# Discrete Differential Geometry of Curves and Surfaces 

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## Preface

This book is the result of a 2 hour a week course I gave at the faculty of mathematics at Kyushu University, Fukuoka in summer 2008. It started out as a typeset version of my handwritten preparations that I made available to the students as a supplement and a compensation for my lazy blackboard hand writing. But after a couple of weeks I was offered the opportunity of turning them into these lecture notes.

Discrete differential geometry has its roots in the 1950s when mathematicians like Robert Sauer and Walter Wunderlich started to investigate difference analogs of curves and surfaces. Later in the 1990s Ulrich Pinkall and Alexander Bobenko made crucial connections between these discrete objects and discrete integrable equations studied in mathematical physics. These integrable equations can be viewed as the second source of discrete differential geometry in its modern form. From then on the focus somewhat shifted, when applications of the discrete structures became apparent in computer graphics and industrial mathematics.

The material covered in this book is by no means a comprehensive overview of the emerging field of discrete differential geometry but I hope that it can serve as an introduction. Choosing material in a rapidly growing field like this is difficult. On the one hand there is already too much to cover in one course on the other hand, the chosen material can become out of date rather fast and interesting results can even appear during the ongoing semester.

The choice I made was to roughly follow what one usually finds in introductionary courses in differential geometry of curves and surfaces. This of course means that many aspects of discrete differential geometry are very much underrepresented. In the introduction I will give some suggestions for further reading that will broaden the view on the subject in several directions.

Discrete differential geometry investigates discrete analogs of objects of smooth differential geometry. Thus, through the notes I refer to various notions of classical differential geometry. But while knowledge of basic differential geometry is of course helpful, most of the material should be understandable without knowing the smooth origin of the various notions.

I decided to leave the original segmentation in individual lectures visible. Although it might not always align with the contents, this is how I prepared and taught the coures. You will find the number of the current lecture in the heading of each page.

I am grateful for all the hospitality and encouragement I got from the
department of mathematics at Kyushu university and I feel that my stay there in 2008 was way too short. I owe many thanks to my students who attended the course as well: They gave valuable feedback and found many errors in the first versions of this text. It goes without saying that I am responsible for any errors - orthographic and contend wise.

Tim Hoffmann - Munich, 21.3.2008

## 1 Introduction

## Motivation

## Name of the game

The name "Discrete differential geometry" (DDG) sounds like a "contradictio in res" ${ }^{1}$. The term differential in differential geometry (DG) stands for the usage of calculus and analysis for studying geometry and this clearly needs smooth and not discrete objects. But differential geometry is not only a set of techniques but it rather stands for notions and constructions or definitions as well and discrete differential geometry aims for finding discrete equivalents for these objects of investigation.

Of course there is not only one valid (or correct) discretization of, say, a surface. So which is the right one or the best one? There is of course no answer to this and in discrete differential geometry one often has several discretizations of the same thing coexisting and which to prefer is a question of what one intends to do with it. However, originally discrete differential geometry arose from the observation that certain discretizations are more than a mere approximation of a smooth object and that they possess interesting properties of their own.

In some sense discrete differential geometry can be considered more fundamental than differential geometry since the later can be obtained form the former as a limit. Discrete differential geometry is richer since it deals with more ingredients (like combinatorics) and since some structures get lost in the smooth limit (e. g. the tangential flow on a smooth regular curve is trivial while on a discrete curve it is not, or the symmetry between surfaces and their transformations).

The recent interest in discrete differential geometry is partly due to its applications in computer graphics (texture maps, re-meshing,...), architecture (glass and steel constructions for freeform models usually need planar face meshes), numerics (partial difference equations), simulations (flows, deformations), and physics (lattice models).

Since this course can not cover the many facets of discrete differential geometry further reading is highly recommended: For a very structural presentation of the subject that emphasizes the underlying integrable structure I recommend the book of Bobenko and Suris [BS08]. An introduction into

[^0]the architectural aspects can be found in the beautiful book by Pottmann et al [PAHA07]. The most classical book on the subject should be mentioned here as well: It is the book "Differenzengeometrie" by R. Sauer [Sau70] which can be considered as one of the roots of discrete differential geometry.

## 2 discrete curves in $\mathbb{R}^{2}$ and $\mathbb{C P}^{1}$

## 2.1 basic notions

We will start very simple, by discretizing the notion of a smooth curve. That is, we want to define a discrete analog to a smooth map from an interval $I \subset \mathbb{R}$ to $\mathbb{R}^{n}$. By discrete we mean here that the map should not be defined on an interval in $\mathbb{R}$ but on a discrete (ordered) set of points therein. It turns out that this is basically all we need to demand in this case:

Definition 2.1 Let $I \subset \mathbb{Z}$ be an intervat (possibly infinite). A map $\gamma: I \rightarrow$ $\mathbb{R}^{n}$ is called a discrete curve. It is in fact a polygon. $\gamma$ is said to be periodic (or closed) if $I=\mathbb{Z}$ and if there is a $p \in \mathbb{Z}$ such that $\gamma(k)=\gamma(k+p)$ for all $k \in I$.

We will write $\gamma_{k}=\gamma(k)$ and even $\gamma=\gamma_{k}, \gamma_{1}=\gamma_{k+1}$, and $\gamma_{\overline{1}}=\gamma_{k-1}$. This looks silly at first but it will turn out handy once we denote shifts in different directions with $\gamma_{1}(=\gamma(k+1, l)), \gamma_{2}=(\gamma(k, l+1)), \gamma_{1 \overline{2}}(=\gamma(k+1, l-1))$, etc.

In classical differential geometry the next notions one would define for a curve would be the tangent vector (its derivative) and the arc-length of a curve. So let us do this for our discrete curve as well.

Definition 2.2 The edge tangent vector of a discrete curve $\gamma: I \rightarrow \mathbb{R}^{n}$ is defined as the forward difference

$$
S_{k}:=\gamma_{k+1}-\gamma_{k}
$$

(we could have written $S:=\gamma_{1}-\gamma$ as well).
Sometimes we will write $\Delta \gamma_{k}:=\gamma_{k+1}-\gamma_{k}$ for the forward differences.
Remark. Note that while the edge tangent vector $S_{k}$ ( $S$ for segment) is labeled like the point $\gamma_{k}$ it should be thought as attached to the edge between $\gamma_{k}$ and $\gamma_{k+1}$.

[^1]Also note that we called the forward differences edge tangent vectors since we will introduce a second type of tangent vectors (located at the vertices) later on.

The arc-length of a curve is usually introduced as the integral of the length of the tangent vector and in our discrete case this turns out to be exactly the length of the polygon: :

Definition 2.3 arc-length of a discrete curve $\gamma: I \rightarrow \mathbb{R}^{n}$ is defined as

$$
L(\gamma):=\sum_{k \in I}\left\|\Delta \gamma_{k}\right\|
$$

Example 2.1 The discrete curve $\gamma: \mathbb{N} \rightarrow \mathbb{R}^{2} \cong \mathbb{C}$,

$$
\gamma_{k}=e^{k(-\epsilon+i \delta)}, \quad \epsilon, \delta>0
$$

is a discretization of the logarithmic spiral (see Fig. 1 ): $t \mapsto e^{t(-\epsilon+i \delta)}$. It is (like its smooth counterpart) invariant under a homothety $x \mapsto e^{(-\epsilon+i \delta)} x$. Let us calculate its arc-length; we set $\omega=(-\epsilon+i \delta)$.

$$
\begin{aligned}
L(\gamma) & =\sum_{k=0}^{\infty}\left|e^{(k+1) \omega}-e^{k \omega}=\left|e^{\omega}-1\right| \sum_{k=0}^{\infty}\right| e^{k \omega} \mid \\
& =\left|e^{\omega}-1\right| \sum_{k=0}^{\infty}\left(e^{-\epsilon}\right)^{k}=\left|e^{-\epsilon+i \delta}-1\right| \frac{1}{1-e^{-\epsilon}}
\end{aligned}
$$



Figure 1: A discrete logarithmic spiral.
We see that the arc-length of the discrete logarithmic spiral is finite.

A smooth curve is called parameterized by arc-length, if its derivative is of unit length everywhere. We adopt this definition as well:

Definition 2.4 $A$ discrete curve $\gamma: I \rightarrow \mathbb{R}^{n}$ is called parameterized by arc-length if $\left\|\Delta \gamma_{k}\right\| \equiv 1$.

Remark.

- Unlike the smooth case we can not reparameterize a curve. A discrete curve is parameterized by arc-length or it is not.
- Sometimes it is convenient to call a curve with $\left\|\Delta \gamma_{k}\right\| \equiv c \neq 0$ parameterized by arc-length as well.

Example 2.2 A regular $n-g o n$ inscribed in a circle of radius $r=\frac{1}{2 \sin \frac{\pi}{n}}$ is parameterized by arc-length. It can of course be viewed as a discretized circle. Note however, that a closed arc-length parameterized discrete circle is not possible for all radii.

Example 2.3 $A$ cycloide is a curve obtained by tracing a point on a circle, while rolling the circle on an axis. Again we will discretize this in the most naive way (see Fig. 2):


Figure 2: A discrete cycloide.
We "roll" a regular n-gon on the $x$-axis and mark the position of a distinguished vertex whenever an edge of the $n$-gon lies on the axis. The resulting discrete curve is given by:

$$
\gamma_{k}=\sum_{l=0}^{k}\left(1-e^{-i l \frac{2 \pi}{n}}\right) .
$$

In the above example, whenever $e^{-i k \frac{2 \pi}{n}}=1$, we have $\Delta \gamma_{k-1}=0$ ! This corresponds to the cusps in the smooth cycloide, where it is not regular ${ }^{3}$. This motivates the following definition:

Definition 2.5 $A$ discrete curve $\gamma: I \rightarrow \mathbb{R}^{n}$ is called regular if any three successive points are pairwise disjoined.

Remark.

- This implies that for regular discrete curves the edge tangent vectors do not vanish.
- To archive this it would have been sufficient to ask for any two successive points to be disjoined. However, a little later we will define vertex tangent vectors as well and this definition will ensure their nondegeneracy as well.

Reparameterization of a curve can be understood in different ways: Firstly $\tilde{\gamma}: \tilde{I} \rightarrow \mathbb{R}^{n}$ is a reparameterization of $\gamma: I \rightarrow \mathbb{R}^{n}$ if there is a diffeomorphism $\phi: \tilde{I} \rightarrow I$ such that $\tilde{\gamma}=\gamma \circ \phi$. Secondly, as a (trivial) tangential flow. If $\gamma$ is arc-length parameterized we can look at the flow given by $\dot{\gamma}=\gamma^{\prime}$ (Here "dot" denotes the time derivative while "prime" stands for the derivative with respect to arc-length). The flow for this deformation is simply a reparameterization $\gamma(s, t)=\gamma(s+t)$, that even preserves arc-length.

The first interpretation has only a trivial discrete analog: If $I$ and $\tilde{I}$ are two discrete intervals of the same cardinality, we can pre-compose $\gamma: I \rightarrow \mathbb{R}^{n}$ with the order preserving or the order reversing bijection $\phi: \tilde{I} \rightarrow I$ to form $\tilde{\gamma}=\gamma \circ \phi$. The second interpretation however, is interesting:

Definition 2.6 $A$ flow on a discrete curve $\gamma: I \rightarrow \mathbb{R}^{n}$ is a continuous deformation

$$
\gamma_{t}: I \times \mathbb{R} \supseteq I \times J \rightarrow \mathbb{R}^{n}
$$

of $\gamma$ with $\gamma_{0}=\gamma$, that is given by a vector field $v:=\dot{\gamma}=\frac{d}{d t} \gamma: I \times J \rightarrow \mathbb{R}^{n}$ describing the evolution

$$
\dot{\gamma}=\frac{d}{d t} \gamma .
$$

[^2]Thus, to formulate the notion of a tangential flow, we will need the notion of a tangential vector at the vertices of a discrete curve (to prescribe the direction). If the discrete curve $\gamma$ is arc-length parameterized, the simplest guess here is to average the edge tangent vectors:

$$
T=\frac{1}{2}\left(\Delta \gamma+\Delta \gamma_{\overline{1}}\right) \quad\left(\text { or } T_{k}=\frac{1}{2}\left(\Delta \gamma_{k}+\Delta \gamma_{k-1}\right)\right)
$$

For arbitrary curves there is a better choice:
Definition 2.7 The vertex tangent vector of a discrete curve $\gamma: I \rightarrow \mathbb{R}^{2} \cong$ $\mathbb{C}$ is given by

$$
T:=\Delta^{h} \gamma:=2 \frac{\Delta \gamma \Delta \gamma_{\overline{1}}}{\Delta \gamma+\Delta \gamma_{\overline{1}}} .
$$

$T$ is the harmonic mear $4^{4}$ of the edge tangent vectors.
Remark.

- The definition works in $\mathbb{R}^{n}$ as well, since any three points will lie in a common $\mathbb{R}^{2}$.
- The definition is the reason for our definition of regularity. For a regular discrete curve no vertex tangent vector will neither be zero nor infinite.
- If $\gamma$ is arc-length parameterized one has

$$
\begin{aligned}
\Delta^{h} \gamma & =2 \frac{\Delta \gamma \Delta \gamma_{\overline{1}}}{\Delta \gamma+\Delta \gamma_{\overline{1}}}=2 \frac{\Delta \gamma \Delta \gamma_{\overline{1}}\left(\Delta \bar{\gamma}+\Delta \bar{\gamma}_{\overline{1}}\right)}{\left\|\Delta \gamma+\Delta \gamma_{\overline{1}}\right\|^{2}}=2 \frac{\Delta \gamma_{\overline{1}}+\Delta \gamma}{\left\|\Delta \gamma+\Delta \gamma_{\overline{1}}\right\|^{2}} \\
& =\frac{S+S_{\overline{1}}}{1+\left\langle S, S_{\overline{1}}\right\rangle}
\end{aligned}
$$

So in case of an arc-length parameterized curve the vertex tangent vector points in the same direction as the averaged edge tangent vectors.

- Note however, that an arc-length parameterized curve usually does not have vertex tangent vectors of constant length!

Definition 2.8 A tangential flow for an arc-length parameterized discrete curve $\gamma: I \rightarrow \mathbb{C}$ is a flow whose vector field points in direction of the vertex tangent vector and that preserves the arc-length parameterization.
${ }^{4}$ The harmonic mean is the inverse of the arithmetic mean of the inverse quantities.

Proposition 2.9 Every tangential flow is a constant multiple of the vertex tangent vector field:

$$
\dot{\gamma}=\alpha \Delta^{h} \gamma
$$

Proof. The condition for the curve to stay arc-length parameterized under the flow $\dot{\gamma}$ reads

$$
0=\frac{\partial}{\partial t}\langle\Delta \gamma, \Delta \gamma\rangle .
$$

Expanding this gives

$$
0=2\langle\Delta \gamma, \Delta \dot{\gamma}\rangle=2\left\langle\Delta \gamma, \alpha_{1} \Delta^{h} \gamma_{1}-\alpha \Delta^{h} \gamma\right\rangle
$$

But

$$
\begin{aligned}
\left\langle\Delta \gamma, \Delta^{h} \gamma\right\rangle & =\left\langle\Delta \gamma, \frac{\Delta \gamma+\Delta \gamma_{\overline{1}}}{1+\left\langle\Delta \gamma, \Delta \gamma_{\overline{1}}\right\rangle}\right\rangle \\
& =\frac{\left\langle\Delta \gamma, \Delta \gamma \gamma+\left\langle\Delta \gamma, \Delta \gamma_{\overline{1}}\right\rangle\right.}{1+\left\langle\Delta \gamma, \Delta \gamma_{\overline{1}}\right\rangle}=1 \quad \text { and } \\
\left\langle\Delta \gamma, \Delta^{h} \gamma_{1}\right\rangle & =\left\langle\Delta \gamma, \frac{\Delta \gamma_{1}+\Delta \gamma}{1+\left\langle\Delta \gamma_{1}, \Delta \gamma\right\rangle}\right\rangle \\
& =\frac{\left\langle\Delta \gamma, \Delta \gamma_{1}\right\rangle+\langle\Delta \gamma, \Delta \gamma\rangle}{1+\left\langle\Delta \gamma_{1}, \Delta \gamma\right\rangle}=1 .
\end{aligned}
$$

Thus one can deduce $0=\alpha_{1}-\alpha(=\Delta \alpha)$ and $\alpha \cong$ const.

