14]. It turned out that the time developments of the ordering of items on these online systems are found to follow the predictions of the model.

One may wonder why such a simple model as introduced in Section 2 could be observed in actual social activities. An explanation is that the ranking numbers on the web (such as those representing the books, in the case of online bookstores) usually seek to align the web pages in the order of *current popularity* of the pages. A social impact of the development of web-based activities is that it has become possible to catalog a huge amount of unpopular items [1]. In fact, a majority of books catalogued on an online bookstore are sold less than one copy a month. For such books, any reasonable order reflecting the current popularity would be equal to the order of the time of most recent sales, because the second recent sale of such book would be long ago, hence would not reflect current popularity. Thus the move-to-front rule will provide a simple but *universal* model in the rankings on the web.

Note that (2) implies the Markov property

$$X_i^{(N)}(t+u) = X_i^{(N)}(u) + \sum_{k=1}^{N} \int_0^t \mathbf{1}_{X_k^{(N)}(s+u-0) > X_i^{(N)}(s+u-0)}\ \tilde{\nu}_k^{(N)}(ds)$$
$$+ \int_0^t (1 - X_i^{(N)}(s+u-0))\,\tilde{\nu}_i^{(N)}(ds),$$

where we put $\tilde{\nu}_i^{(N)}(A) = \nu_i^{(N)}(A+u)$. In practical application, this property enables us to shift the time origin $t = 0$ to the time that a particle we observe jumps to the top, namely, we may set $X_i^{(N)}(0) = x_i^{(N)} = 1$, by adjusting the 'clock' for the intensity measure accordingly. (See Fig. 5 and Fig. 7, as well as [14, 15].)

Note also that if $x_i^{(N)} = 1$, then up to the first jump of i to the top, namely, for $t < \tau_{i,1}^{(N)}$, comparison of (4) and (5) leads to

$$X_i^{(N)}(t) = X_C^{(N)}(t) + 1,$$

Therefore, in practical application in Section 3.2, we may proceed with observing a trajectory (time development) of a single particle, putting the time of its first jump to top as $t = 0$ and observing until its next jump to top, and then apply Proposition 1.

Concerning the explicit time dependence of intensity measures for the Poisson random measures, one should note that data from an online bookstore and from a collected web bulletin board arise as results of social activities, which are expected to contain day-night difference in the intensity. In Section 3.1, we summarize a simple method of [12, §A, §5], to factorize the time dependence and the distribution of relative jump rates among different particles. Then we show the data from amazon.co.jp in Section 3.2, and the data from 2ch.net in Section 3.3, together with statistical applications of the theoretical results.

3.1 Intensities with common time dependence.

In practical situation, intensity measures $\rho_i^{(N)}$ are usually unknown quantities to be determined statistically from observed data. This is usually a difficult task if intensity measures have time dependence, because then we have to consider both particle dependence and time dependence at once in the statistical analysis. Explicit

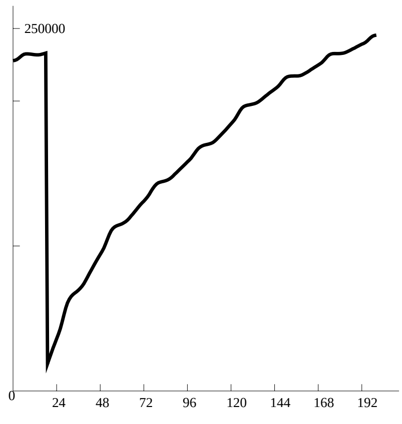

Fig 1:

time dependence, reflecting day-night difference of social activities, are observed in actual data. Fig. 1 is an 8 days plot of Amazon.co.jp rankings for a book. (See [15] for basic fact about Amazon.co.jp ranking and its relation to the stochastic ranking process.) The vertical axis stands for the ranking number, and the horizontal axis is the time axis labelled in the unit of hour. The large discontinuous drops near to the top ranking correspond to the point of sales of the book. Note the 24 hours periodic time dependence. Without explicit time dependence of activities (i.e., the homogeneous case), (28) implies that the curve in Fig. 1 should be concave,

$$y''_C(t) < 0. \tag{23}$$

However, the curve in Fig. 1 actually has convex intervals every 24 hours, proving explicit time dependence, which naturally can be interpreted as day-night difference in social activities.

A simplest way to take day-night-difference of social activity into account, is to assume a common time dependence. Assume that there exist $\tilde{a} \in L^1_{loc}(\mathbb{R}_+)$ and positive constants $w_i^{(N)} > 0,\ i = 1, 2, \cdots, N,\ N = 1, 2, \cdots$, such that the intensity measure (1) is given by

$$\rho_i^{(N)}((s,t]) = w_i^{(N)} \int_s^t \tilde{a}(u)\, du, \quad i = 1, 2, \cdots, N,\ N = 1, 2, \cdots. \tag{24}$$

Proposition 4 *Let $\tilde{a} \in L^1_{loc}(\mathbb{R}_+)$. If there exists a probability distribution λ on $\mathbb{R}_+$ such that*

$$\lambda^{(N)} := \frac{1}{N} \sum_{i=1}^{N} \delta_{w_i^{(N)}} \to \lambda, \quad \textit{weakly, as} \quad N \to \infty, \tag{25}$$

then Proposition 1 holds with (24), and $y_C(t)$ of (8) is given by

$$y_C(t) = 1 - \int_{\mathbb{R}_+} e^{-w\,A(t)}\, \lambda(dw), \tag{26}$$

where

$$A(t) = \int_0^t \tilde{a}(u)\, du. \tag{27}$$

◇

The formula (26) is to be compared with the case of the (homogeneous) Poisson process in [13, Proposition 2], where we have

$$y_C(t) = 1 - \int_{\mathbb{R}_+} e^{-wt} \lambda(dw). \tag{28}$$

3.2 Factorization of day-night social activity difference, and sales ranks of Amazon.co.jp .

We can show that under the common time dependence assumption (24), periodic time dependence of $\tilde{a}$ can be factorized, and that the use of (28) is justified in obtaining λ statistically from data. Assume that there exists a positive constant T such that $\tilde{a}(t+T) = \tilde{a}(t),\ t \geqq 0$. We may normalize $w_i^{(N)}$'s in (24) so that $\dfrac{1}{T}\displaystyle\int_0^T \tilde{a}(u)\, du = 1$ holds. Then $A_p(t) := A(t) - t = \displaystyle\int_0^t (\tilde{a}(u) - 1)\, du$ is a periodic function with period T, and (26) is

$$y_C(t) = 1 - \int_{\mathbb{R}_+} e^{-w\,(t + A_p(t))}\, \lambda(dw). \tag{29}$$