## Remark.

- This is the first justification for our peculiar choice of the vertex tangent vector.
- In the smooth case a straight line can be characterized by the fact, that its tangential flow is a translation. This is no longer true in the discrete case. A regular zig-zag constitutes a counter example. It is a common phenomenon in discrete differential geometry, that the discretization has some extra freedom which will vanish in a smooth limit (but which might allow for non-smooth limits as well).


Figure 3: For this zig-zag the tangential flow is a translation.

## 2.2 curvature

The curvature of a plane curve is defined to be the inverse of the radius of its osculating circle ${ }^{5}$. Thus a way to establish a notion of curvature for a discrete curve is to define a discrete osculating circle.

We will consider three choices here:
vertex osculating circles:
The vertex osculating circle of a discrete curve at a point $\gamma_{k}$ is given by the unique circl $\underbrace{6}$ through the point and its two nearest neighbours $\gamma_{k \pm 1}$ (see Fig 44).

Defining the osculating circle is through three neighbouring points gives just the "right" non-locality since it corresponds to involving second derivatives. This choice for an osculating circle also matches our choice for the vertex tangent vector as the following lemma emphasizes:

Lemma 2.10 The vertex tangent vector at a point $\gamma$ is always tangential to the vertex osculating circle at that point.

[^3]

Figure 4: The vertex osculating circle.

To show this we first prove the following elementary lemma:

Lemma 2.11 Given a triangle $A B C$, let $M$ be the circumcenter (the center of the circumscribing circle), and let $\alpha=\angle(A B, A C), \beta=\angle(B C, B A)$, and $\gamma=\angle(C A, C B)$. Then

$$
\angle(M B, M C)=2 \alpha
$$

as illustrated in Fig. 5.


Figure 5: Angles in a triangle.

Proof. The triangle $M C A$ is isosceles so $\angle(M C, M A)=\pi-2 \angle(A M, A C)$. The same argument holds for $M A B$ giving $\angle(M A, M B)=\pi-2 \angle(A B, A M)$.

But since $\angle(A B, A M)+\angle(A M, A C)=\alpha$ holds, we have

$$
\begin{aligned}
2 \alpha & =2 \angle(A B, A M)+2 \angle(A M, A C)=2 \pi-\angle(M C, M A)-\angle(M A, M B) \\
& =\angle(M B, M C)
\end{aligned}
$$

Corollary 2.12 For the radius $r$ of the circumscribing circle and the edges of the triangle $a=|B-C|, b=|C-A|$, and $c=|A-B|, 2 r=\frac{a}{\sin \alpha}=\frac{b}{\sin \beta}=\frac{c}{\sin \gamma}$ holds.

Proof. (of Lemma 2.10) Given $\gamma_{k-1}, \gamma_{k}$, and $\gamma_{k+1}$ of a regular discrete curve. We set $B=\gamma_{k-1}, A=\gamma_{k}$, and $C=\gamma_{k+1}$. Assume the edge $B-C$ is parallel to the real axis, then the angle of the tangent to the circumcircle at $A$ is $\angle(-i, M A)=\alpha+2 \beta$ using the previous lemma. On the other hand the argument of $\Delta^{h} \gamma_{k}=2 \frac{\Delta \gamma \Delta \gamma_{\overline{1}}}{\Delta \gamma+\Delta \gamma_{\overline{1}}}$ is $\angle(B C, B A)-\angle(B C, A C)=\beta-\gamma=$ $\beta-\pi+(\alpha+\beta)=2 \beta+\alpha-\pi$ since $\Delta \gamma+\Delta \gamma_{\overline{1}}$ is parallel to the real axis. So the tangent and the vertex tangent vector are parallel.
Remark. This is the second justification for our choice of the vertex tangent vector.

Let us calculate the curvature that arises from this definition of an osculating circle: The radius of the vertex osculating circle is given by $r=\frac{\left|\Delta \gamma+\Delta \gamma_{\overline{1}}\right|}{2 \sin \alpha}$, so we get for the curvature

$$
\kappa_{k}=\frac{2 \sin \phi_{k}}{\left|\Delta \gamma_{k}+\Delta \gamma_{k-1}\right|}
$$

where $\phi_{k}=\angle\left(\Delta \gamma_{k-1}, \Delta \gamma_{k}\right)$.
Note that the sign of the curvature agrees with what one would expect from the smooth case (a curve bending to the left has positive curvature). We saw already the the definition has a flaw, namely that arc-length parameterized discrete curves cannot have arbitrarily large curvature (at most 2 actually, since the radius is always bigger or equal to $\frac{1}{2}$ ).

## edge osculating circles:

Another choice of osculating circle over coming the above problem is given by the circle touching three successive edges $S_{k-1}, S_{k}$, and $S_{k+1}$ (or their extensions) with matching orientations (see Fig. 6). More precisely it is


Figure 6: The edge osculating circle.
the circle that has its center at the intersection of the angular bisectors of $\angle\left(-S_{k-1}, S_{k}\right)$ and $\angle\left(-S_{k}, S_{k+1}\right)$ and that touches the straight line through $S_{k}$.

We will call this the edge osculating circle. It is not as local as one would like since it involves four successive points. In addition the definition works only for planar curves since the three successive edges need to lie in a plane. Its radius is given by

$$
r=\frac{|\Delta \gamma|}{\tan \frac{\phi}{2}+\tan \frac{\phi_{1}}{2}} .
$$

## osculating circles for arc-length parameterized discrete curves:

Best would be a mixture of the two notions and in fact for arc-length parameterized discrete curves we can take the vertex centre of curvature but the circle around it that touches the two corresponding edges in their midpoints (see Fig. 7). Here we can calculate the the radius to be

$$
r=\frac{1}{2 \tan \frac{\phi}{2}} .
$$

Definition 2.13 The curvature of a arc-length parameterized discrete curve $\gamma$ is given by

$$
\kappa=2 \tan \frac{\phi}{2}
$$

with $\phi=\angle\left(\Delta \gamma, \Delta \gamma_{\overline{1}}\right)$.
We will now make a slight contact with integrable equations by looking at how our curvature evolves with the tangential flow - mainly because the


Figure 7: The osculating circle for discrete arc-length parameterized curves.
proof is instructive. Exhaustive treatment of the integrable background of discrete differential geometry can be found in [BS08].

Lemma 2.14 The curvature $\kappa$ of an arc-length parameterized discrete curve evolves with the tangential flow $\dot{\gamma}=\alpha \Delta^{h} \gamma$ as

$$
\frac{\dot{\kappa}}{1+\frac{\kappa^{2}}{4}}=\frac{\alpha}{2}\left(\kappa_{1}-\kappa_{\overline{1}}\right) .
$$

Proof. For the tangential flow $\Delta \dot{\gamma}=i \mu \Delta \gamma$ for some real function $\mu$ since $\Delta \dot{\gamma} \perp \Delta \gamma$ and

$$
\begin{aligned}
\mu & =\left\langle\alpha \frac{\Delta \gamma_{1}+\Delta \gamma}{1+\left\langle\Delta \gamma_{1}, \Delta \gamma\right\rangle}-\alpha \frac{\Delta \gamma+\Delta \gamma_{\overline{1}}}{1+\left\langle\Delta \gamma, \Delta \gamma_{\overline{1}}\right.}, i \Delta \gamma\right\rangle \\
& =\alpha\left(\frac{\left\langle\Delta \gamma_{1}, i \Delta \gamma\right\rangle}{1+\left\langle\Delta \gamma_{1}, \Delta \gamma\right\rangle}-\frac{\left\langle\Delta \gamma_{\overline{1}}, i \Delta \gamma\right\rangle}{1+\left\langle\Delta \gamma, \Delta \gamma_{\overline{1}}\right\rangle}\right) \\
& =\alpha\left(\frac{\sin \phi_{1}}{1+\cos \phi_{1}}+\frac{\sin \phi}{1+\cos \phi}\right)=\alpha \frac{1}{2}\left(\kappa_{1}+\kappa\right) .
\end{aligned}
$$

Now $\frac{\Delta \gamma}{\Delta \gamma_{\overline{1}}}=e^{i \phi}=\frac{1+i \tan \frac{\phi}{2}}{1-i \tan \frac{\phi}{2}}=\frac{2+i \kappa}{2-i \kappa}$, so $\kappa=\frac{2}{i} \frac{\Delta \gamma-\Delta \gamma_{\overline{1}}}{\Delta \gamma+\Delta \gamma_{\overline{1}}}$ and with that at hand one can calculate

$$
\dot{\kappa}=4\left(\mu-\mu_{\overline{1}}\right) \frac{\Delta \gamma \Delta \gamma_{\overline{1}}}{\left(\Delta \gamma+\Delta \gamma_{\overline{1}}\right)^{2}}=\left(\mu-\mu_{\overline{1}}\right)\left(1+\frac{\kappa^{2}}{4}\right) .
$$

Remark. Note that $\phi=2 \arctan \frac{\kappa}{2}$ and $\dot{\phi}=\frac{\dot{\kappa}}{1+\left(\frac{\kappa}{2}\right)^{2}}$. Thus

$$
\dot{\phi}=\frac{\dot{\kappa}}{1+\frac{\kappa^{2}}{4}}=\frac{\alpha}{2}\left(\kappa_{1}-\kappa_{\overline{1}}\right)=\alpha\left(\tan \frac{\phi_{1}}{2}-\tan \frac{\phi_{\overline{1}}}{2}\right) .
$$

Both versions of the equation can be treated as a (space) discrete integrable equation - the " 0 -th" flow of the mKdV hierarchy.

Remark. Some words about normals: For planar curves we can easily define vertex and edge normals by taking $i$ times the corresponding tangent vectors but our definition of edge osculating circle adds a second version of vertex normal vector ${ }^{7}$. The angular bisector n $^{8}$. In the smooth setup $\gamma^{\prime \prime}=i \kappa \gamma^{\prime}$ holds for arc-length parameterized curves. If we discretize $\gamma^{\prime \prime}$ as $\Delta \Delta \gamma$ we find

- $\Delta \Delta \gamma=i \sin \phi \Delta^{h} \gamma$
- $\Delta \Delta \gamma=i \kappa \frac{1}{2}\left(\Delta \gamma+\Delta \gamma_{\overline{1}}\right)$
- The vertex centre of curvature $m$ is given by $m=\gamma+i \frac{1}{\kappa} \Delta^{h} \gamma$.

Theorem 2.15 The discrete curvature function $\kappa$ determines a arc-length parameterized discrete curve up to euclidean motion.

Proof. Given $\kappa$ fix $\gamma_{0}$ and the direction of $\Delta \gamma_{0}$. Then $\gamma$ is determined by the recurrent relation

$$
\gamma_{k+1}=\gamma_{k}+\Delta \gamma_{k-1} \frac{2+i \kappa}{2-i \kappa}
$$

## 2.3 new curves from old ones: evolutes and involutes

We will now look into generating new discrete curves from old ones. First we will look into the classical notions of evolutes and involutes, which have strong connections to discrete versions of the four vertex theorem. The by far most important construction - the discrete Tractrix - will follow.

Definition 2.16 The sequence of vertex/edge centres of curvature of a regular discrete curve $\gamma$ gives rise to a new curve the vertex/edge evolute of $\gamma$.

Remark.

[^4]- the evolute need not be regular
- for the vertex evolute $\tilde{\gamma}$ holds $\Delta \tilde{\gamma} \perp \Delta \gamma$
- for the edge evolute vertices get mapped to edges and vice versa

Example 2.4 The evolutes of the logarithmic spiral $\gamma$ are similar logarithmic spirals: Since $\gamma$ is invariant under the homothety $z \mapsto e^{-\epsilon+i \delta} z$ so are its evolutes.
the next example shows that both notions of evolute are on equal footing.
Example 2.5 • The edge evolute of a cycloide with even " $n$ " is a translated cycloide again.

- The vertex evolute of a cycloide with odd " $n$ " is a translated cycloide again.


in the " $n$ " is even case the idea of the proof is the following (consult Fig. 8 for an illustration):


Figure 8: Intersection points of the edges.
The argument of an edge of a cycloide is $\arg \left(\Delta \gamma_{k-1}\right)=\arg \left(1-e^{-i k \frac{2 \pi}{n}}\right)=$ $\frac{\pi}{2}-\frac{k \pi}{n}$ for $k \neq n$ but we will take a"continuous continuation" and think of the argument of an edge of length 0 to be $\frac{\pi}{2}$. So for even $n \Delta \gamma_{k} \perp \Delta \gamma_{k+\frac{n}{2}}$ and the direction of the perpendicular bisectors coincide with the directions of the edges of the cycloide half a period away. Then we can show, that the (edge) tangent line through an edge of the cycloide will always go through the upper left vertex of the rolling $n$-gon (see above image) and the perpendicular bisectors will always hit the lower left vertices of the rolling n-gon (again see above). Thus one can place a copy of the cycloide translated by half a period below the original one and the perpendicular bisectors of the first will coincide with the tangent lines of the second since both lines have the same direction and a common point.

For the odd case one can observe that $\angle\left(-\Delta \gamma_{k-1}, \Delta \gamma_{k}\right)=\pi-\frac{\pi}{n}$. Since this is clearly a constant the edges of the cycloide form isosceles triangles with their corresponding points of the evolute and since again angular bisectors and edges half a period away have same argument, what is left to show is that the edges of the cycloide and of the evolute have same lengths.

Definition 2.17 Let $\gamma$ be a regular discrete curve. $\tilde{\gamma}$ is called a (edge/vertex) involute of $\gamma$ if $\gamma$ is (edge/vertex) evolute of $\tilde{\gamma}$.

Remark. Any regular discrete curve is involute of a 2-parameter family of discrete curves. To see this for vertex involutes start with any point $\tilde{\gamma}_{0}$ (this freedom gives rise to the 2-parameter family) and mirror it at the first edge of $\gamma$ to get $\tilde{\gamma}_{1}$. The curve $\tilde{\gamma}$ obtained by iterating this is clearly a curve that has $\gamma$ as evolute. A similar construction holds for the edge involute.

Note that this is different in the smooth case where one only sees a 1parameter family of involutes:

$$
\tilde{\gamma}(s)=\gamma(s)-(a+s) \gamma^{\prime}(s)
$$

### 2.4 Four vertex theorems

The following classical global result was first proved by Mukhopadhyaya 1909 (see [Muk09]):

Theorem 2.18 (four vertex theorem) Any regular simply closed convex curve has at least four vertices (points where $\kappa^{\prime}=0$ ).

This remarkable result has seen many generalizations and many different proofs (and many discretizations as well) since then. One of them is a theorem by Bose (1932) (O. Musin [Mus04] notes that Kneser had the theorem ten years before Bose [Bos32]):

Theorem 2.19 Let $\gamma$ be an oval with no four points lying on a common circle. Denote by $s_{+}\left(s_{-}\right)$the number of osculating circles that lie outside (inside) $\gamma$ and with $t_{+}\left(t_{-}\right)$the number of circles that touch $\gamma$ in three distinct points from the outside (inside). Then

$$
s_{+}-t_{+}=s_{-}-t_{-}=2
$$

holds.

If one defines a vertex as a point where the osculating circle lies outside or inside the curve, then a four vertex theorem follows.

The literature on four vertex theorems is vast. We will now look at two discretizations the first of which (by O. Musin [Mus04]) is a discrete version of the Bose/Kneser theorem. But first we have to define what a discrete vertex should be (simply asking for $\Delta \kappa=0$ will not do of course).

Definition 2.20 The (vertex/edge) curvature (or curvature radius) of a regular discrete curve is said to have a true critical point (vertex) if $\Delta \kappa$ (or $\Delta r$ ) changes sign at that point.

Remark. Note that "point" can be an edge, depending which notion of curvature is considered.

Definition $2.21 \gamma: I \rightarrow \mathbb{C}$ is said to be simply closed if it bounds a topological disc (its interior). It is called strictly convex if for all non successive $\gamma_{i}, \gamma_{j}$ the connecting line (without the endpoints) lies completely in the interior of $\gamma$.

In the following we will say that a circle is full if it contains all the points of the curve and we will say that it is empty if it contains none in its interior.

Theorem 2.22 (O. Musin's discrete Kneser Theorem) Let $\gamma: \mathbb{Z} \rightarrow$ $\mathbb{C}$ be a strictly convex regular discrete curve with more than 3 vertices no four of which lie on a common circle. Denote by $s_{+}\left(s_{-}\right)$the number of vertex osculating circles that are full (empty) and by $t_{+}$( $t_{-}$) the full (empty) circles that go through 3 non neighbouring points of $\gamma$. then

$$
s_{+}-t_{+}=s_{-}-t_{-}=2
$$

holds.

Proof. Claim 1: A convex $n$-gon can be triangulated and there are $n-2$ triangles in each triangulation.
proof: Cut the $n$-gon along an inner edge. Iterate with both halves until all polygons are triangles. In particular it follows from this, that each triangulation has at least one (in fact even two) triangles with two border edges. So given a triangulated $n$-gon one can cut away such a border-triangle to obtain a triangulated $n-1$-gon. This can be done iteratively $n-3$ times until only a triangle is left. So the triangulation of the $n$-gon contained $n-2$ triangles.

Claim 2 (geometry): The set of full (empty) circles through 3 points of a convex $n$-gon as in the theorem gives rise to a triangulation.
proof: Start with any full circle through 3 points of the $n$-gon. The three points give rise to a triangle. Now for any inner edge of the triangle, the pencil of circles through its two endpoints contains a full circle (the one we started with) so it must contain a second one (these two circles are the border cases of all the full circles that contain the two points only). The second circle gives rise to a second triangle ${ }^{9}$ sharing an edge with the first. Thus we always find a neighbouring triangle that comes from a full circle. Likewise if we have two full circles through three points, their triangles will not intersect. The two circles necessarily will intersect (if one lies completely inside the other, the outer could not have touched the polygon) and the pencil of circles through the two intersection points will contain only two that touch the polygon. Moreover the points of touch for them will be separated by the line through the two intersection points (since the polygon is convex and the circles must touch in 3 points simultaneously). Thus the triangles formed by them can not intersect.

Together we can conclude that the triangles given by the full circles through three points will form a triangulation.

Claim 3: Given a triangulation of a polygon. Denote by $s$ the number of triangles with two border edges and by $t$ the number of triangles with no border edge, then $s-t=2$ holds.
proof: Denote by $r$ the number of triangles with one border edge. Since the triangulation contains $n-2$ triangles we have $s+r+t=n-2$. Since the triangulation has $n$ border edges we have in addition $n=2 s+r$ or $r=n-2 s$.

[^5]together we know $n-2=s+n-2 s+t$ which gives the claim
$$
2=s-t
$$

Applying the third claim to the triangulation(s) from the second gives a proof of the theorem.

Corollary 2.23 If one defines a vertex to be a point of the curve that has a full or empty osculating circle a four-vertex-theorem follows immediately.

Corollary 2.24 If in addition the vertex centres of curvature all lie on the same side of their neighbouring oriented edges, the vertex curvature has at least 4 vertices.

This extra condition is automatically satisfied for arc-length parameterized curves.
Proof. Let the vertex osculating circle at $\gamma$ be full. Then the radii $r_{\overline{1}}, r$, and $r_{1}$ of the vertex osculating circles at $\gamma_{\overline{1}}, \gamma$, and $\gamma_{1}$ satisfy

$$
\frac{\left|\Delta \gamma_{\overline{1}}\right|}{2} \leq r_{\overline{1}}<r, \quad \text { and } \quad \frac{|\Delta \gamma|}{2} \leq r_{1}<r
$$

(The centres lie somewhere between the centre of the full circle and the edge's midpoint). Thus $r-r_{\overline{1}}>0$ and $r_{1}-r<0$ and $\Delta r$ changes sign (and so does $\Delta \kappa)$. A similar argument can be carried out for an empty circle.

We will now turn to a notion of vertex, that is defined by the cusps in the curves evolute. The notions and the theorem together with its proof are essentially from Tabachnikov [Tab00].

Definition 2.25 A set of oriented lines is called generic if no three consecutive lines intersect in one point. The caustic of a generic set of lines $l_{1}, \ldots, l_{n}$ is the discrete curve given by the intersection points of $l_{k}$ with $l_{k+1}$ (for all $k$ ). A cusp vertex of the caustic is a point where the orientation of the edge tangent vectors of the caustic changes with respect to the orientation inherited form the lines.

Example 2.6 The vertex evolute of a discrete curve is the caustic of its edge normals while the edge evolute is the caustic of the angular bisectors.


Figure 9: A closed discrete curve and the caustic of its angular bisectors.

Corollary 2.26 With the conditions of the last theorem, the vertex evolute of the curve has at least four cusp vertices.

Proof. The proof uses the same argument as the one of the last corollary. Namely that the centres of curvature neighbouring a full curvature circle's centre lie on the side towards the edge on the edge normal lines.

Definition 2.27 Let $\gamma$ be a closed convex discrete curve of period $n$. A set of lines $l_{k}$ passing through the $\gamma_{k}$ is called exact if for the angles $\alpha=\angle\left(l, \Delta \gamma_{\overline{1}}\right)$ and $\beta=\angle(-\Delta \gamma, l)$

$$
\prod_{k=1}^{n} \sin \alpha_{k}=\prod_{k=1}^{n} \sin \beta_{k}
$$

holds.
Example 2.7 The angular bisectors of the inner angles are trivially exact.
The next theorem in its full extend is due to Tabachnikov while the special case of angular bisectors is due to Wegner.

Theorem 2.28 (Tabachnikov) Let $\gamma$ be a regular closed convex discrete curve of period $n>3$. Given a generic exact set of lines $l_{1}, \ldots, l_{n}$, then its caustic has at least four cusp vertices.

Corollary 2.29 (Wegner) A four-vertex-theorem for the edge curvature radii.

Proof. Given a curve $\gamma$ and its edge evolute $\tilde{\gamma}$ we have for the curvature radii $r_{\overline{1}}=\left|\tilde{\gamma}_{\overline{1}}-\gamma\right| \sin \frac{\pi-\phi}{2}$ and $r=|\tilde{\gamma}-\gamma| \sin \frac{\pi-\phi}{2}$. Thus $\Delta r_{\overline{1}}=r-r_{\overline{1}}=$ $\left(|\tilde{\gamma}-\gamma|-\left|\tilde{\gamma}_{\overline{1}}-\gamma\right|\right) \sin \frac{\pi-\phi}{2}$ holds and the signum of $\Delta r$ depends only on the signum of the caustic of the angular bisectors.

Proof. Let $\tilde{\gamma}$ be the caustic of $\gamma$. The condition for a set of lines to be exact has a simple geometric background: The set of lines $l_{k}$ is exact if and only if a curve $\delta$ with its points $\delta_{k} \in l_{k}$ and edges parallel to $\gamma$ is a closed curve as well. This is easy to see because $\left|\delta_{k}-\gamma_{k}\right| /\left|\delta_{k+1}-\gamma_{k+1}\right|=\sin \left(\alpha_{k+1}\right) / \sin \left(\beta_{k}\right)$ by construction. Thus $\delta_{n+1}=\delta_{1} \Leftrightarrow \prod_{k} \sin \alpha_{k} / \prod_{k} \sin \beta_{k}=1$. Now take a parallel curve $\delta$ (which we know is closed) and translate the edges between $\gamma$ and $\delta$ to the origin forming a third curve $\nu_{k}=\delta_{k}-\gamma_{k}$. Again the edges of $\nu$ are parallel to $\gamma$. Moreover the Triangles $\triangle \nu_{k} \nu_{k+1} 0$ are similar to $\triangle \gamma_{k} \gamma_{k+1} \tilde{\gamma}_{k}$ with some factor of similarity $\lambda_{k}$. One can read of the cusps of $\tilde{\gamma}$ from the differences $\Delta \lambda$ : $\tilde{\gamma}_{k}$ is a cusp vertex iff $\Delta \lambda_{k-1}$ and $\Delta \lambda_{k}$ differ in sign. Since $\sum_{k} \Delta \lambda_{k}=0$, we know that it changes sign at least twice. Now assume that it changes sign exactly twice and suppose that $\Delta \lambda_{k}>0$ for $k=1, \ldots, j-1$ and $\Delta \lambda_{k}<0$ for $k=j, \ldots, n$ (it can never be 0 since the set of lines is generic by assumption). By "integration by parts" we find

$$
\begin{aligned}
\sum_{k} \Delta \lambda_{k} \gamma_{k+1} & =\sum_{k} \lambda_{k+1} \gamma_{k+1}-\sum_{k} \lambda_{k} \gamma_{k+1}=\sum_{k} \lambda_{k} \gamma_{k}-\sum_{k} \lambda_{k} \gamma_{k+1} \\
& =-\sum \lambda_{k} \Delta \gamma_{k-1}=-\sum_{k} \Delta \nu_{k}=0 .
\end{aligned}
$$

But if we choose the origin on a line crossing the edges $\gamma_{1}, \gamma_{2}$ and $\gamma_{j}, \gamma_{j+1}$ then all $\Delta \lambda_{k} \gamma_{k+1}$ lie on one side of the line and can therefore not sum up to zero. Thus the assumption was wrong and there is one more change of sign. Since the $\Delta \lambda$ sum to zero the number must be even which forces the number to be at least four.
Remark. Unlike the smooth case, we can not generalize the discrete four vertex theorems easily to non convex curves. Wegner has counter examples here.

## 2.5 curves in $\mathbb{C P}^{1}$

We will now turn to Möbius geometry for a moment, meaning we want to do geometry invariant under Möbius transformations. They are the group of transformations generated by (an even number of) inversions on circles ${ }^{10}$.

$$
z \mapsto r^{2} \frac{z-c}{|z-c|^{2}}+c
$$

[^6]gives the inversion on a circle with centre $c$ and radius $r$. Choosing $c=0$ and $r=a$ leaves us with $z \mapsto a^{2} / \bar{z}$ and choosing $c=i r$ gives in the limit $r \rightarrow \infty$ the inversion on the real axis:
$$
\lim _{r \rightarrow \infty} r^{2} \frac{z-i r}{|z-i r|^{2}}+i r=\lim _{r \rightarrow \infty} \frac{r^{2}}{\bar{z}+i r}+i r=\lim _{r \rightarrow \infty} \frac{r^{2}+i r \bar{z}-r^{2}}{\bar{z}+i r}=\bar{z}
$$

In a similar manner we can treat the inversion on any other line. The composition of the inversions on two lines will give a rotation around their intersection point and if they are parallel it will give a translation.

Combining the above maps gives in general maps of the form

$$
z \mapsto \frac{a z+b}{c z+d}, \quad \text { with } a d-b c \neq 0
$$

(and an additional complex conjugate if the number of inversions was odd). In complex analysis Möbius transformations are usually introduced in this form as fractional linear maps. The zero of the denominator generates a pole and in turn one can think of $a / c$ as the image of $\infty$ under the map. This shows that Möbius transformations are in fact mappings from $\mathbb{C}^{*}=\mathbb{C} \cup\{\infty\}$ onto itself. But $C^{*}$ can be seen as $\mathbb{C P}{ }^{1}$ - the complex projective line or the space of all 1-dim linear subspaces in $\mathbb{C}^{2}$ : Each line $t\binom{a}{b}$ in $\mathbb{C}^{2}$ can be identified with the complex number $a / b$ except for the line $t\binom{a}{0}$ which one identifies with $\infty$ (think of each line getting identified with its intersection with the line $\binom{t}{1}$ which is unique and possible for all lines except the one parallel to $\binom{t}{1}$.


Figure 10: Points in $\mathbb{R} P^{1}$ correspond to lines in $\mathbb{R}^{2}$.

If we now identify $z$ with $\binom{z}{1}$ in $\mathbb{C}^{2}$ if $z \neq \infty$ and $z=\infty$ with $\binom{1}{0}$ then a matrix $M=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right) \in G L(2, \mathbb{C})$ acts on $z \in \mathbb{C P}^{1}$ by $M(z)=$ $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)\binom{z}{1}=\binom{a z+b}{c z+d} \cong \frac{a z+b}{c z+d}$, which is a Möbius transformation. Since scaling does not matter here we can normalize $M$ to be in $S L(2, \mathbb{C})$. In homogenous coordinates, Möbius transformations are just linear maps form $S L(2, \mathbb{C})$.

Now let us find the simplest invariant in Möbius geometry. For $\binom{\alpha}{\beta}$ and $\binom{\gamma}{\delta}$ in $\mathbb{C}^{2}$

$$
\operatorname{det}\left(\binom{\alpha}{\beta},\binom{\gamma}{\delta}\right)=\operatorname{det}\left(M\binom{\alpha}{\beta}, M\binom{\gamma}{\delta}\right)
$$

but this expression is not invariant, since scaling the vectors in $\mathbb{C}^{2}$ will change the value of the determinant. In fact there is no invariant for two or even three points in $\mathbb{C P}^{1}$ in general position, since up to scaling one can always map three distinct vectors to three other distinct ones by a Möbius transformation: $M v_{1}=\lambda_{1} w_{1}, M v_{2}=\lambda_{2} w_{2}, M v_{3}=\lambda_{3} w_{3}$ gives 6 equations for 6 unknowns $\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right.$ and 3 entries in $\left.M\right)$ and can in general be solved.

For 4 vectors in general position we can form an invariant:
Definition 2.30 The cross-ratio of four vectors $v_{1}, \ldots, v_{4}$ in $\mathbb{C}^{2} \backslash\{0\} \cong \mathbb{C P}^{1}$ is given by

$$
\frac{\operatorname{det}\left(v_{1}, v_{2}\right)}{\operatorname{det}\left(v_{2}, v_{3}\right)} \frac{\operatorname{det}\left(v_{3}, v_{4}\right)}{\operatorname{det}\left(v_{4}, v_{1}\right)}=: \operatorname{cr}\left(v_{1}, v_{2}, v_{3}, v_{4}\right) .
$$

Since $\operatorname{det}\left(\binom{a}{1},\binom{b}{1}\right)=b-a$, for four finite points $a, b, c, d$ the cross-ratio reads

$$
c r(a, b, c, d) \frac{a-b}{b-c} \frac{c-d}{d-a} .
$$

This is well-defined and invariant under scaling since all four vectors appear linear in the numerator and in the denominator.