If we collect data at each fixed time of the day, at $t_n = t_0 + nT$, $n = 0, 1, 2, \cdots$, then (29) implies

$$y_C(t_n) = 1 - \int_{\mathbb{R}_+} e^{-w(nT+t_0+A_p(t_0))}\,\lambda(dw). \tag{30}$$

Hence the effect of day-night difference in $\tilde{a}$ is absorbed in the translation of origin of time $t_0 \mapsto t_0 + A_p(t_0)$, and the use of formula (28) is justified.

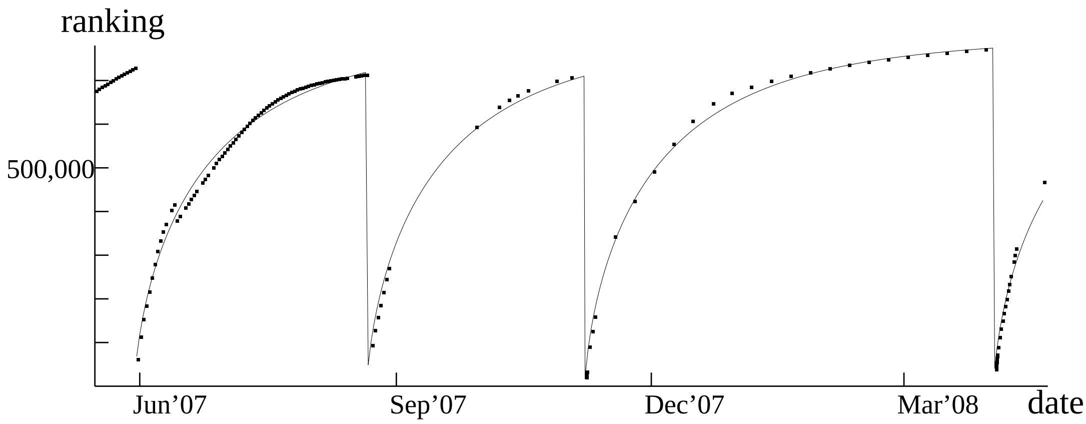

Fig 2:

Fig. 2 is a plot of Amazon.co.jp rankings for a book over a year [15]. The data was taken manually for a year starting in May 2007, at 21:00 each day. As seen in Fig. 2, for a relatively unpopular book, a book which sells less than a copy per week, ranking fall (increase in number) steadily and smoothly at several hundred thousands for much of the time, but once in a while they make sudden jumps to numbers around ten thousand. These occasional large discontinuous jumps near to the top ranking correspond to the point of sales of the book [15].

To apply (28) or (30) to the data, we need to specify λ. A standard choice in social and economic studies seems to be the Zipf's law, defined by

$$w_i^{(N)} = a\left(\frac{N}{i}\right)^{1/b}, \quad i = 1, 2, \cdots, N, \tag{31}$$

for positive constants a and b. The corresponding $N \to \infty$ weak limit is the (generalized) Pareto distribution, defined by

$$\lambda([w,\infty)) = \begin{cases} \left(\dfrac{a}{w}\right)^b & w \geqq a, \\ 1 & w < a. \end{cases} \tag{32}$$

Substituting (32) in (28), we have

$$y_C(t) = 1 - e^{-at} + (at)^b\Gamma(1-b, at). \tag{33}$$

where Γ is the incomplete Gamma function defined by $\Gamma(z,p) = \displaystyle\int_p^\infty e^{-x}x^{z-1}dx$.

Using the data $\{x_i \mid i = 1, 2, \cdots, n_d\}$ of size $n_d = 77$ at the leftmost arc in Fig. 2, taken between May, 2007 and August, 2007, at 21:00 each day, and choosing to minimize

$$E = E(N,a,b) = \sum_{i=1}^{n_d} \frac{(x_i - N\,y_C(t_i))^2}{x_i}, \tag{34}$$

we obtained the best fit for the parameter set

$$(N^*,\ a^*,\ b^*) = (8.15 \times 10^5,\ 5.30 \times 10^{-4},\ 0.767), \tag{35}$$

with $E_{min} = E(N^*, a^*, b^*) = 4.17 \times 10^4$. In particular, we have $b^* < 1$, which implies that amazon.co.jp earns dominantly from a small number of best hit books [15], rather than the majority of books in the long tail, in contrast to the Amazon bookstores fame as a successful long tail business model [1].

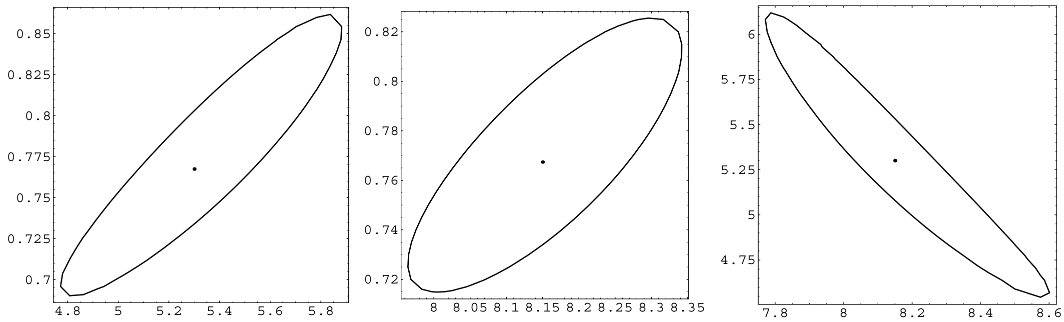

Fig 3:

Fig. 3 shows contour plots of E in (34), representing error estimates (confidence intervals) for the parameters in (35): $E(N, a, b) = \dfrac{\kappa}{n_d} E_{min}$, with κ defined (as usual) by $p = \mathrm{P}[\ \chi^2_{n_d} \leqq \kappa\ ]$, where $\chi^2_{n_d}$ is a random variable with chi-square distribution of degree of freedom n_d. The curves in the graphs correspond to the confidence level of 90% , namely, $p = 0.9$. The three figures are cross sections of $N = N^*$, $a = a^*$, $b = b^*$, respectively, in the 3-dimensional parameter space (N, a, b). Horizontal and vertical axes are respectively $a \times 10^4$ and b for the first figure, $N \times 10^{-5}$ and b for the second figure, and $N \times 10^{-5}$ and $a \times 10^4$ for the third figure. The dot in the center of each figure is the best fit (35). Fig. 3 supports $b < 1$, a standard best hit business model, rather than a long tail business model.

To see the stability of the parameters, a similar fit by adding to above mentioned data of size 77 a data of size 21 at the rightmost arc in Fig. 2, taken between November, 2007 and March, 2008, at 21:00 on every Saturday. The best fit is

$$(N^*,\ a^*,\ b^*) = (7.97 \times 10^5,\ 5.93 \times 10^{-4},\ 0.809). \tag{36}$$

The solid curve in Fig. 2 shows the theoretical curve $N\, y_C(t)$ with $y_C(t)$ as in (33) with parameters (36).

We have less data for amazon.com, the original Amazon online bookstore in USA. Fig. 4 shows a data from Amazon.com, which obviously shows a similar behavior as amazon.co.jp data Fig. 2.

3.3 Time change according to intensity measure, and title listings of 2ch.net .

Fig. 5 shows a data from a web page for a list of subject titles at a collected bulletin board 2ch.net. Each curve corresponds to the position of the title of a subject ('thread') in the page of list of titles. See [14] for a description of how the order of

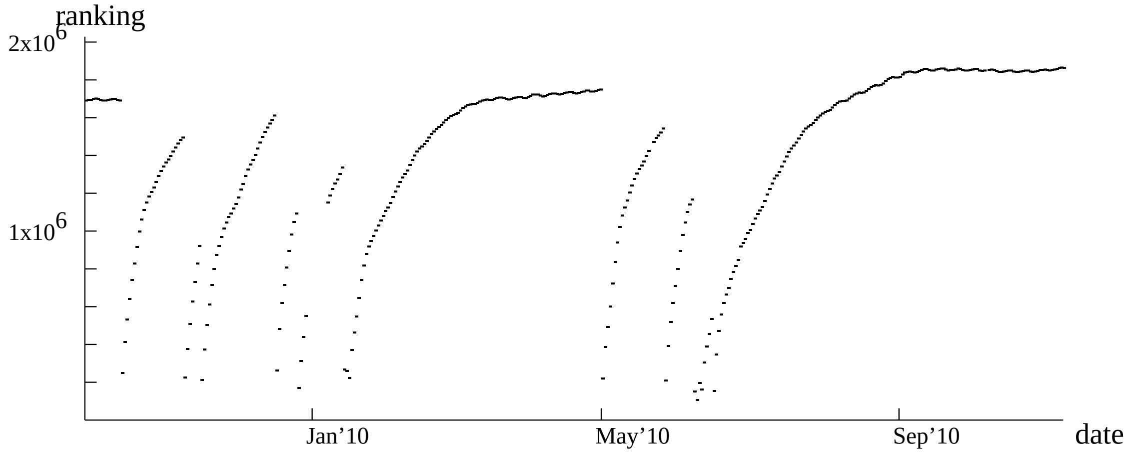

Fig 4:

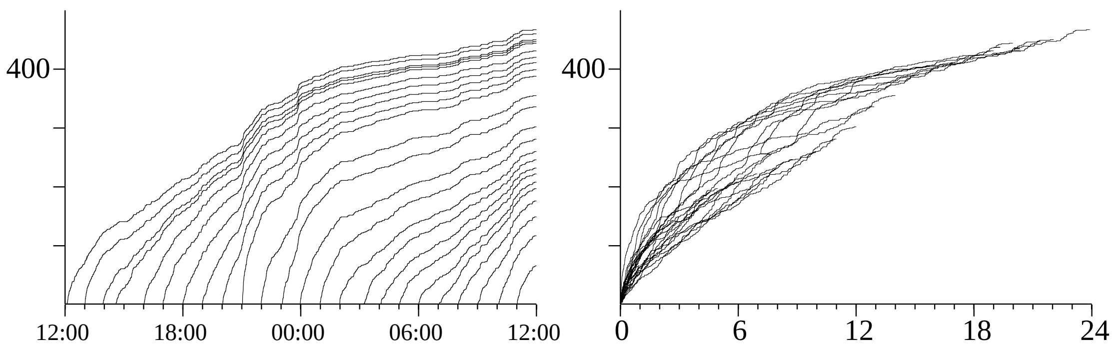

Fig 5:

the list of subject titles is organized at 2ch.net. In short, a thread title jumps to the top of the list if and only if someone writes on the thread (a 'response'), and the jump occurs instantaneously. If one assumes that responses are independent and random, then the time dependence of the thread list is a sample of the stochastic ranking process. The data is taken by Y. Takeshima for 24 hours starting on Oct. 18, 2008, 12:00 JST, using his original data collection program (master thesis, [12]). The vertical axis stands for the position in the list (the horizontal axis at the bottom stands for the top of the list), and the horizontal axis is the time axis labelled in the unit of hour. For clarity of the figure, 24 threads are chosen out of 697 threads, and for each thread, shown is the part from the last jump of the thread until the end of data collection. The second figure is a plot of same data as the first figure, but each curve is shifted in the horizontal direction, so that all the curves starts from the origin $(0, 0)$.

If the jump rate of threads are constant, then the formula (28) for the homogeneous case should be applicable, and the curves, when shifted in the horizontal direction so that the curves start from the origin $(0, 0)$, should follow a single curve defined by (28). As seen from the second figure in Fig. 5, the time dependence of position of threads do not follow a single curve. Also, the first figure in Fig. 5 clearly indicates a violation of convexity (23) at around 21:00 and 00:00. We may interpret the result as internet activities at 2ch.net being more active at night before midnight, compared to deep in the night until early in the morning.