We will now give the cross-ratio some interpretation in the setup of our curves. If $\gamma$ is arc-length parameterized we can calculate

$$
\begin{aligned}
\operatorname{cr}\left(\gamma_{\overline{1}}, \gamma, \gamma_{11}, \gamma_{1}\right) & =\frac{S_{\overline{1}} S_{1}}{\left(S_{\overline{1}}+S\right)\left(S+S_{1}\right)}=\left(\left(1+\frac{S}{S_{\overline{1}}}\right)\left(\frac{S}{S_{1}}+1\right)\right)^{-1} \\
& =\left(\left(1+\frac{2 i-\kappa}{2 i+\kappa}\right)\left(1+\frac{2 i+\kappa_{1}}{2 i-\kappa_{1}}\right)\right)^{-1} \\
& =\frac{-1}{16}(2 i+\kappa)\left(2 i-\kappa_{1}\right)=\frac{1}{16}\left(2 i\left(\kappa_{1}-\kappa\right)+\kappa \kappa_{1}+4\right)
\end{aligned}
$$

So $\operatorname{cr}\left(\gamma_{\overline{1}}, \gamma, \gamma_{11}, \gamma_{1}\right)-1 / 4=(1 / 16)\left(2 i \Delta \kappa+\kappa \kappa_{1}\right)$. Now for a smooth curve $\kappa=\frac{\gamma^{\prime \prime}}{i \gamma^{\prime}}$ and thus

$$
2 i \kappa^{\prime}+\kappa^{2}=2 \frac{\gamma^{\prime \prime \prime}}{\gamma^{\prime}}-3\left(\frac{\gamma^{\prime \prime}}{\gamma^{\prime}}\right)^{2}=: 2 S(\gamma) .
$$

Here $S(\gamma)$ denotes the Schwarzian derivative. It is not really a derivative but it measures how far a function is from being a Möbius transformation $(S(M)=0$ and $S(M(f))=S(f)$ for all Möbius transformations $M)$. We will define the cross-ratio minus $1 / 4$ to be a discrete Schwarzian derivative. Note that since $\operatorname{cr}(1,2,4,3)=1 / 4$ the discrete Schwarzian derivative of an arc-length parameterized discrete straight line (or the identity) will be 0 and since the cross-ratio is invariant under Möbius transformations so will be our discrete Schwarzian derivative.

Next we will show that our vertex tangent vector is in fact a Möbius geometric notion. For this we calculate the Möbius transformation that sends $-1,0$, and 1 to $\gamma_{\overline{1}}, \gamma$, and $\gamma_{1}$. In other words we want to find $a, b, c$, and $d$ such that

$$
\frac{-a+b}{-c+d}=\gamma_{\overline{1}}, \quad \frac{b}{d}=\gamma, \quad \frac{a+b}{c+d}=\gamma_{1} .
$$

It turns out that $c \neq 0$ and thus we can choose $d=1$ giving us $a=\gamma-\gamma_{\overline{1}}-$ $\gamma_{\overline{1}} \frac{\gamma_{1}-2 \gamma+\gamma_{\overline{1}}}{\gamma_{1}-\gamma_{\overline{\mathrm{I}}}}, b=\gamma$, and $c=-\frac{\gamma_{1}-2 \gamma+\gamma_{\overline{1}}}{\gamma_{1}-\gamma_{\overline{1}}}$.

Now the derivative of $M(z)=\frac{a z+b}{c z+d}$ is $M^{\prime}(z)=\frac{a d-b c}{(c z+d)^{2}}$ and its value at 0 is

$$
\begin{aligned}
M^{\prime}(0) & =\frac{\left(\gamma-\gamma_{\overline{1}}\right)\left(\gamma_{1}-\gamma_{\overline{1}}\right)-\gamma_{\overline{1}}\left(\gamma_{1}-2 \gamma+\gamma_{\overline{1}}\right)+\gamma\left(\gamma_{1}-2 \gamma+\gamma_{\overline{1}}\right)}{\gamma_{1}-\gamma_{\overline{1}}} \\
& =2 \frac{\left(\gamma_{1}-\gamma\right)\left(\gamma-\gamma_{\overline{1}}\right)}{\gamma_{1}-\gamma_{\overline{1}}}=\Delta^{h} \gamma .
\end{aligned}
$$

Reading this in a differential geometric way it says that the unit tangent vector of the real axis at 0 is mapped to our vertex tangent vector at $\gamma$ and by composition of Möbius transformations we see that the differential $d M$ of a Möbius transformation $M$ sends the vertex tangent vectors of a discrete curve $\gamma$ to the vertex tangent vectors of $M(\gamma)$.

To investigate discrete curves $\gamma \in \mathbb{C P}^{1}$ (and in particular flows on them) we will usually lift them into $\mathbb{C}^{2}: \Gamma_{k}=\lambda_{k}\binom{\gamma_{k}}{1}$ (in case $\left.\gamma_{k}=\infty \operatorname{set} \Gamma_{k}=\lambda_{k}\binom{1}{0}\right)$ and in order to fix the scale freedom $\lambda$ we will require $\operatorname{det}\left(\Gamma_{k}, \Gamma_{k+1}\right)=1 .{ }^{11}$ Define

$$
u_{k}:=\operatorname{det}\left(\Gamma_{k-1}, \Gamma_{k+1}\right) .
$$

Then one finds $\Gamma_{k+1}=u_{k} \Gamma_{k}-\Gamma_{k-1}$ and

$$
Q_{k}:=\operatorname{cr}\left(\gamma_{k-1}, \gamma_{k}, \gamma_{k+2}, \gamma_{k+1}\right)=\frac{1}{u_{k} u_{k+1}}
$$

Since $\operatorname{det}\left(\Gamma_{,} \Gamma_{1}-\Gamma_{\overline{1}}\right)=2, \Gamma$ and $\Gamma_{1}-\Gamma_{\overline{1}}$ are linear independent and any flow $\dot{\Gamma}$ on $\Gamma$ can be written as

$$
\dot{\Gamma}=\alpha \Gamma+\frac{\beta}{u}\left(\Gamma_{1}-\Gamma_{\overline{1}}\right) .
$$

[^7]the condition for such a flow to preserve our normalization reads
$$
0=\frac{d}{d t} \operatorname{det}\left(\Gamma, \Gamma_{1}\right)=\alpha-\beta+\alpha_{1}+\beta_{1}
$$

So we can require $\left(\alpha_{1}+\alpha\right)=-\left(\beta_{1}-\beta\right)$.
Lemma 2.31 The quantities $u$ and $Q$ evolve under a flow $\dot{\Gamma}=\alpha \Gamma+\frac{\beta}{u}\left(\Gamma_{1}-\right.$ $\Gamma_{\overline{1}}$ ) with

$$
\begin{aligned}
\dot{u} & =u\left(\alpha_{1}+\alpha_{\overline{1}}+\beta_{1}-\beta_{\overline{1}}\right)+2\left(\frac{\beta_{\overline{1}}}{u_{\overline{1}}}-\frac{\beta_{1}}{u_{1}}\right) \\
\frac{\dot{Q}}{Q} & =2\left((Q-1)\left(\beta_{1}-\beta\right)+Q_{1} \beta_{11}-Q_{\overline{1}} \beta_{\overline{1}}\right) .
\end{aligned}
$$

Proof. A straight forward calculation.
Lemma 2.32 Given a discrete curve $\gamma$, if its lift $\Gamma$ evolves with $\dot{\Gamma}=\alpha \Gamma+$ $\frac{\beta}{u}\left(\Gamma_{1}-\Gamma_{\overline{1}}\right)$, then

$$
\dot{\gamma}=\beta \Delta^{h} \gamma .
$$

Proof. Insert and expand using the fact that $\gamma$ is the quotient of the two components of the vector $\Gamma=\left(\frac{\Gamma^{(1)}}{\Gamma^{(2)}}\right): \gamma=\Gamma^{(1)} / \Gamma^{(2)}$.

The simplest flow one can think of is $\beta \equiv 0$ resulting in $\alpha_{1}=-\alpha$. This flow corresponds to the initial freedom of choosing the scaling of $\Gamma_{0}$ and is clearly not visible on the curve $\gamma$ in $\mathbb{C P}^{1}$. Next we can choose $\beta=$ const, say $\beta=1 / 2$ and for simplicity $\alpha=0$. This gives for the curve $\Gamma$ and the quantities $u$ and $Q$

$$
\begin{aligned}
\dot{\Gamma} & =\frac{1}{2 u}\left(\Gamma_{1}-\Gamma_{\overline{1}}\right) \\
\dot{u} & =\frac{1}{u_{\overline{1}}}-\frac{1}{u_{1}}-2 \alpha u \\
\dot{Q} & =Q\left(Q_{1}-Q_{\overline{1}}\right)
\end{aligned}
$$

the evolution equation for $Q$ being the well known Volterra model (see [FT86]). For the curve $\gamma$ in $\mathbb{C P}{ }^{1}$ the flow reads $\dot{\gamma}=1 / 2 \Delta^{h} \gamma$. In case of an arc-length parameterized curve this is our tangential flow ${ }^{12}$

[^8]Next we choose $\beta=\left(Q+Q_{\overline{1}}\right)$ and get

$$
\begin{aligned}
\frac{\dot{Q}}{Q} & =2\left((Q-1)\left(Q_{1}-Q_{\overline{1}}\right)+Q_{1}\left(Q_{11}+Q_{1}\right)-Q_{\overline{1}}\left(Q_{\overline{1}}+Q_{\overline{1} \overline{1}}\right)\right) \\
\dot{\gamma} & \left.=\frac{1}{16}\left(2 i\left(\kappa_{1}-\kappa\right)+\kappa \kappa_{1}+4\right)+\left(2 i\left(\kappa-\kappa_{\overline{1}}\right)+\kappa_{\overline{1}} \kappa+4\right)\right) \Delta^{h} \gamma .
\end{aligned}
$$

Here we assumed again that $\gamma$ is arc-length parameterized and up to some scaling and a constant tangential flow part one can think of this as a discrete version of $\dot{\gamma}=S(\gamma) \gamma^{\prime}$ where $S(\gamma)$ denotes the Schwarzian derivative. In the smooth case the curvature then solves the mKdV equation $\dot{\kappa}=\kappa^{\prime \prime \prime}+\frac{3}{2} \kappa^{2} \kappa^{\prime}$ : With $\dot{\gamma}=S(\gamma) \gamma^{\prime}=\left(\frac{\kappa^{2}}{2}+i \kappa^{\prime}\right) \gamma^{\prime}$ and $\kappa=\frac{\gamma^{\prime \prime}}{i \gamma^{\prime}}$ we find:

$$
\begin{aligned}
\dot{\gamma}^{\prime} & =\left(\kappa^{\prime} \kappa+i \kappa^{\prime \prime}\right) \gamma^{\prime}+\left(\frac{\kappa^{2}}{2}+i \kappa^{\prime}\right) \gamma^{\prime \prime} \\
& =i\left(\kappa^{\prime \prime}+\frac{\kappa^{3}}{2}\right) \gamma^{\prime}, \\
\dot{\gamma}^{\prime \prime} & =i\left(\kappa^{\prime \prime \prime}+\frac{3}{2} \kappa^{2} \kappa^{\prime}\right) \gamma^{\prime}-\left(\kappa^{\prime \prime}+\frac{\kappa^{3}}{2}\right) \kappa \gamma^{\prime}, \\
\dot{\kappa} & =-i\left(\frac{\dot{\gamma}^{\prime \prime} \gamma^{\prime}-\gamma^{\prime \prime} \dot{j}^{\prime}}{\left(\gamma^{\prime}\right)^{2}}\right) \\
& =-i\left(i\left(\kappa^{\prime \prime \prime}+\frac{3}{2} \kappa^{2} \kappa^{\prime}\right)-\left(\kappa^{\prime \prime} \kappa+\frac{\kappa^{4}}{2}\right)+\kappa\left(\kappa^{\prime \prime}+\frac{\kappa^{3}}{2}\right)\right) \\
& =\kappa^{\prime \prime \prime}+\frac{3}{2} \kappa^{2} \kappa^{\prime}
\end{aligned}
$$

Therefore one can think of the evolution of the discrete $\kappa$ as a discrete $m K d V$ equation [HK04]. It can be computed to be

$$
\frac{\dot{\kappa}}{1+\frac{\kappa^{2}}{4}}=\frac{1}{4}\left(\kappa_{1}-\kappa_{\overline{1}}\right)+\frac{1}{6}\left(\left(\frac{\kappa^{2}}{4}+1\right)\left(\kappa_{11}+\kappa\right)-\left(\frac{\kappa^{2}}{4}+1\right)\left(\kappa+\kappa_{\overline{1} \overline{1}}\right)\right) .
$$

Remark. One can conjecture with good reason (again see [HK04]) that a hierarchy of flows can be generated by iterating $\beta_{1}^{\text {new }}-\beta^{\text {new }}=\frac{\dot{Q}^{\text {old }}}{Q^{\text {old }}}$ as we have in essence done above. the conjecture here is that $\frac{\dot{Q}^{\text {old }}}{Q^{\text {old }}}$ can always be "integrated" to give a closed (locally defined) expression for $\beta^{n e w}$.

We are now about to introduce a Lax-pair for the equations that arise from the flow evolutions. That is, we will describe them as compatibility conditions for a linear matrix problem.

The matrix $F^{T}=\left(\Gamma \Gamma_{\overline{1}}\right)(T$ stands for the matrix transposed here) clearly evolves by

$$
F_{1}^{t}=F^{T}\left(\begin{array}{cc}
u & 1 \\
-1 & 0
\end{array}\right)=: F^{T} L^{T}
$$

since $\Gamma_{1}=u \Gamma-\Gamma_{\overline{1}}$ and for a flow $\dot{\Gamma}=\alpha \Gamma+\frac{\beta}{u}\left(\Gamma_{1}-\Gamma_{\overline{1}}\right)$ we get

$$
\dot{F}^{T}=F^{T}\left(\begin{array}{cc}
\alpha+\beta & 2 \frac{\beta_{\overline{1}}}{u_{\overline{1}}} \\
-2 \frac{\beta}{u} & \alpha_{\overline{1}}-\beta_{\overline{1}}
\end{array}\right)=: F^{T} V^{T} .
$$

So we have the matrix problem $F_{k+1}=L_{k} F_{k}$ and $\dot{F}_{k}=V_{k} F_{k}$ and the compatibility condition for this system is (using $L_{k}=F_{k+1} F_{k}^{-1}$ ):

$$
\begin{aligned}
\dot{L}_{k} & =\dot{F}_{k+1} F_{k}^{-1}-F_{k+1} F_{k}^{-1} \dot{F}_{k} F_{k}^{-1}=V_{k+1} F_{k+1} F_{k}^{-1}-F_{k+1} F_{k}^{-1} V_{k} F_{k} F_{k}^{-1} \\
& =V_{k+1} L_{k}-L_{k} V_{k} .
\end{aligned}
$$

This set of equations in essence gives us back the condition on $\alpha$ and the time evolution of $u$. the matrices $L$ and $V$ are called a Lax-pair.

### 2.6 Darboux transformations and time discrete flows

Lemma 2.33 Let $a, b$, and $d \in \mathbb{C P}^{1}$ in general position be given and choose $\mu \in \mathbb{C}$. Then there is a unique $c \in \mathbb{C P}^{1}$ with $\operatorname{cr}(a, b, c, d)=\mu$ and the map sending $d \mapsto c$ given $a$ and $b$ is a Möbius transformation.

Proof. Since $\operatorname{cr}(0, \infty, x, d)=\frac{-\infty}{\infty-x} \frac{x-d}{d-0}=\frac{d-x}{d}$ there is a unique $x=1-\mu$ that satisfies $\operatorname{cr}(0, \infty, x, 1)=\mu$ and since there is a unique Möbius transformation $M$ sending $a, b$, and $d$ to $0, \infty$, and $1, c=M^{-1}(x)$ is the unique number satisfying $\operatorname{cr}(a, b, c, d)=\operatorname{cr}(M(a), M(b), M(c), M(d))=c r(0, \infty, x, 1)=\mu$. If one fixes any Möbius transformation $M$ sending $a$ and $b$ to 0 and $\infty$ independent of $d$ then $\mu=\frac{d-x}{d}$ can again be solved for $x$ giving the Möbius transformation $N: d \mapsto d(1-\mu)$. Thus the map sending $d$ to $c$ is $M^{-1} \circ N \circ M$ and indeed a Möbius transformation. Note that it is unique although we had freedom in the choice of $M$.

Corollary 2.34 The quantity $Q_{k}=\operatorname{cr}\left(\gamma_{k-1}, \gamma_{k}, \gamma_{k+2}, \gamma_{k+1}\right)$ determines the discrete curve $\gamma$ uniquely up to Möbius transformations.

Proof. Given $\gamma_{0}, \gamma_{1}$, and $\gamma_{2}$ We can iteratively reconstruct $\gamma$ by the above lemma given $Q$. Choosing different initial points $\tilde{\gamma}_{0}, \tilde{\gamma}_{1}$, and $\tilde{\gamma}_{2}$ gives a curve that is related to the first one by applying the Möbius transformation that is fixed by the condition $M\left(\gamma_{i}\right)=\tilde{\gamma}_{i}, i=1,2,3$.

Definition 2.35 Let $\gamma$ be a regular discrete curve in $\mathbb{C P}^{1}$. Given $\mu \in \mathbb{C}$ and an initial point $\tilde{\gamma}_{0}$ the unique discrete curve $\tilde{\gamma}$ satisfying

$$
\operatorname{cr}\left(\gamma_{k}, \gamma_{k+1}, \tilde{\gamma}_{k+1}, \tilde{\gamma}_{k}\right)=\mu
$$

for all $k$ is called a Darboux transform of $\gamma$.
Remark. There is a 4 (real) parameter family of Darboux transforms for a given curve.
Now assume $\gamma$ is a closed curve with period $n$. Then for any Darboux transform $\tilde{\gamma}$, the map sending $\tilde{\gamma}_{0}$ to $\tilde{\gamma}_{n}$ is a Möbius transformation $H$. It depends on the curve $\gamma$ and the parameter $\mu$ of the Darboux transformation but not on the initial point $\tilde{\gamma}_{0}$. Generically a Möbius transformation has two fixpoints (corresponding to the two eigenlines of the $S L(2, \mathbb{C})$-matrix) $\cdot{ }^{13}$ So if (and only if) we choose $\tilde{\gamma}_{0}$ to be one of the two fix-points the Darboux transform $\tilde{\gamma}$ will be closed again. All together:

[^9]Lemma 2.36 For each parameter $\mu \in \mathbb{C}$ there are in general two distinct Darboux transforms. ${ }^{14}$

We can read them as history and future of a time-discrete evolution.
The next lemma shows how the quantity $Q$ changes under this evolution.

Lemma 2.37 Let $\tilde{\gamma}$ be a Darboux transform of $\gamma$ with parameter $\mu$ and define $s_{k}=\operatorname{cr}\left(\gamma_{k-1}, \tilde{\gamma}_{k}, \gamma_{k+1}, \gamma_{k}\right)$. Then

$$
\tilde{Q}_{k}=Q_{k} \frac{s_{k+1}}{s_{k+1}}
$$

and

$$
(1-\mu) Q_{k}=\frac{s_{k+1}}{\left(1-s_{k}\right)\left(s_{k+1}-1\right)}
$$

For use in the proof we will state some identities for the cross-ratio under permutation of the arguments in Fig. 11 .
It should be read in the following way: the four vertices are symbolizing the four arguments, the arrows show their order as arguments in the cross-ratio, the dashed line and the not drawn edge connecting the start and end points will form the differences in the denominator, while the two solid lines give rise to the differences in the numerator of the cross-ratio expression. Proof. Using the above identities one finds

$$
\begin{aligned}
(1-\mu) & =\operatorname{cr}\left(\gamma_{k}, \tilde{\gamma}_{k+1}, \gamma_{k+1}, \tilde{\gamma}_{k}\right) \\
Q_{k} & =\operatorname{cr}\left(\gamma_{k-1}, \gamma_{k}, \gamma_{k+2}, \gamma_{k+1}\right) \\
\tilde{Q}_{k} & =\operatorname{cr}\left(\tilde{\gamma}_{k-1}, \tilde{\gamma}_{k}, \tilde{\gamma}_{k+2}, \tilde{\gamma}_{k+1}\right) \\
\frac{1}{1-s_{k}} & =\operatorname{cr}\left(\gamma_{k-1}, \gamma_{k}, \tilde{\gamma}_{k}, \gamma_{k+1}\right) \\
\frac{s_{k+1}}{s_{k+1}-1} & =\operatorname{cr}\left(\gamma_{k}, \tilde{\gamma}_{k+1}, \gamma_{k+1}, \gamma_{k+2}\right)
\end{aligned}
$$

Multiplying the first two and the last two equations gives the second claim. For the first claim define $\tilde{s}_{k}=\operatorname{cr}\left(\tilde{\gamma}_{k-1}, \gamma_{k}, \tilde{\gamma}_{k+1}, \tilde{\gamma}_{k}\right)$. One finds $s_{k} \tilde{s}_{k}=1$ and

[^10]

Figure 11: Cross-ratios for permutations of the arguments.
by symmetry

$$
\begin{aligned}
(1-\mu) \tilde{Q}_{k} & =\frac{\tilde{s}_{k+1}}{\left(1-\tilde{s}_{k}\right)\left(\tilde{s}_{k+1}-1\right)}=\frac{\frac{1}{s_{k+1}}}{\left(1-\frac{1}{s_{k}}\right)\left(\frac{1}{s_{k+1}}-1\right)}=\frac{1}{\left(1-\frac{1}{s_{k}}\right)\left(1-s_{k+1}\right)} \\
& =\frac{s_{k}}{\left(s_{k}-1\right)\left(1-s_{k+1}\right)}=\frac{s_{k}}{s_{k+1}} \frac{s_{k+1}}{\left(1-s_{k}\right)\left(s_{k+1}-1\right)}=\frac{s_{k}}{s_{k+1}} Q_{k}(1-\mu) .
\end{aligned}
$$

Remark.

- This time discrete evolution can easily be identified with the time discrete Volterra model [HK04].
- $\mu \cong 1$ should be viewed as a time discrete version of our $\beta \equiv$ const flow (the $C P^{1}$ version of the tangential flow).
- We can create lattices with prescribed cross-ratio from given Cauchy data (like a stair case path or a pair of lines with $m=$ const and $n=$ const).

The next lemma [HJHP99] will give us a Bianchi permutability theorem as well as a way to Darboux transform whole lattices.

Lemma 2.38 (Hexahedron lemma) Given a quadrilateral $x_{1}, \ldots, x_{4} \in \mathbb{C}$ with cross-ratio $\operatorname{cr}\left(x_{1}, x_{2}, x_{3}, x_{4}\right)=\mu$. Then for any $\lambda \in \mathbb{C}$ there is a unique quadrilateral $y_{1}, y_{2}, y_{3}, y_{4} \in \mathbb{C}$ to each initial $y_{1}$ such that

$$
\begin{aligned}
& \operatorname{cr}\left(y_{1}, y_{2}, y_{3}, y_{4}\right)=\mu=\operatorname{cr}\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \\
& \operatorname{cr}\left(y_{1}, y_{2}, x_{2}, x_{1}\right)=\lambda \mu=\operatorname{cr}\left(y_{3}, y_{4}, x_{4}, x_{3}\right) \\
& \operatorname{cr}\left(y_{2}, y_{3}, x_{3}, x_{2}\right)=\lambda=\operatorname{cr}\left(y_{4}, y_{1}, x_{1}, x_{4}\right) .
\end{aligned}
$$

Proof. Evolve $y_{1}$ around $x_{1}, x_{2}, x_{3}, x_{4}$ using the unique cross-ratio evolution to generate $y_{2}, y_{3}$, and $y_{4}$ and check for the two unused cross-ratios.

Corollary 2.39 (Bianchi permutability) Given a regular discrete curve $\gamma$ in $\mathbb{C P}^{1}$ let $\hat{\gamma}$ be a Darboux transform of $\gamma$ to the parameter $\lambda$ and $\tilde{\gamma}$ be a Darboux transform of $\gamma$ to the parameter $\mu$. Then there is a unique curve $\hat{\tilde{\gamma}}$ that is a $\lambda$-Darboux transform of $\tilde{\gamma}$ and a $\mu$-Darboux transform of $\hat{\gamma}$.

Proof. Choose $\hat{\tilde{\gamma}}_{0}$ to have $\operatorname{cr}\left(\gamma_{0}, \tilde{\gamma}_{0}, \hat{\tilde{\gamma}}_{0}, \hat{\gamma}_{0}\right)=\frac{\lambda}{\mu}$. Then by the previous lemma there is a unique $\hat{\tilde{\gamma}}_{1}$ that gives $\operatorname{cr}\left(\gamma_{1}, \tilde{\gamma}_{1}, \hat{\tilde{\gamma}}_{1}, \hat{\gamma}_{1}\right)=\frac{\lambda}{\mu}$ and $\operatorname{cr}\left(\hat{\gamma}_{0}, \hat{\gamma}_{1}, \hat{\tilde{\gamma}}_{1}, \hat{\tilde{\gamma}}_{0}\right)=\mu$ and $\operatorname{cr}\left(\tilde{\gamma}_{0}, \tilde{\gamma}_{1}, \hat{\tilde{\gamma}}_{1}, \hat{\tilde{\gamma}}_{0}\right)=\lambda$. This clearly can be extended along the whole triplet of curves.

Corollary 2.40 (Darboux transformation for meshes) Given a map $z$ : $\mathbb{Z}^{2} \rightarrow \mathbb{C}$ with all elementary quadrilaterals having $\operatorname{cr}\left(z, z_{1}, z_{12}, z_{2}\right)=\mu$ then for each initial point $\tilde{z}_{0,0}$ and each $\lambda \in \mathbb{C}$ there is a map $\tilde{z}: \mathbb{Z}^{2} \rightarrow \mathbb{C}$ such that

$$
\begin{aligned}
\operatorname{cr}\left(\tilde{z}, \tilde{z}_{1}, \tilde{z}_{12}, \tilde{z}_{2}\right) & =\mu, \\
\operatorname{cr}\left(z, z_{1}, \tilde{z}_{1}, \tilde{z}\right) & =\lambda, \\
\operatorname{cr}\left(z, z_{2}, \tilde{z}_{2}, \tilde{z}\right) & =\lambda \mu .
\end{aligned}
$$

Proof. One can interpret the sequence $\gamma_{k}=z_{k, 0}$ as a discrete curve and $\hat{\gamma}_{k}=z_{k, 1}$ as a $\mu$-Darboux transform of it. Then starting with $\tilde{z}_{0,0}$ we can create a $\lambda$-Darboux transform for $\gamma$. The Binachi permutability now states that $\hat{\tilde{\gamma}}_{k}=\tilde{z}_{k, 1}$ is given uniquely such that $\hat{\tilde{\gamma}}_{k}=\tilde{z}_{k, 1}$ is a $\lambda$-Darboux transform of $\hat{\gamma}_{k}=z_{k, 1}$ and a $\mu$-Darboux transform of $\tilde{\gamma}_{k}$. Iterating this defines the map $\tilde{z}$ uniquely and satisfies the stated cross-ratio conditions.


Figure 12: A Tractrix and Darboux transform of the straight line.

Remark. We can relax the conditions here: if $z: \mathbb{Z}^{2} \rightarrow \mathbb{C}$ has cross-ratios $\operatorname{cr}\left(z_{k, l}, z_{k+1, l}, z_{k+1, l+1}, z_{k, l+1}\right)=\frac{\alpha_{k}}{\beta_{l}}$ then for $\lambda \in \mathbb{C}$ and an initial $\tilde{z}_{0,0}$ we find a unique map $\tilde{z}: \mathbb{Z}^{2} \rightarrow \mathbb{C}$ with

$$
\begin{aligned}
\operatorname{cr}\left(\tilde{z}_{k, l}, \tilde{z}_{k+1, l}, \tilde{z}_{k+1, l+1}, \tilde{z}_{k, l+1}\right) & =\frac{\alpha_{k}}{\beta_{l}} \\
\operatorname{cr}\left(z_{k, l}, z_{k+1, l}, \tilde{z}_{k+1, l}, \tilde{z}_{k, l}\right) & =\frac{\beta_{l}}{\lambda} \\
\operatorname{cr}\left(z_{k, l}, z_{k, l+1}, \tilde{z}_{k, l+1}, \tilde{z}_{k, l}\right) & =\frac{\alpha_{k}}{\lambda}
\end{aligned}
$$

Now we will turn back and discuss what the Darboux transformation is in the euclidian picture.

Definition 2.41 Let $\gamma$ be a smooth arc-length parameterized curve. $\hat{\gamma}$ is called a Tractrix of $\gamma$ if $v:=\hat{\gamma}-\gamma$ satisfies

- $|v|=$ const,
- $\hat{\gamma}^{\prime} \| v$.

If $\hat{\gamma}$ is a Tractrix of $\gamma$ the curve $\tilde{\gamma}:=\gamma+2 v=\gamma+2(\hat{\gamma}-\gamma)$ is called a Darboux transform of $\gamma$.

Fig. 12 shows the Tractrix and Darboux transform of a straight line.
Lemma 2.42 The Darboux transform of an arc-length parameterized curve is again parameterized by arc-length.

Proof. Since $|v|=$ const, $v \perp v^{\prime}$ holds. Now

$$
\left\langle\tilde{\gamma}^{\prime}, \tilde{\gamma}^{\prime}\right\rangle=1+4\left\langle v^{\prime}, \gamma^{\prime}\right\rangle+4\left\langle v^{\prime}, v^{\prime}\right\rangle=1+4\left\langle v^{\prime}, \hat{\gamma}^{\prime}\right\rangle=1
$$

since $\hat{\gamma}^{\prime} \| v \perp v^{\prime}$.
Note that $\hat{\gamma}$ is a Tractrix of $\tilde{\gamma}$ as well.

Definition 2.43 Let $\gamma$ be a arc-length parameterized discrete curve, Then $\tilde{\gamma}$ is called a Darboux transform of $\gamma$ if $\left|\tilde{\gamma}_{1}-\tilde{\gamma}\right|=1,|\tilde{\gamma}-\gamma|=l=$ const and $\gamma, \gamma_{1}, \tilde{\gamma}_{1}$, and $\tilde{\gamma}$ do not form a parallelogram. $\hat{\gamma}=1 / 2(\tilde{\gamma}+\gamma)$ is then called a discrete Tractrix of $\gamma$ (and $\tilde{\gamma}$ ).

Remark.

- It is easy to see that the definition implies $\Delta \gamma+\Delta \tilde{\gamma} \|(\tilde{\gamma}-\gamma)+\left(\tilde{\gamma}_{1}-\gamma_{1}\right)$.
- $\operatorname{cr}\left(\gamma, \gamma_{1}, \tilde{\gamma}_{1}, \tilde{\gamma}\right)=\frac{1}{l^{2}}$. For the absolute value of the cross-ratio this is obvious. For its argument, observe that the triangles $\left(\gamma, \gamma_{1}, \tilde{\gamma}\right)$ and $\left(\tilde{\gamma}, \gamma_{1}, \tilde{\gamma}_{1}\right)$ are similar giving that the arguments of $\left(\gamma-\gamma_{1}\right) /\left(\gamma_{1}-\tilde{\gamma}_{1}\right)$ and $\left(\tilde{\gamma}_{1}-\tilde{\gamma}\right) /(\tilde{\gamma}-\gamma)$ sum to a multiple of $2 \pi$.
So indeed the euclidean Darboux transform is a special case of the one we formulated for the $\mathbb{C P}{ }^{1}$ picture.
- One can show that the Darboux transformation commutes with the tangential and (m)KdV flows: If $\gamma$ and a Darboux transform $\tilde{\gamma}$ evolve with one of these flows they stay related by a Darboux transformation for all times.


## 3 discrete curves in $\mathbb{R}^{3}$

When describing curves and surfaces in $\mathbb{R}^{3}$ we will frequently switch models for $\mathbb{R}^{3}$ depending on whether we do euclidean or Möbius (or other) geometry. when doing euclidean geometry a quaternionic description turns out to be the most useful.