Let us consider the factorization assumption of (24). Since the assumption is neither of logical consequence of the model nor the established social fact, such assumption should be tested by actual application of the formula to the data. Since 2ch.net is very 'transparent', concerning basic facts such as the number of threads in a board or the records of response (jump) times, it is a useful website to test (24), or any other possible practical assumptions.

For $t \geqq 0$, let

$$S^{(N)}(t) = \sum_{i=1}^{N} \nu_i^{(N)}((0,t]) \tag{37}$$

and denote its right continuous inverse by

$$s^{(N)}(t) = \inf\{s \geqq 0 \mid S^{(N)}(s) > t\}. \tag{38}$$

Let $\tilde{a} \in L^1_{loc}(\mathbb{R}_+)$. For simplicity, assume further that

$$\tilde{a}(t) > 0, \; t \geqq 0. \tag{39}$$

Then $A(t)$ of (27) is strictly increasing, and the inverse function A^{-1} is continuous.

Theorem 5 ([12, Thm 5.3, Lem. 5.4]) *Let $\tilde{a} \in L^1_{loc}(\mathbb{R}_+)$, and assume (39). Put*

$$Z(N) = \sum_{i=1}^{N} w_i^{(N)} \tag{40}$$

and assume $\lim_{N\to\infty} Z(N) = \infty$. If, as in Proposition 4, there exists a probability distribution λ on $\mathbb{R}_+$ such that (25) holds, then for each $t \geqq 0$,

$$\frac{1}{Z(N)} S^{(N)}(t) \to A(t), \quad \text{and} \quad s^{(N)}(Z(N)\,t) \to A^{-1}(t), \quad \text{in probability, as} \quad N \to \infty, \tag{41}$$

and

$$Y_C^{(N)}(s^{(N)}(Z(N)\,t)) \to y_C(A^{-1}(t)) = 1 - \int_{\mathbb{R}_+} e^{-w\,t} \lambda(dw), \tag{42}$$

in probability, as $N \to \infty$, where $Y_C^{(N)}$ is defined in (7). $\diamond$

Fig. 6 shows the cumulative total number of jumps $S^{(N)}(t)$ in (37) up to time t, for $N = 697$ threads at 2ch.net. The data is from the same board at same time as the data for Fig. 5, collected by Y. Takeshima, and Fig. 6 is accumulated by T. Kobayashi (master thesis, [12]). The dashed line denotes the hypothetical case of constant jump rates. The data is consistent with the observation made for Fig. 5 that the activities (responses) are high at night before midnight, and low between deep in the night to early in the morning.

Fig. 7 is a plot of the same data as Fig. 5, except that the horizontal axis is measured by $S^{(N)}(t)$ of Fig. 6. Fig. 7 is a revised plot of the original one by T. Kobayashi (master thesis, [12]). Compared with the second figure in Fig. 5, the second figure in Fig. 7 is apparently closer to a single curve, which supports an approximate validity of the common time dependence assumption (24).

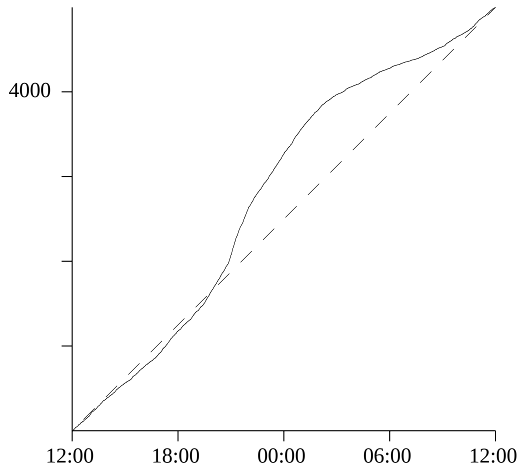

Fig 6:

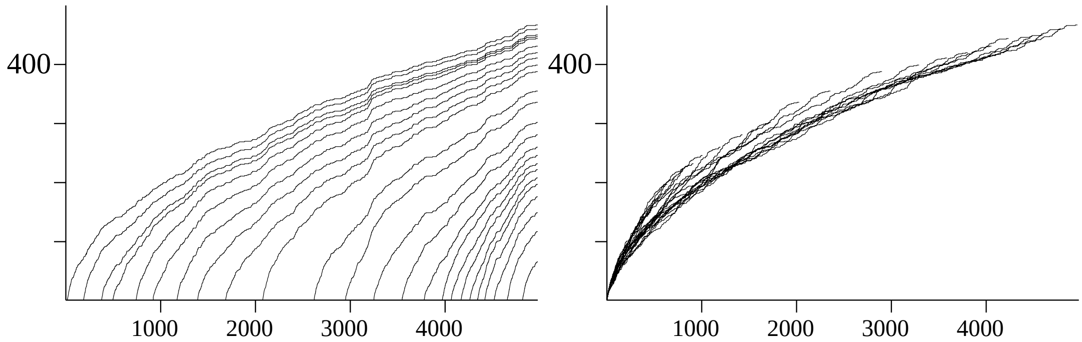

Fig 7:

Using (41) in (26), with the Pareto distribution (32) for λ,

$$x_C(t) = Ny_C(t) + 1 \simeq N - N\int_{\mathbb{R}_+} e^{-w\, S^{(N)}(t)/Z(N)}\,\lambda(dw) =$$
$$N - Ne^{-S^{(N)}(t)/(N^{1/b}\zeta_N(1/b))} + (\frac{S^{(N)}(t)}{\zeta_N(1/b)})^b\,\Gamma(1-b, \frac{S^{(N)}(t)}{N^{1/b}\zeta_N(1/b)}) =: x_b^{(N)}(S^{(N)}(t)), \tag{43}$$

where $\zeta_N(z) = \displaystyle\sum_{i=1}^{N}\frac{1}{i^z}$. Denote the data of size $n_d = 70140$ given in Fig. 7 by (s_i, x_i), $i = 1, 2, \cdots, n_d$. We performed a statistical fit of the data to (43), by minimizing $E = \displaystyle\sum_{i=1}^{n_d}\frac{(x_i - x_b^{(N)}(s_i))^2}{x_b^{(N)}(s_i)}$, with $N = 697$, and obtained $b = 0.872 \pm 0.002$ (90% CL). Apparently, we have a good single parameter fit to the data, which suggests that the practical assumption (24) is good.

We note that a smaller value of b was obtained for 2ch.net in [14] (with a different set of data). The data used in [14] was small in size, because the data was collected manually in those times, and also, to avoid influence of day-night activity difference, the data was for a short time period, hence the result in [14] is less reliable compared to the present result.

We also note that we have $b < 1$, consistently with observation for Amazon.co.jp,

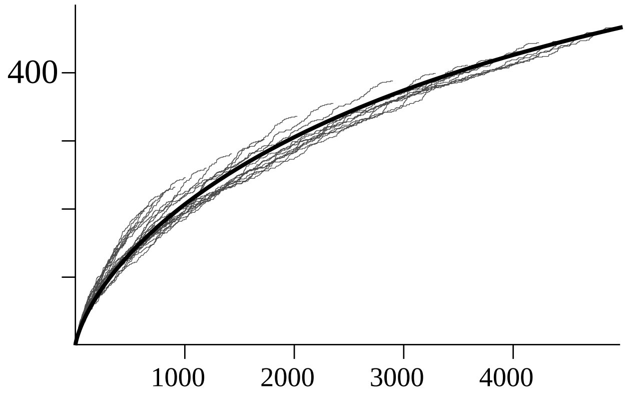

Fig 8:

where we obtained $b = 0.809$. This shows that, as in Amazon.co.jp, the popularity of subjects is concentrated on a small number of threads in 2ch.net.

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A Singular Limit Theorem in Statistical Learning Theory

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Abstract

Statistical learning theory and statistical mechanics have common mathematical structure, where the log likelihood ratio function corresponds to the random Hamiltonian. However, the log likelihood function has singularities which can not be approximated by any quadratic form, resulting that it has been difficult to analyze its partition function. In this paper, we show a singular limit theorem in statistical learning theory by using algebraic geometrical method and introduce its application to a regression problem. In the main theorem we prove that the asymptotic behaviors of generalization and training errors are determined by the two birational invariants, the log canonical threshold and the singular fluctuation. In an application, we show that the log canonical threshold can be obtained by recursive blow-ups and that the singular fluctuation can be estimated by the empirical samples.

1 Introduction

It is well known that statistical learning theory and statistical mechanics have the common mathematical structure, where the log likelihood function corresponds to random Hamiltonian. However, there are two main differences between them. Firstly, Hamiltonian in statistical learning theory is not a function but a random process. Hence the partition function is a random variable. Secondly, the ground state of the average Hamiltonian in statistical learning theory is not a single point but an analytic set with singularities. By these mathematical differences, the behaviors of the random variables in statistical learning theory have been left unknown [8, 19].

Recently, we proposed that several problems in statistical learning theory can be resolved using algebraic geometrical method [20, 22]. It was difficult to study the Boltzmann distribution in the original parameter space, however, resolution of singularities [9, 4] enables us to construct statistical mechanics in the resolution space.

In this paper, we show a singular limit theorem and its application to a concrete regression problem. In the singular limit theorem, it is proved that asymptotic behaviors of the generalization and training errors are determined by two birational invariants. In an application, we show that the concrete value of the log canonical threshold is obtained by recursive blow-ups and that the singular fluctuation can be estimated from random samples.

2 Statistical Learning Theory

Let X be an $\mathbb{R}^N$ valued random variable which is subject to the probability distribution $q(x)dx$. Assume that $D_n = (X_1, X_2, ..., X_n)$ is a set of random variables which are independently subject to the same probability distribution as X. A statistical model $p(x|w)$ is defined as a probability density function of $x \in \mathbb{R}^N$ for a given parameter $w \in W \subset \mathbb{R}^d$. Let $\varphi(w)$ be a probability density function on an open set W with compact support. The posterior distribution for $0 < \beta < \infty$ is defined by

$$p(w|D_n)dw = \frac{1}{Z}\ \exp(-\beta n H_n(w))\ \varphi(w)\ dw,$$

where $H_n(w)$ is a random Hamiltonian

$$H_n(w) = -\frac{1}{n}\sum_{i=1}^{n} \log p(X_i|w),$$

and Z is a normalizing constant. Let $E_w[\quad]$ be the expectation value using $p(w|D_n)dw$. The generalization error G and the training error T are respectively defined by

$$G = -E_X\Big[\log E_w[p(X|w)]\Big], \tag{1}$$

$$T = -\frac{1}{n}\sum_{i=1}^{n} \log E_w[p(X_i|w)]. \tag{2}$$

Since $E_w[\ \ ]$ is an expectation operator using random samples D_n, both G and T are random variables. The functional variance is defined by

$$V = \sum_{i=1}^{n} \Big\{ E_w[(\log p(X_i|w))^2] - E_w[\log p(X_i|w)]^2 \Big\}. \tag{3}$$

In this paper, we show that G, T and V are asymptotically determined by two birational invariants. Let

$$f(x, w) = \log(q(x)/p(x|w)).$$

Also let

$$S = -E_X[\log q(X)],$$

$$S_n = -\frac{1}{n}\sum_{i=1}^{n} \log q(X_i).$$

Then the relative entropy

$$K(w) = \int q(x) f(x, w) dx$$

is a nonnegative function and

$$K(w) = 0 \Longleftrightarrow q(x) = p(x|w).$$

Moreover, the generalization and training errors are given by

$$G = S - E_X\Big[\log E_w[\exp(-f(X,w))]\Big], \tag{4}$$

$$T = S_n - \frac{1}{n}\sum_{i=1}^{n}\log E_w[\exp(-f(X_i,w))]. \tag{5}$$

The functional variance is also rewritten as

$$V = \sum_{i=1}^{n}\Big\{E_w[f(X_i,w)^2] - E_w[f(X_i,w)]^2\Big\}. \tag{6}$$

Therefore asymptotic behaviors of G, T, and V are given by the statistical mechanical structure determined by $f(x,w)$. In this paper, we assume that the set $\{w \in W\ ;\ K(w) = 0\}$ is a nonempty analytic set with singularities, resulting that $\exp(-\beta n H_n(w))$ cannot be approximated by any gaussian distribution in general.