## $3.1 \mathbb{R}^{3}$ and the Quaternions

Definition 3.1 The Quaternions $\mathbb{H}$ are the 4-dimensional real vector space spanned by $1, \dot{1}, q j$, and $\mathbb{k}$ furnished with a multiplication given by

$$
\dot{\mathrm{i} j}=\mathbb{k}, \quad \dot{j} \mathbb{k}=\dot{\mathrm{i}}, \quad \mathbb{k} \dot{\mathrm{i}}=\dot{\mathrm{j}}, \quad \dot{\mathrm{i}}^{2}=\dot{\mathrm{j}}^{2}=\mathbb{k}^{2}=-1 .
$$

There is a representation of $\mathbb{H}$ in the complex $2 \times 2$-matrices $g l(2, \mathbb{C})=$ $\operatorname{Mat}(2, \mathbb{C})$ via

$$
1 \cong\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right), \quad \dot{\mathrm{i}} \cong-i \sigma_{1}=-i\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right), \quad \dot{\mathrm{j}} \cong-i \sigma_{2}=-i\left(\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right)
$$

$$
\mathbb{k} \cong-i \sigma_{3}=-i\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) .
$$

The matrices $\sigma_{k}$ are sometimes called Pauli spin matrices. In analogy to the complex numbers one defines for $q \in \mathbb{H}, q=\alpha+\beta \dot{\mathrm{i}}+\gamma \dot{\mathrm{j}}+\delta \mathbb{k}$

- $\operatorname{Re}(q)=\alpha$
- $\operatorname{Im}(q)=q-\operatorname{Re}(q)=\beta \dot{\mathbb{1}}+\gamma \dot{\mathrm{j}}+\delta \mathbb{k}$
- $\bar{q}=q-2 \operatorname{Im}(q)=\alpha-\beta \dot{\mathrm{i}}-\gamma \dot{\mathrm{j}}-\delta \mathbb{k}$
- $|q|=\sqrt{q \bar{q}}=\sqrt{\alpha^{2}+\beta^{2}+\gamma^{2}+\delta^{2}}$
and finds

$$
q^{-1}=\frac{\bar{q}}{|q|^{2}} .
$$

Note that in contrast to the complex case $\operatorname{Im}(q)$ is not a real number (unless it is 0 ) but lies in the imaginary quaternions $\operatorname{ImH}:=\operatorname{span}\{\dot{\mathrm{i}}, \dot{j}, \mathbb{k}\}$.

We will identify the real 3 -dim vector space $\operatorname{Im} \mathbb{H}$ with $\mathbb{R}^{3}$. For $v, w \in \mathbb{R}^{3}$ we will denote the usual euclidean inner product with $\langle v, w\rangle=v_{1} w_{1}+v_{2} w_{2}+$ $v_{3} w_{3}$ and the vector product with $v \times w=\left(v_{2} w_{3}-v_{3} w_{2}, v_{3} w_{1}-v_{1} w_{3}, v_{1} w_{2}-\right.$ $v_{2} w_{1}$ ). Using the identification we now find for $v, w \in \operatorname{Im} \mathbb{H} \cong \mathbb{R}^{3}$

$$
v w=-\langle v, w\rangle+v \times w
$$

thus the quaternionic product incorporates both scalar and vector product in $\mathbb{R}^{3}$ in one multiplication formula.

Clearly $\mathbb{R}^{4} \supset S^{3}=\{a \in \mathbb{H}| | a \mid=1\}$ and using our matrix representation of $\mathbb{H}$ we also see $\{a \in \mathbb{H}||a|=1\}=S U(2)$ Now for a unit quaternion $a=\cos \frac{\omega}{2}+\sin \frac{\omega}{2} v$ with $v \in \operatorname{Im} \mathbb{H},|v|=1$ the map

$$
x \mapsto a x a^{-1}
$$

on $\operatorname{Im} \mathbb{H}$ is a rotation around the axis $v$ by the angle $\omega$

$$
a x a^{-1}=\cos \omega x+\sin \omega v \times x+(1-\cos \omega)\langle v, x\rangle v
$$

However the correspondence is not one-to-one since $a$ and $-a$ represent the same rotation. The map furnishes a group homomorphism $S^{3}=S U(2) \rightarrow$ $S O(3)$ with kernel $\{\mp 1\}$ showing that $S U(2)$ is a double cover of $S O(3)$ (see e.g. $\left.\left[\mathrm{EHH}^{+} 92\right]\right)$. More generally one finds

Theorem 3.2 (Hamilton) For any map $F \in O(3)$ there is $a \in S^{3}$ such that $F(x)=a x a^{-1}$ if $F$ is orientation preserving and $F(x)=a \bar{x} a^{-1}$ if $F$ is orientation reversing.

Again we only cite the analog theorem for $O(4)$ by Cayley:
Theorem 3.3 (Cayley) Any orthogonal transformation $F \in O(4)$ can be written as either $F(x)=a x b$ or $F(x)=x \mapsto a \bar{x} b, a, b \in S^{3}$ depending on whether $F$ is orientation preserving or orientation reversing.

Example 3.1 We will introduce the notion of a moving frame for curves in this example. The idea is to attach an (orientation preserving) orthogonal transformation to every point of the curve that maps the first unit basis vector to the tangent vector of the curve. The image of the remaining vectors of the standard orthonormal basis will then span the normal space at that point. We will make use of our above considerations and describe the transformations with quaternions:

Let $\gamma: I \rightarrow \mathbb{R}^{3}$ be a smooth arc-length parameterized curve. $F: I \rightarrow S^{3}$ is called $a$ (smooth) frame for $\gamma$ if

$$
F^{-1} \dot{\mathrm{i}} F=\gamma^{\prime}
$$

It is called parallel if in addition

$$
\left(F^{-1} \dot{j} F\right)^{\prime} \| \gamma^{\prime}
$$

holds. If we write $A:=F^{\prime} F^{-1}$ then the parallelity condition gives that $A \in$ $\operatorname{span}\{\dot{j}, \mathbb{k}\}$ :

$$
F^{-1} \dot{\mathrm{i}} F \|\left(F^{-1} \dot{\mathrm{j}} F\right)^{\prime}=-F^{-1} F^{\prime} F^{-1} \dot{\mathrm{j}} F+F^{-1} \dot{\mathrm{j}} F^{\prime} F^{-1} F=F^{-1}(\dot{\mathrm{j}} A-A \dot{\mathrm{j}}) F .
$$

So ii $\| \dot{j} A-A \dot{j}$ or $A \in \operatorname{span}\{\dot{j}, \mathbb{k}\}$. Knowing this we can write $A=\Psi \mathbb{k}$ with $\Psi \in \mathbb{C} \cong \operatorname{span}\{1, \dot{\mathrm{i}}\}$.

Definition $3.4 \Psi$ is called the complex curvature of $\gamma$.
The same can be done in the discrete domain:
Definition 3.5 Let $\gamma: I \rightarrow \mathbb{R}^{3}$ be an arc-length parameterized discrete curve. $F: I \rightarrow S^{3}$ with $F^{-1} \dot{\mathrm{i}} F=\Delta \gamma$ is called a discrete frame. $F$ is called a parallel frame if in addition $\operatorname{Im}\left(F_{k-1}^{-1} \dot{\mathrm{j}} F_{k-1} F_{k}^{-1} \dot{\mathrm{j}} F_{k}\right) \| \operatorname{Im}\left(\Delta \gamma_{k-1} \Delta \gamma_{k}\right)$ holds.

Again we can define $A$ by $F_{k+1}=A_{k} F_{k}$ and one finds that

$$
A=\cos \frac{\phi}{2}-\sin \frac{\phi}{2} e^{\sum_{n} \dot{\mathrm{i}} \tau_{n}} \mathbb{k}
$$

with $\tau=\angle\left(B_{1}, B\right), B=\frac{\Delta \gamma_{\overline{1}} \times \Delta \gamma}{\left\|\Delta \gamma_{\overline{1}} \times \Delta \gamma\right\|}$ and $\phi=\angle\left(\Delta \gamma_{\overline{1}}, \Delta \gamma\right)$ as before. Now we can renormalize $F$ such that $A=1+\frac{\Psi}{2} \mathbb{k}$.

Definition 3.6 $\Psi$ is called the (discrete) complex curvature of $\gamma . \tau$ is the discrete torsion.

The absolute value if the discrete complex curvature is the curvature we already defined for plane curves ${ }^{15}$ Note that both smooth and discrete complex curvature are defined up to a unitary factor only (the constant of integration /summation for the exponent).

[^11]
### 3.2 Möbius transformations in higher dimensions

We will now collect a few facts about Möbius geometry in space. Therefore we will for now treat hyperplanes as hyperspheres in the same way we already did with lines and circles in the plane.

Definition 3.7 The Möbius group is the group of transformations of $\mathbb{R}^{n} \cup$ $\{\infty\}$ that is generated by inversions on hyperspheres (or planes).

Remark. Note that the stereographic projection can be viewed as an inversion on a sphere restricted to $S^{2}$.

The cross-ratio was the simplest invariant in plane M/"obius geometry and we can almost transfer this result to arbitrary dimensions: there is always a 2 -sphere through 4 points in $\mathbb{R}^{n}$. It is unique if the points are in general position otherwise the points lie on a circle already. Now this sphere (or any sphere containing the circle) can be viewed as the Riemann sphere of complex numbers. On that we can compute a cross-ratio and assign it to the 4 points. The only ambiguity is the orientation of that 2 -sphere: It is not fixed by the 4 points and a change of orientation will change the cross-ratio into its complex conjugate.

Definition 3.8 The cross-ratio of four points in $\mathbb{R}^{n}$ is given by the real part and absolute value of the complex cross-ratio of the four points on the sphere through them interpreted as the Riemann sphere.

In $\mathbb{R}^{4}=\mathbb{H}$ the cross-ratio for $q_{1}, q_{2}, q_{3}$, and $q_{4}$ can be computed via real part and norm of

$$
\operatorname{qcr}\left(q_{1}, q_{2}, q_{3}, q_{4}\right):=\left(q_{1}-q_{2}\right)\left(q_{2}-q_{3}\right)^{-1}\left(q_{3}-q_{4}\right)\left(q_{4}-q_{1}\right)^{-1} .
$$

Lemma 3.9 qcr is invariant under $M /$ "obius transformations up to a conjugation with a non vanishing quaternion.

Proof. $q c r$ is clearly invariant under translations and scaling. So we can restrict our proof to the case of the inversion $q_{i} \mapsto q_{i}^{-1}$ and $q_{1} \neq 0$. But

$$
\begin{aligned}
\operatorname{qcr}\left(q_{1}^{-1}, q_{2}^{-1}, q_{3}^{-1}, q_{4}^{-1}\right) & =\left(q_{1}^{-1}-q_{2}^{-1}\right)\left(q_{2}^{-1}-q_{3}^{-1}\right)^{-1}\left(q_{3}^{-1}-q_{4}^{-1}\right)\left(q_{4}^{-1}-q_{1}^{-1}\right)^{-1} \\
& =q_{1}^{-1}\left(q_{1}-q_{2}\right)\left(q_{2}-q_{3}\right)^{-1}\left(q_{3}-q_{4}\right)\left(q_{4}-q_{1}\right)^{-1} q_{1} \\
& =q_{1}^{-1} q \operatorname{cr}\left(q_{1}, q_{2}, q_{3}, q_{4}\right) q_{1} .
\end{aligned}
$$

Remark. If $q_{1}, q_{2}, q_{3}$, and $q_{4}$ are co-planar we can rotate and translate the plane into span $\{1, \dot{i}\}$. Here the quaternionic cross-ratio coincides with the complex one.

## 4 discrete surfaces in $\mathbb{R}^{3}$

## 4.1 isothermic surfaces

We will now consider isothermic surfaces. The class of isothermic surfaces covers a wide range of classical surfaces including quadrics, surfaces of revolution, and minimal as well as cmc (constant mean curvature) surfaces.

When discretizing isothermic surfaces we will follow the "historic route" (as in [BP96]) first and then give an alternative description that allows a more direct formulation of these discrete surfaces. But first we will collect some basic facts about isothermic surfaces in the smooth setup.

Definition 4.1 Let $f: \mathbb{R}^{2} \ni G \rightarrow \mathbb{R}^{n}$ be an immersion ${ }^{16}$. $f$ is called $a$ conformal if $f_{x}=\frac{\partial f}{\partial x}$ and $f_{y}=\frac{\partial f}{\partial y}$ satisfy $f_{x} \perp f_{y}$ and $\left\|f_{x}\right\|=\left\|f_{y}\right\|$.
$f$ is called isothermal if in addition $f_{x y}=\frac{\partial^{2} f}{\partial x \partial y} \in \operatorname{span}\left\{f_{x}, f_{y}\right\}$.

## Remark.

- A surface is called isothermic if it admitsisothermal parameterization ${ }^{17}$.
- One can rephrase the above definition of isothermal as a conformal parameterization by curvature lines.
- since all directions in the plane (or on a sphere) are curvature directions any conformal parameterization of a 2-plane or 2-sphere is isothermal.

Lemma 4.2 If $f$ is a surface in isothermal parameterization and $M$ is a Möbius transformation, then $M \circ f$ is isothermal again.

Proof. We only have to check that inverting $f$ leaves it isothermal, since the conditions to be isothermal are invariant with respect to scaling and translations.

Let $\tilde{f}=\frac{f}{\|f\|^{2}}$ then we find

$$
\tilde{f}_{x}=\frac{f_{x}\|f\|^{2}-2 f\left\langle f, f_{x}\right\rangle}{\|f\|^{4}}, \quad \text { and } \quad \tilde{f}_{y}=\frac{f_{y}\|f\|^{2}-2 f\left\langle f, f_{y}\right\rangle}{\|f\|^{4}}
$$

[^12]Since $\left\|\tilde{f}_{x}\right\|^{2}=\frac{1}{\|f\|^{8}}\left(\|f\|^{4}\left\|f_{x}\right\|^{2}-4\|f\|^{2}\left\langle f, f_{x}\right\rangle^{2}+4\|f\|^{2}\left\langle f, f_{x}\right\rangle^{2}\right)=\frac{\left\|f_{x}\right\|^{2}}{\|f\|^{4}}=$ $\frac{\left\|f_{y}\right\|^{2}}{\|f\|^{4}}=\left\|\tilde{f}_{y}\right\|^{2}$ we see that $\tilde{f}$ is conformal. The fact that $\tilde{f}_{x y} \in \operatorname{span}\left\{\tilde{f}_{x}, \tilde{f}_{y}\right\}$ can be computed easily as well.

Lemma 4.3 If $f$ is isothermal then $f^{*}$ given by

$$
f_{x}^{*}:=\frac{f_{x}}{\left\|f_{x}\right\|^{2}}, \quad f_{y}^{*}:=-\frac{f_{y}}{\left\|f_{y}\right\|^{2}}
$$

is isothermal again.
Proof. let us set $e^{2 u}:=\left\|f_{x}\right\|^{2}=\left\|f_{y}\right\|^{2}$. Then

$$
2 u_{y} e^{2 u}=\left(e^{2 u}\right)_{y}=2\left\langle f_{x}, f_{x y}\right\rangle .
$$

and

$$
2 u_{x} e^{2 u}=\left(e^{2 u}\right)_{x}=2\left\langle f_{y}, f_{x y}\right\rangle .
$$

giving $f_{x y}=u_{y} f_{x}+u_{x} f_{y}$. Now

$$
f_{x y}^{*}=\left(e^{-2 u} f_{x}\right)_{y}=e^{-2 u}\left(u_{x} f_{y}-u_{y} f_{x}\right)=f_{y x}^{*}
$$

showing that we can indeed locally integrate $f_{x}^{*}$ and $f_{y}^{*}$ into a surface $f^{*}$. This $f^{*}$ is automatically conformal and the above formula shows that $f_{x y}^{*} \in$ $\operatorname{span}\left\{f_{x}^{*}, f_{y}^{*}\right\}=\operatorname{span}\left\{f_{x}, f_{y}\right\}$ as well.

Definition $4.4 f^{*}$ is called the dual isothermic surface of $f$.
The next lemma gives us the crucial information that will lead to a discretization of isothermal parameterized surfaces [BP96]:

Lemma 4.5 Let $f$ be a smooth immersion and

$$
\begin{aligned}
& f_{1}=f+\epsilon\left(-f_{x}-f_{y}\right)+\frac{\epsilon^{2}}{2}\left(f_{x x}+f_{y y}+2 f_{x y}\right) \\
& f_{2}=f+\epsilon\left(+f_{x}-f_{y}\right)+\frac{\epsilon^{2}}{2}\left(f_{x x}+f_{y y}-2 f_{x y}\right) \\
& f_{3}=f+\epsilon\left(+f_{x}+f_{y}\right)+\frac{\epsilon^{\epsilon}}{2}\left(f_{x x}+f_{y y}+2 f_{x y}\right) \\
& f_{4}=f+\epsilon\left(-f_{x}+f_{y}\right)+\frac{\epsilon^{2}}{2}\left(f_{x x}+f_{y y}-2 f_{x y}\right)
\end{aligned}
$$

be the quadrilateral given by the truncated Taylor expansion of $f(x \pm \epsilon, y \pm \epsilon)$ at some point $(x, y)$. Then

- $\operatorname{cr}\left(f_{1}, f_{2}, f_{3}, f_{4}\right)=-1+O(\epsilon) \Leftrightarrow f$ is conformal
- $\operatorname{cr}\left(f_{1}, f_{2}, f_{3}, f_{4}\right)=-1+O\left(\epsilon^{2}\right) \Leftrightarrow f$ is isothermal.

Proof. First we can translate the quadrilateral to have $f_{1}=0$ and scale it by $1 /(2 \epsilon)$ giving $f_{1}=0, f_{2}=f_{x}-\epsilon f_{x y}, f_{3}=f_{x}+f_{y}, f_{4}=f_{y}-\epsilon f_{x y}$, then we can invert sending $f_{1}$ to infinity and leaving

$$
f_{2}=\frac{f_{x}-\epsilon f_{x y}}{\left\|f_{x}-\epsilon f_{x y}\right\|^{2}}, f_{3}=\frac{f_{x}+f_{y}}{\left\|f_{x}+f_{y}\right\|^{2}}, f_{4}=\frac{f_{y}-\epsilon f_{x y}}{\left\|f_{y}-\epsilon f_{x y}\right\|^{2}} .
$$

Now $\operatorname{cr}\left(f_{1}, f_{2}, f_{3}, f_{4}\right)=-\frac{f_{3}-f_{4}}{f_{2}-f_{3}}$ and we need to find out what the conditions on the parameterization are if $f_{2}-f_{3}=f_{3}-f_{4}+O\left(\epsilon^{k}\right)$ or equivalently $2 f_{3}=f_{2}+f_{4}+O\left(\epsilon^{k}\right), k \in 1,2$. Expanding with the denominators of the right hand side of

$$
2 \frac{f_{x}+f_{y}}{\left\|f_{x}+f_{y}\right\|^{2}}=\frac{f_{x}-\epsilon f_{x y}}{\left\|f_{x}-\epsilon f_{x y}\right\|^{2}}+\frac{f_{y}-\epsilon f_{x y}}{\left\|f_{y}-\epsilon f_{x y}\right\|^{2}}+O\left(\epsilon^{k}\right)
$$

and omitting terms that are of $\epsilon$-order greater than 1 gives

$$
\begin{gathered}
2 \frac{f_{x}+f_{y}}{\left\|f_{x}+f_{y}\right\|^{2}}\left(\left\|f_{x}\right\|^{2}-2 \epsilon\left\langle f_{x}, f_{x y}\right\rangle\right)\left(\left\|f_{y}\right\|^{2}-2 \epsilon\left\langle f_{y}, f_{x y}\right\rangle\right) \\
=\left(f_{x}-\epsilon f_{x y}\right)\left(\left\|f_{y}\right\|^{2}-2 \epsilon\left\langle f_{y}, f_{x y}\right\rangle\right)+\left(f_{y}-\epsilon f_{x y}\right)\left(\left\|f_{x}\right\|^{2}-2 \epsilon\left\langle f_{x}, f_{x y}\right\rangle\right)+O\left(\epsilon^{k}\right)
\end{gathered}
$$

. The constant part yields

$$
2 \frac{f_{x}+f_{y}}{\left\|f_{x}+f_{y}\right\|^{2}}\left\|f_{x}\right\|^{2}\left\|f_{y}\right\|^{2}=f_{x}\left\|f_{y}\right\|^{2}+f_{y}\left\|f_{x}\right\|^{2}
$$

which gives that $f$ must indeed be conformal. Using this the linear term reads

$$
-2\left(f_{x}+f_{y}\right)\left\langle f_{x}+f_{y}, f_{x y}\right\rangle=-2 f_{x}\left\langle f_{y}, f_{x y}\right\rangle-f_{x y}\left(\left\|f_{y}\right\|^{2}+\left\|f_{x}\right\|^{2}\right)-2 f_{y}\left\langle f_{x}, f_{x y}\right\rangle
$$

giving that $f_{x y} \in \operatorname{span}\left\{f_{x}, f_{y}\right\}$ and thus showing that $f$ needs to be isothermal.

Since all essential properties of isothermic surfaces are phrased in terms of their isothermal parameterization it seems natural to discretize this parameterization and not arbitrarily parameterized isothermic surfaces. This basically fixes our combinatorics to quad-meshes (planar graphs with all faces
being quadrilaterals) or at a regular neighbourhood to pieces of $\mathbb{Z}^{2}$. This is a design decision led by the hope that we will be able to find discrete analogs of all the above properties (and more) of smooth isothermic surfaces.

The last lemma then motivates the definition of discrete isothermic surfaces as maps with cross-ratio equal to -1 for all elementary quadrilaterals.

Definition 4.6 $F: \mathbb{Z}^{2} \rightarrow \mathbb{R}^{n}$ is called discrete isothermic if the cross-ratio of all elementary quadrilaterals is -1 .

Remark. We will also call $F: \mathbb{Z}^{2} \rightarrow \mathbb{R}^{n}$ discrete isothermic if

$$
\operatorname{cr}\left(F_{k, l}, F_{k+1, l}, F_{k+1, l+1}, F_{k, l+1}\right)=\frac{\alpha_{k}}{\beta_{l}}<0, \quad \alpha_{k}, \beta_{l} \in \mathbb{R} .
$$

This corresponds to a parameterization $f$ with $\left|f_{x}\right|=\frac{\alpha}{s},\left|f_{y}\right|=\frac{\beta}{s}, f_{x} \perp f_{y}, s$ a real function, and $f_{x y} \in \operatorname{span}\left\{f_{x}, f_{y}\right\}$.

## Example 4.1

- Any map $z: \mathbb{Z}^{2} \rightarrow \mathbb{C}$ with cross-ratio -1 as discussed earlier constitutes a discrete isothermic map. Since isothermal maps in the plane are just conformal maps these discrete maps are called discrete conformal or discrete holomorphic maps. Examples include the identity and $z(k, l)=$ $e^{\alpha k+i \beta l}, \beta=\frac{2 \pi}{n} ; \alpha=2 \operatorname{arcsinh}\left(\sin \frac{\beta}{2}\right)$.
- Starting with $F_{k, 0}$ and $F_{0, l}$ the cross-ratio evolution gives rise to a unique $F: \mathbb{Z}^{2} \rightarrow \mathbb{R}^{n}$.

The following two lemmata show that our discretization indeed fulfills what we would expect from a proper discretization: It shares essential properties with the smooth counterpart.

Lemma 4.7 Let $F$ be discrete isothermic and $M$ be a Möbius transformation. Then $M \circ F$ is discrete isothermic again.

Proof. This is obvious, since the defining property of given cross-ratios is invariant with respect to Möbius transformations.

Lemma 4.8 Let $F$ be discrete isothermic. Then $F^{*}$ given by

$$
\begin{aligned}
& F_{k+1, l}^{*}-F_{k, l}^{*}:=\frac{\alpha_{k}}{\left\|F_{k+1, l}-F_{k, l}\right\|^{2}}\left(F_{k+1, l}-F_{k, l}\right) \\
& F_{k, l+1}^{*}-F_{k, l}^{*}:=\frac{\beta_{l}}{\left\|F_{k, l+1}-F_{k, l}\right\|^{2}}\left(F_{k, l+1}-F_{k, l}\right)
\end{aligned}
$$

is discrete isothermic again. It is called the dual.

Proof. Looking at one quadrilateral we can identify the plane containing it with $\mathbb{C}$ and subsequently work in the complex numbers. Set $a=F_{1}-F$, $b=F_{12}-F_{1}, c=F_{2}-F_{12}$, and $d=F-F_{2}$. Then $\frac{a c}{b d}=\frac{\alpha}{\beta}$ and $a+b+c+d=0$. Using $a^{*}=F_{1}^{*}-F^{*}=\frac{\alpha}{\bar{a}}$, and likewise $b^{*}=\frac{\beta}{\bar{b}}, c^{*}=\frac{\alpha}{\bar{c}}$, and $d^{*}=\beta \bar{d}$ we find

$$
\frac{a^{*} c^{*}}{b^{*} d^{*}}=\frac{\alpha^{2}}{\beta^{2}} \frac{\bar{b} \bar{d}}{\bar{a} \bar{c}}=\frac{\alpha}{\beta} .
$$

If the dual edges $a^{*}, b^{*}, c^{*}$, and $d^{*}$ form a quadrilateral its cross-ratio will be the same as the one of the original quadrilateral. What is left to show is that $a^{*}+b^{*}+c^{*}+d^{*}=0$. Conjugating the left hand side and multiplying it with $a c$ results in

$$
\alpha c+\beta \frac{a c}{b}+\alpha a+\beta \frac{a c}{d}=\alpha c+\beta \frac{\alpha}{\beta} d+\alpha a+\beta \frac{\alpha}{\beta} b=0 .
$$

This shows that we can integrate the dual edges into closing quadrilaterals.

Remark. As in the smooth case dualizing is a duality: $F^{* *}=F$ up to translation.

Example 4.2 Minimal surfaces are known to be isothermic and the dual surface is their Gauß map (it maps into the unit sphere and thus is a conformal map into $S^{2}$ ). In fact minimal surfaces can be characterized by this property and the dual of any conformal map into $S^{2}$ is minimal. We can use this characterization to construct discrete minimal surfaces:

1. Start with a discrete holomorphic map.
2. map it onto $S^{2}$ with a stereographic projection
3. dualize

The resulting discrete isothermic surface is called a discrete minimal surface. At the moment we can justify this only by construction but we will soon see that there is a notion of discrete mean curvature that vanishes for these surfaces as well.). The above mentioned identity map will lead to a discrete Enneper surfac ${ }^{181}$ (see Fig. 13) and the discrete exponential map gives a discrete Catenoid (see Fig. 14):


Figure 13: A discrete Enneper surface.

Note that the procedure outlined here is essentially a discrete version of the well known Weierstraß representation for minimal surfaces (see e. g. [EJ07]) and as in the smooth case one can write the whole procedure in one formula: given $z: \mathbb{Z}^{2} \rightarrow \mathbb{C}$ discrete holomorphic, set

$$
\begin{aligned}
& \left.F_{1}-F=\frac{1}{2} \operatorname{Re}\left(\frac{1}{z_{1}-z}\left(1-z_{1} z\right), i\left(1+z_{1} z\right), z_{1}+z\right)\right) \\
& \left.F_{2}-F=\frac{1}{2} \operatorname{Re}\left(\frac{1}{z_{2}-z}\left(1-z_{2} z\right), i\left(1+z_{2} z\right), z_{2}+z\right)\right)
\end{aligned}
$$

### 4.2 The classical model for Möbius geometry

We will now turn to an interesting description of Möbius geometry in space. In the plane we turned to the projective description in homogeneous coordinates to linearize the Möbius group and now we again seek for a model where the Möbius transformations become linear maps. Avery thorough treatment

[^13]

Figure 14: A discrete Catenoid surface.
of various models for Möbius geometry can be found in the Book of Udo Hertrich-Jeromin [HJ03].

We will identify points $p \in \mathbb{R}^{n}$ with light-like lines (1-dim subspaces) in a Minkowsky $\mathbb{R}^{n+2}$. But we will also find that hyperspheres (or planes) have a simple description in this model (as space-like unit vectors).

Consider $\mathbb{R}^{n+2}$ together with an inner product $\langle x, y\rangle=-x_{0} y_{0}+\sum_{k=1}^{n+1} x_{k} y_{k}$. Then $v \in \mathbb{R}^{n+2}$ is called space-like / light-like / time-like if $\langle v, v\rangle$ is greater than / equal to / less than 0 . The light-like vectors form a double cone the light cone. They can not be normalized, since they have length 0 .

Now identify $p \in \mathbb{R}^{n}$ with

$$
\hat{p}:=\left(\frac{1+\|p\|^{2}}{2}, p, \frac{1-\|p\|^{2}}{2}\right) .
$$

It is easy to check that $\langle\hat{p}, \hat{p}\rangle=0$ and $\hat{p}$ is indeed light-like. If on the other hand $\langle q, q\rangle=0$ for some $q \in \mathbb{R}^{n+2}$ we find a $p \in \mathbb{R}^{n}$ with $\hat{p}=\lambda q$ by

$$
p=\frac{1}{q_{0}+q_{n+1}}\left(q_{1}, \ldots, q_{n}\right) .
$$

(note that $p$ might be $\infty$ ).

Next let $s$ be an oriented hypersphere $s$ in $\mathbb{R}^{n}$ with center $c$ and radius $r$. Oriented means here that the radius can be negative. We will identify $s$ with

$$
\hat{s}=\frac{1}{2 r}\left(1+\left(\|c\|^{2}-r^{2}\right), 2 c, 1-\left(1-\|c\|^{2}-r^{2}\right)\right)
$$

and a plane with the normal form $\langle v, n\rangle=d,\|n\|=1$ is mapped to

$$
\hat{s}=(d, n,-d) .
$$

Again it is easy to check that $\langle\hat{s}, \hat{s}\rangle=1$ and $\hat{s}$ is a space-like unit vector in both cases.

If $q \in \mathbb{R}^{n+2}$ is a space-like unit vector we can find the radius of the sphere it represents by $r=\frac{1}{q_{0}+q_{n+1}}$ and its center by $c=r\left(q_{1}, \ldots, q_{n}\right)\left(\right.$ if $q_{0}+q_{n+1}=0$ the sphere is actually a plane with normal $n=\left(q_{1}, \ldots, q_{n}\right)$ and normal form $\left.\langle v, n\rangle=q_{0}\right)$.
Remark. The geometry behind these identifications is the following: Using a stereographic projection one can map $\mathbb{R}^{n}$ into $S^{n} \subset \mathbb{R}^{n+1}$. Thus points and hyperspheres in $\mathbb{R}^{n}$ correspond to points and hyperspheres in $S^{n}$ but the hyperspheres in $S^{n}$ can be identified with the tip of the cone in $\mathbb{R}^{n+1}$ that touches $S^{n}$ in that hypersphere. Now embed $\mathbb{R}^{n+1}$ in $\mathbb{R}^{n+2}$ by as $\{1\} \times$ $\mathbb{R}^{n+1}$. Then $S^{n}$ gets mapped into the light cone in $\mathbb{R}^{n+2}$ and the points that represent hyperspheres get mapped to vectors outside the light cone (and thus can be normalized to length 1).