3 Two Birational Invariants

Let $L^s(q)$ $(s \geq 2)$ be a real Banach space

$$L^s(q) = \{f(x)\ ;\ \int |f(x)|^s q(x)dx < \infty\}.$$

Assume that $w \mapsto f(x,w)$ is an $L^s(q)$-valued analytic function on W. Then $K(w)$ is a nonnegative analytic function. By using resolution of singularities [9, 4], there exist both a manifold $\mathcal{M}$ and a real analytic map $g : \mathcal{M} \to W$ such that, in each local coordinate of $\mathcal{M}$,

$$K(g(u)) = u^{2k} \equiv u_1^{2k_1}u_2^{2k_2}\cdots u_d^{2k_d}, \tag{7}$$

$$\varphi(g(u))|g'(u)| = u^h b(u) \equiv u_1^{h_1}u_2^{h_2}\cdots u_d^{h_d}b(u), \tag{8}$$

where $k = (k_1, k_2, ..., k_d)$ and $h = (h_1, h_2, ..., h_d)$ are sets of nonnegative integers, $|g'(u)|$ is the Jacobian determinant of $w = g(u)$, and $b(u) > 0$. Let $\{\alpha\}$ be a set of all local coordinates of $\mathcal{M}$. The *log canonical threshold* λ is defined by

$$\lambda = \min_{\alpha}\min_{j=1}^{d}\Big(\frac{h_j+1}{2k_j}\Big), \tag{9}$$

where we put $(h_j+1)/k_j = \infty$ for $k_j = 0$. Let $\{\alpha^*\}$ be the set of all local coordinates in which the above minimum is attained. By using

$$K(g(u)) = \int S(x,g(u))\, q(x)dx,$$

$$S(x,g(u)) \equiv e^{-f(x,g(u))} + f(x,g(u)) - 1 \geq 0,$$

and $f(x,g(u))$ is an analytic function of u, $f(x,g(u))^2$ can be divided by u^{2k}. In other words, there exists a function-valued analytic function $a(x,u)$ such that

$$f(x,g(u)) = a(x,u)u^k.$$

Moreover, from $K(w) = E_X[f(X,w)]$, we have $E_X[a(X,u)] = u^k$. Let $\xi(u)$ be a gaussian random process on $\mathcal{M}$ which is uniquely determined by its expectation and covariance,

$$\begin{aligned} E_\xi[\xi(u)] &= 0, \\ E_\xi[\xi(u)\xi(v)] &= E_X[a(X,u)a(X,v)] - E_X[a(X,u)]E_X[a(X,v)]. \end{aligned}$$

The *singular fluctuation* ν is defined by

$$\nu = \frac{\beta}{2} E_\xi E_X \Big[\langle a(X,u)^2 t \rangle - \langle a(X,u)\sqrt{t} \rangle^2 \Big],$$

where $\langle \ \rangle$ shows the expetation value over a renormalized posterior distribution,

$$\langle F(u,t) \rangle = \frac{\sum_{\alpha^*} \int dt \int du^* \ F(u,t) \ t^{\lambda-1} \exp(-\beta t - \beta\sqrt{t}\xi(u))}{\sum_{\alpha^*} \int dt \int du^* \ t^{\lambda-1} \exp(-\beta t - \beta\sqrt{t}\xi(u))},$$

where du^* is a measure whose support is contained in the set $\{u \in \mathcal{M}; K(g(u)) = 0\}$. Note that neither λ nor ν depends on the choice of desingularization $(\mathcal{M}, g)$, hence they are birational invariants.

4 Main Theorem

The following is the main theorem of this paper and its short proof. The mathematically rigorous proof is shown in [23, 22].

Theorem 1 *The following asymptotic expansions hold as $n \to \infty$,*

$$\begin{aligned} E[G] &= S + \Big(\frac{\lambda-\nu}{\beta} + \nu\Big)\frac{1}{n} + o(\frac{1}{n}), \\ E[T] &= S + \Big(\frac{\lambda-\nu}{\beta} - \nu\Big)\frac{1}{n} + o(\frac{1}{n}), \\ E[V] &= \frac{2\nu}{\beta} + o(1). \end{aligned}$$

where $o(n^{-\alpha})$ shows a function of n which satisfies

$$\lim_{n\to\infty} \sup n \ o(n^{-\alpha}) < \infty.$$

(Short Proof) Let us introduce a generating function,

$$F_n(\alpha) = E\Big[-\log \int \exp(-\alpha f(X,w) - \beta n H_n(w))\varphi(w)dw\Big],$$

where $E[\ ]$ shows the expectation value over $X_1, X_2, .., X_n$ and X. Then, by using eqs.(4),(5),(6), it immediately follows that

$$E[G] = S + F_n(1) - F_n(0), \quad (10)$$

$$E[T] = S + F_{n-1}(1+\beta) - F_{n-1}(\beta), \quad (11)$$

$$E[V] = -nF''_{n-1}(\beta). \quad (12)$$

Therefore, there exist $0 < \alpha^*, \alpha^{**}, \alpha^{***} < 1 + \beta$ such that

$$E[G] = S + F_n'(0) + \frac{1}{2}F_n''(0) + \frac{1}{6}F^{(3)}(\alpha^*), \tag{13}$$

$$\begin{aligned} E[T] &= S + F_{n-1}'(0) + \frac{2\beta+1}{2}F_{n-1}''(0) \\ &+\frac{1}{6}\Big((1+\beta)^3 F_{n-1}^{(3)}(\alpha^{**}) - \beta^3 F_{n-1}^{(3)}(\alpha^{***})\Big). \end{aligned} \tag{14}$$

Since $K(w)$ is an analytic function, we can apply resolution of singularities [9, 4] to $K(w)$, and obtain eq.(7),(8).

Let us define an empirical process on $\mathcal{M}$,

$$\xi_n(u) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{a(X_i, u) - u^k\}.$$

Then the probability distribution of $\xi_n(u)$ converges to that of the gaussian process $\xi(u)$ by using Prohorov's theorem [16]. The gaussian process $\xi(u)$ can be represented by

$$\xi(u) = \sum_{j=1}^{\infty} c_j(u) g_j$$

where $\{g_j\}$ are independent random variables and each g_j is subject to the standard normal distribution. Then

$$E_\xi[\xi(u)\xi(u')] = \sum_{j=1}^{\infty} c_j(u) c_j(u').$$

The random Hamiltonian on the manifold $\mathcal{M}$ is represented by

$$nH_n(g(u)) = nu^{2k} - \sqrt{n}u^k \xi_n(u).$$

To study the generatining function $F_n(\alpha)$, we need the asymptotic behavior of

$$Z_n(s) = \int f(x, w)^s \exp(-\beta n H_n(w)) \varphi(w) dw,$$

where $s \geq 0$ is a real value. For example,

$$F_n'(0) = E\Big[\frac{Z_n(1)}{Z_n(0)}\Big], \tag{15}$$

$$F_n''(0) = -E\Big[\frac{Z_n(2)}{Z_n(0)}\Big] + E\Big[\frac{Z_n(1)}{Z_n(0)}\Big]^2. \tag{16}$$

Then by using the function $w = g(u)$,

$$\begin{aligned} Z_n(s) &= \sum_\alpha \int du\, a(x,u)^s u^{sk+h} \exp(-\beta n u^{2k} + \beta\sqrt{n}u^k \xi_n(u)) b_\alpha(u) \\ &= \sum_\alpha \int_0^\infty dt \int du\, \frac{1}{n}\, \delta\Big(\frac{t}{n} - u^{2k}\Big)\, a(x,u)^s u^{sk+h} \\ &\quad \exp(-\beta t + \beta\sqrt{t}\xi_n(u)) b_\alpha(u), \end{aligned}$$

where $\sum_\alpha$ shows the sum over all local coordinates and $b_\alpha(u) \geq 0$ satisfies $\sum_\alpha b_\alpha(u) = b(u)$. By using the asymptotic expansion of the Schwartz distribution $\delta(t/n - u^{2k})$ for $n \to \infty$ [20, 22, 23, 7, 10, 14, 15], there exists a Schwartz distribution $D_\alpha(u)$ such that

$$\sum_\alpha \frac{1}{n}\,\delta\Big(\frac{t}{n} - u^{2k}\Big)\; u^{sk+h}\; b_\alpha(u) \cong \frac{(\log n)^{m-1}}{n^{\lambda+s/2}}\; t^{\lambda-1+s/2}\Big(\sum_{\alpha^*} D_{\alpha^*}(u)\Big),$$

where $\lambda > 0$ is the log canonical threshold and m is the maximum number of j which attains the minimum in eq.(9). Also $\sum_{\alpha^*}$ shows the sum over all local coordinates that attain the above minimum and the support of $D_{\alpha^*}(u) \equiv du^*$ is contained in the set $\{u \in \mathcal{M}; K(g(u)) = 0\}$. Hence

$$Z_n(s) \cong \frac{(\log n)^{m-1}}{n^{\lambda+s/2}}\Big(\int \mathcal{D}(u,t) t^{s/2} \exp(\beta\sqrt{t}\xi(u))\Big).$$

where $\int \mathcal{D}(u,t)$ is defined by the integration over the manifold,

$$\int \mathcal{D}(u,t) = \sum_{\alpha^*}\int_0^\infty dt \int du D_{\alpha^*}(u)\; t^{\lambda-1}\; \exp(-\beta t).$$

Let us define

$$\hat{Z}(q,r,s) = \int \mathcal{D}(u,t)\; \xi(u)^q\; t^{r/2}\; a(x,u)^s\; \exp(\beta\sqrt{t}\xi(u)).$$

Then

$$Z_n(s) \cong \frac{(\log n)^{m-1}}{n^{\lambda+s/2}} \hat{Z}(0,s,s). \tag{17}$$

Firstly, since $E_X[a(X,u)] = u^k$,

$$E_X[\hat{Z}(0,1,1)] = \hat{Z}(0,2,0).$$

Secondly, by using the partial integration of t

$$\begin{aligned}\int_0^\infty dt\; t^\lambda e^{-\beta t+\beta\sqrt{t}\xi(u)} &= \frac{\lambda}{\beta}\int_0^\infty dt\; t^{\lambda-1} e^{-\beta t+\beta\sqrt{t}\xi(u)} \\ &\quad + \frac{1}{2}\int_0^\infty dt\; t^{\lambda-1/2}\xi(u) e^{-\beta t+\beta\sqrt{t}\xi(u)},\end{aligned}$$

it follows that

$$\hat{Z}(0,2,0) = \frac{\lambda}{\beta}\hat{Z}(0,0,0) + \frac{1}{2}\hat{Z}(1,1,0).$$

And lastly, by using the partial integration over the gaussian process $\xi(u)$,

$$\begin{aligned} E_\xi\Big[\frac{\hat{Z}(1,1,0)}{\hat{Z}(0,0,0)}\Big] &= E_\xi\Big[\int \mathcal{D}(u,t)\Big(\sum_{j=1}^\infty c_j(u) g_j\Big)\frac{t^{1/2}\exp(\beta\sqrt{t}\xi(u))}{\int \mathcal{D}(u',t')\exp(\beta\sqrt{t'}\xi(u'))}\Big] \\ &= E_\xi\Big[\int \mathcal{D}(u,t)\Big(\sum_{j=1}^\infty c_j(u)\frac{\partial}{\partial g_j}\Big)\frac{t^{1/2}\exp(\beta\sqrt{t}\xi(u))}{\int \mathcal{D}(u',t')\exp(\beta\sqrt{t'}\xi(u'))}\Big] \\ &= \beta E_X E_\xi\Big[\frac{\hat{Z}(0,2,2)}{\hat{Z}(0,0,0)}\Big] - \beta E_X E_\xi\Big[\frac{\hat{Z}(0,1,1)}{\hat{Z}(0,0,0)}\Big]^2 \qquad (18) \\ &= 2\nu, \qquad (19) \end{aligned}$$

where we used $E_\xi[\xi(u)\xi(u')] = E_X[a(X,u)a(X,u')]$ on the set $\{u; K(g(u)) = 0\}$. Then by using eqs.(15),(16),(17), we can show

$$\begin{aligned} F_n'(0) &\cong (\frac{\lambda}{\beta}+\nu)\cdot\frac{1}{n}, \\ F_n''(0) &\cong -\frac{2\nu}{\beta}\cdot\frac{1}{n}, \\ |F_n'''(\alpha)| &\cong \frac{1}{n^{3/2}}. \end{aligned}$$

Therefore, we obtain

$$E[G] = S + (\frac{\lambda-\nu}{\beta}+\nu)\frac{1}{n} + o(\frac{1}{n}), \tag{20}$$

$$E[T] = S + (\frac{\lambda-\nu}{\beta}-\nu)\frac{1}{n} + o(\frac{1}{n}), \tag{21}$$

$$E[V] = \frac{2\nu}{\beta} + o(1), \tag{22}$$

which completes the proof (Q.E.D.)

5 Application to Statistical Learning Theory

5.1 Log Canonical Threshold

Let us study an application to statistical learning theory. A statistical model of $x = (x_1, y_1)$ for a parameter $w = (a, b, c, d)$ is defined by

$$p(x|w) = \frac{q_0(x_1)}{\sqrt{2\pi}}\exp(-\frac{1}{2}(y_1 - F(x_1,w))^2),$$

where $q_0(x_1)$ is a probability density function of x_1 which satisfies

$$\int x_1^n q_0(x_1)dx_1 < \infty \quad (n = 0, 1, 2, 3, \cdots).$$

and

$$F(x_1, w) = a\sigma(bx) + c\sigma(dx),$$

where

$$\sigma(x) = e^x - 1.$$

Assume that the true distribution is given by

$$q(x) = p(x|0,0,0,0).$$

Then it is easy to show that there exist $c_1, c_2 > 0$ such that

$$c_1K_0(w) \le K(w) \le c_2K_0(w),$$

where

$$K_0(w) = (ab + cd)^2 + (ab^2 + cd^2)^2.$$

Therefore the log canonical threshold of $K(w)$ is equal to that of $K_0(w)$. Resolution of singularities can be found as follows.
(1) Firstly a blow-up with center $\mathbb{V}(b,d) \subset \mathbb{V}(K_0)$ is tried. By $b = b_1 d$,

$$K_0(w) = d^2\{(ab_1 + c)^2 + d^2(ab_1^2 + c)^2\}.$$

Since K_0 is symmetric for (b,d), we need not try $d = bd_1$. The transform $c_1 = ab_1 + c$ is an analytic isomorphim and its Jacobian determinant is equal to one.

$$K_0(w) = d^2\{c_1^2 + d^2(ab_1^2 + c_1 - ab_1)^2\}.$$

(3) The second step is the blow-up with center $\mathbb{V}(c_1,d) \subset \mathbb{V}(K_0)$. In the first local coordinate, by $d = c_1 d_1$, it follows that

$$K_0(w) = c_1^4 d_1^2\{1 + d_1^2(ab_1^2 + c_1 - ab_1)^2\},$$

which is normal crossing. In the second local coordinate, by $c_1 = c_2 d$, it follows that

$$K_0(w) = d^4\{c_2^2 + (ab_1^2 + c_2 d - ab_1)^2\},$$

which is not normal crossing.
(4) The third step is the blow-up with center $\mathbb{V}(a,c_2) \subset \mathbb{V}(K_0)$. By $a = c_2 a_1$, K_0 is made normal crossing. By $c_2 = ac_3$, it follows that

$$K_0(w) = d^4 a^2\{c_3^2 + (b_1^2 + c_3 d - b_1)^2\}$$

which is not yet normal crossing.
(5) The fourth step is the blow-up with center $\mathbb{V}(b_1,c_3) \subset \mathbb{V}(K_0)$. By $b_1 = c_3 b_2$, K_0 is made normal crossing. By $c_3 = b_1 c_4$, it follows that

$$K_0(w) = a^2 b_1^2 d^4\{c_4^2 + (b_1 + c_4 d - 1)^2\}, \tag{23}$$

which is not yet normal crossing.
(6) The last step is the blow-up with $\mathbb{V}(b_1 - 1, c_4) \subset \mathbb{V}(K_0)$. $b_1 - 1 = c_4 b_2$ makes KL_0 normal crossing. Also $c_4 = c_5(b_1 - 1)$ results in

$$K_0(w) = d^4 a^2 b_1^2 (b_1 - 1)^2\{c_5^2 + (1 + c_5 d)^2\},$$

which is normal crossing. The last coordinate is given by

$$\begin{aligned} a &= a, \\ b &= b_1 d, \\ c &= a(b_1 - 1)b_1 c_5 d - ab_1, \\ d &= d, \end{aligned}$$

whose Jacobian determinant is given by

$$|g'| = |ab_1(b_1 - 1)d^2|.$$

Therefore the log canonical threshold is $\lambda = 3/4$.

5.2 Application to statistics

Unfortunately, the singular fluctuation of the general still can not be calculated. However, it can be estimated from random samples from V. Hence we can estimate $E[G]$ from $E[T]$ and $E[V]$ without any knowledge of $q(x)$ by *equation of state in statistical learning*,

$$E[G] = E[T] + \frac{\beta}{n}E[V] + o(\frac{1}{n}).$$

This equation holds for an arbitrary $(q(x), p(x|w), \varphi(w))$, which can be understood as the equation of state for Boltzmann distribution $p(w|D_n)$ in statistical learning theory. The equation of state in ideal gas is useful in many sciences, whereas the equation of state in learning theory is useful in statistical sciences.

6 Conclusion

In this paper, a singular limit theorem in statistical learning theory was proved and an application to a concrete regression model was introduced.

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