To see why this model is useful we state the following lemma:
Lemma 4.9 In the above setting

- a point p lies in a sphere s: $p \in S \Leftrightarrow \hat{p} \perp \hat{S}$
- for two spheres $s_{1}$ and $s_{2}: \cos \angle\left(s_{1}, s_{2}\right)=\left\langle\hat{s}_{1}, \hat{s}_{2}\right\rangle$
- For a sphere s the map $\hat{x} \mapsto \hat{x}-2\langle\hat{x}, \hat{s}\rangle \hat{s}$ is the inversion on $s$
- for two points $p$ and $q:\langle\hat{p}, \hat{q}\rangle=-\frac{1}{2}\|p-q\|^{2}$ (note that this only holds for the normalization we have chosen for $\hat{p}$ and $\hat{q}$. In general one can choose the scaling of $\hat{p}$ and $\hat{q}$ freely).

Proof. In all four cases the proof is just a simple calculation:

- For a point $p$ and a sphere $s$ with radii $r$ and centre $c$

$$
\langle\hat{p}, \hat{s}\rangle=\frac{1}{2}\left(r-\frac{\langle c-p, c-p\rangle}{r}\right)=0 \Leftrightarrow\|c-p\|=r \Leftrightarrow p \in s
$$

holds.

- For two intersecting spheres $s_{1}$ and $s_{2}$ with centers $c_{1}$ and $c_{2}$ and radii $r_{1}$ and $r_{2}$ respectively, the intersection angle $\phi$ is given by the cosine law

$$
\left\|c_{1}-c_{2}\right\|^{2}=r_{1}^{2}+r_{2}^{2}-2 r_{1} r_{2} \cos \phi
$$

giving

$$
\cos \phi=\frac{r_{1}^{2}+r_{2}^{2}-\left\|c_{1}-c_{2}\right\|^{2}}{2 r_{1} r_{2}}
$$

On the other hand

$$
\left\langle\hat{s}_{1}, \hat{s}_{2}\right\rangle=\frac{1}{2 r_{1} r_{2}}\left(r_{1}^{2}+r_{2}^{2}-\left\langle c_{1}-c_{2}, c_{1}-c_{2}\right\rangle\right) .
$$

- The inversion on $s$ is given by

$$
x \mapsto r^{2} \frac{x-c}{\|x-c\|^{2}}+c
$$

Now first we find that $I(\hat{x}):=\hat{x}-2\langle\hat{x}, \hat{s}\rangle \hat{s}$ is an orthogonal map:

$$
\langle I(\hat{x}), I(\hat{x})\rangle=4\langle\hat{x}, \hat{s}\rangle^{2}\langle\hat{s}, \hat{s}\rangle-4\langle\hat{x}, \hat{s}\rangle\langle\hat{x}, \hat{s}\rangle+\langle\hat{x}, \hat{x}\rangle=\langle\hat{x}, \hat{x}\rangle .
$$

therefore $I$ sends light-like vectors onto light-like vectors. So

$$
I(\hat{x})=\lambda\left(\frac{1+|q|^{2}}{2}, q, \frac{1-|q|^{2}}{2}\right)
$$

for some point $q$ and a factor $\lambda$. One easily finds that $\lambda q=\frac{|x-c|^{2}}{r^{2}}(c+$ $\left.\frac{r^{2}}{|x-c|^{2}}(x-c)\right)$ and $\lambda=\frac{|x-c|^{2}}{r^{2}}$ and concludes that

$$
q=\frac{r^{2}}{|x-c|^{2}}(x-c)+c
$$

as claimed.

- The last claim is again a direct calculation

$$
\langle\hat{p}, \hat{q}\rangle=-\frac{1+|p|^{2}}{2} \frac{1+|q|^{2}}{2}+\langle p, q\rangle+\frac{1-|p|^{2}}{2} \frac{1-|q|^{2}}{2}=-\frac{1}{2}\langle p-q, p-q\rangle .
$$

Remark.

- Möbius transformations are orthogonal maps in this model.
- Since $I_{s}(\hat{x})=\hat{x}-2\langle\hat{x}, \hat{s}\rangle \hat{s}$ is orthogonal it maps spheres into spheres.
- $I_{s}(\hat{s})=-\hat{s}$.
- $I_{s}(\hat{t})=\hat{t} \Leftrightarrow \hat{s} \perp \hat{t}$
- all the above extends to hyperplanes naturally.
- A circle in $\mathbb{R}^{3}$ can be defined as the intersection of two distinct spheres containing it $c=s_{1} \cap s_{2}$ so $c=\left\{p \in \mathbb{R}^{3} \mid\left\langle\hat{p}, \hat{s}_{1}\right\rangle=\left\langle\hat{p}, \hat{s}_{2}\right\rangle=0\right\}$. Thus a circle is uniquely defined by the time-like 3 -dim subspace that is perpendicular to $\hat{s}_{1}$ and $\hat{s}_{2}$. As a consequence all spheres that contain $c$ are given as linear combinations of $\hat{s}_{1}$ and $\hat{s}_{2}$. the 3 -dim subspace is of course fixed by prescribing 3 light-like vectors in it, showing that a circle is determined by three points on it.

The next theorem shows how this model of Möbius geometry helps rewriting the condition for isothermality.

Theorem 4.10 $A$ smooth surface $f: G \supset \mathbb{R}^{2} \rightarrow \mathbb{R}^{n}$ is isothermic with

$$
\left\|f_{x}\right\|^{2}=\alpha(x) e^{2 u},\left\|f_{y}\right\|^{2}=\beta(x) e^{2 u}, f_{x} \perp f_{y}, f_{x y} \in \operatorname{span}\left\{f_{x}, f_{y}\right\}
$$

if and only if

$$
\left(e^{-u} \hat{f}\right)_{x y}=\lambda e^{-u} \hat{f}
$$

for some function $\lambda$.
Proof. Let $f$ be an isothermic surface in a parameterization as stated in the theorem. Then $f_{x y}=u_{y} f_{x}+u_{x} f_{y}$ since $\left\|f_{x}\right\|^{2}=\left\langle f_{x}, f_{x}\right\rangle=\alpha(x) e^{2 u(x, y)} \Rightarrow$ $2\left\langle f_{x}, f_{x y}\right\rangle=2\left\|f_{x}\right\|^{2} u_{y}$ and the analog for $\left\|f_{y}\right\|$. Now

$$
\begin{aligned}
\hat{f}= & \left(\frac{1+\|f\|^{2}}{2}, f, \frac{1-\|f\|^{2}}{2}\right) \\
\left(e^{-u} \hat{f}\right)_{x}= & -u_{x} e^{-u} \hat{f}+e^{-u}\left(\left\langle f, f_{x}\right\rangle, f_{x},-\left\langle f, f_{x}\right\rangle\right) \\
\left(e^{-u} \hat{f}\right)_{x y}= & -u_{x y} e^{-u} \hat{f}+u_{x} u_{y} e^{-u} \hat{f}-u_{x} e^{-u}\left(\left\langle f, f_{y}\right\rangle, f_{y},-\left\langle f, f_{y}\right\rangle\right) \\
& -u_{y} e^{-u}\left(\left\langle f, f_{x}\right\rangle, f_{x},-\left\langle f, f_{x}\right\rangle\right)+e^{-u}\left(\left\langle f, f_{x y}\right\rangle, f_{x y},-\left\langle f, f_{x y}\right\rangle\right) \\
= & \left(u_{x} u_{y}-u_{x y}\right) e^{-u} \hat{f} .
\end{aligned}
$$

If on the other hand $\hat{g}: G \subset \mathbb{R}^{2} \rightarrow \mathbb{R}^{n+2}$ with $\hat{g}_{x y}=\lambda \hat{g}$ is given, set $e^{-u}:=\hat{g}_{0}+\hat{g}_{n+1}$ (note that we can assume $\hat{g}_{0}+\hat{g}_{n+1}>0$ !). Then $\lambda e^{-u}=\lambda\left(\hat{g}_{0}+\hat{g}_{n+1}\right)=\left(\hat{g}_{0}+\hat{g}_{n+1}\right)_{x y}=\left(u_{x} u_{y}-u_{x y}\right) e^{-u}$. So $\lambda=u_{x} u_{y}-u_{x y}$. Setting $\hat{f}=e^{u} \hat{g}$ we find

$$
\hat{f}_{x y}=u_{y} \hat{f}_{x}+u_{x} \hat{f}_{y}
$$

This implies in particular $f_{x y}=u_{y} f_{x}+u_{x} f_{y}$ and $\left(\hat{f}_{0}\right)_{x y}=\left\langle f_{x}, f_{y}\right\rangle+\left\langle f, f_{x y}\right\rangle=$ $\left\langle f_{x}, f_{y}\right\rangle+u_{y}\left\langle f, f_{x}\right\rangle+u_{x}\left\langle f, f_{y}\right\rangle=u_{y}\left(\hat{f}_{0}\right)_{x}+u_{x}\left(\hat{f}_{0}\right)_{y}$ which gives $f_{x} \perp f_{y}$.

Finally set $\alpha:=\left\|f_{x}\right\|^{2} e^{-2 u}$. Then

$$
\alpha_{y}=\frac{2\left\langle f_{x}, f_{x y}\right\rangle e^{2 u}-2\left\langle f_{x}, f_{x}\right\rangle u_{y} e^{2 u}}{e^{4 u}}=0
$$

so $\alpha=\alpha(x)$ and like wise $\beta=\beta(y)$.
Definition 4.11 $A$ map $f: G \subset \mathbb{R}^{2} \rightarrow \mathbb{R}^{n}$, $n>1$ is said to solve the Moutard equation if

$$
f_{x y}=\lambda f
$$

holds for some function $\lambda$.
We see that isothermic surfaces are in one to one correspondence to solutions to the Moutard equation in the light-cone.

Considering this observation, it seems a natural approach to discretize the Moutard equation and interpret solutions to it that lie in the light-cone as discrete isothermic nets. If we denote $\Delta_{1} F:=F_{1}-F$ and likewise $\Delta_{2} F:=$ $F_{2}-F$ then $\Delta_{1} \Delta_{2} F=\Delta_{2} \Delta_{1} F=\left(F_{12}-F_{2}\right)-\left(F_{1}-F\right)=F_{12}+F-\left(F_{1}+F_{2}\right)$ is clearly a discretization of the mixed second derivative. Since $\Delta_{2} \Delta_{1} F$ is defined on a quadrilateral rather than on a vertex comparing it with $\lambda F$ is not symmetric. A symmetric choice for the right hand side is $\lambda\left(F_{12}+F+F_{1}+F_{2}\right)$. Now a discrete version of the Moutard equation is

$$
\begin{array}{rlrl} 
& & F_{12}+F-\left(F_{1}+F_{2}\right) & =\lambda\left(F_{12}+F+F_{1}+F_{2}\right) \\
\Leftrightarrow & \left(F_{12}+F\right)(1-\lambda) & =\left(F_{1}+F_{2}\right)(1+\lambda) \\
\Leftrightarrow & F_{12}+F & =\frac{1+\lambda}{1-\lambda}\left(F_{1}+F_{2}\right)
\end{array}
$$

The next definition is due to Nimo and Schief (see [NS98]):
Definition $4.12 F: \mathbb{Z}^{2} \rightarrow \mathbb{R}^{n}$, $n>1$ is said to solve the discrete Moutard equation if there exists $\lambda: \mathbb{Z}^{2} \rightarrow \mathbb{R}$ such that

$$
F_{12}+F=\frac{1+\lambda}{1-\lambda}\left(F_{1}+F_{2}\right)
$$

holds (the diagonal averages are parallel).
Lemma 4.13 The restriction of the discrete Moutard equation to quadrics of constant length $c \in \mathbb{R}$ is admissible (and determines $\lambda$ ).

Proof. Let $\langle F, F\rangle=\left\langle F_{1}, F_{1}\right\rangle=\left\langle F_{2}, F_{2}\right\rangle=c$. Then

$$
\left\langle F_{12}, F_{12}\right\rangle=\lambda^{2}\left\langle F_{1}+F_{2}, F_{1}+F_{2}\right\rangle-2 \lambda\left\langle F_{1}+F_{2}, F\right\rangle+c=c
$$

if $\lambda=0$ or

$$
\lambda=2 \frac{\left\langle F_{1}+F_{2}, F\right\rangle}{\left\langle F_{1}+F_{2}, F_{1}+F_{2}\right\rangle}
$$

and

$$
F_{12}=2 \frac{\left\langle F_{1}+F_{2}, F\right\rangle}{\left\langle F_{1}+F_{2}, F_{1}+F_{2}\right\rangle}\left(F_{1}+F_{2}\right)-F
$$

( $F_{12}$ is $F$ mirrored on $F_{1}+F_{2}$ or $-F_{12}$ is $F$ inverted on $\frac{F_{1}+F_{2}}{\left\|F_{1}+F_{2}\right\|}$.)
Note that $c=0$ is allowed here. Note also that $x \mapsto 2 \frac{\left\langle F_{1}+F_{2}, x\right\rangle}{\left\langle F_{1}+F_{2}, F_{1}+F_{2}\right\rangle}\left(F_{1}+F_{2}\right)-x$ sends $F_{1}$ to $F_{2}$ and vice versa and is an orthogonal transformation.

Lemma $4.14 F: \mathbb{Z}^{2} \rightarrow \mathbb{R}^{n}, n>1$ is discrete isothermic with

$$
\operatorname{cr}\left(F_{k, l}, F_{k+1, l}, F_{k+1, l+1}, F_{k, l+1}\right)=-\frac{\alpha_{k}}{\beta_{l}}
$$

with $\alpha \beta>0$, if and only if a scaled version of $\hat{F}$ solves the discrete Moutard equation in one component of the light cone.

Proof. Let $\hat{F}$ be a solution to the discrete Moutard equation in the light cone of $\mathbb{R}^{n+2}$ with first components $>0$ and $F$ its projection to $\mathbb{R}^{n}$. Obviously $\hat{F}_{12}$ is a linear combination of $\hat{F}, \hat{F}_{1}$, and $\hat{F}_{2}$. So the four points lie in a common 3 -space giving that the corresponding points in $\mathbb{R}^{n}$ are concircular (and therefore have a real cross-ratio. Moreover the notes above on orthogonality of the map that constitutes the discrete Moutard equation show that

$$
\left\langle\hat{F}, \hat{F}_{1}\right\rangle=\left\langle\hat{F}_{2}, \hat{F}_{12}\right\rangle, \quad \text { and } \quad\left\langle\hat{F}, \hat{F}_{2}\right\rangle=\left\langle\hat{F}_{1}, \hat{F}_{12}\right\rangle
$$

is true for any solution of the discrete Moutard equation. Thus one can define

$$
\begin{aligned}
\alpha_{k} & :=\left\langle\hat{F}_{k, 0}, \hat{F}_{k+1,0}\right\rangle=\left\langle\hat{F}_{k, l}, \hat{F}_{k+1, l}\right\rangle \\
\beta_{l} & :=\left\langle\hat{F}_{0, l}, \hat{F}_{0, l+1}\right\rangle=\left\langle\hat{F}_{k, l}, \hat{F}_{k, l+1}\right\rangle
\end{aligned}
$$

All $\alpha_{k}$ and $\beta_{l}$ have the same sign, since we assumed $\hat{F}$ to lie in one component of the light-cone. Now

$$
\left|c r\left(F_{k, l}, F_{k+1, l}, F_{k+1, l+1}, F_{k, l+1}\right)\right|^{2}=\frac{\left\langle\hat{F}, \hat{F}_{1}\right\rangle}{\left\langle\hat{F}_{1}, \hat{F}_{12}\right\rangle} \frac{\left\langle\hat{F}_{12}, \hat{F}_{2}\right\rangle}{\left\langle\hat{F}_{2}, \hat{F}\right\rangle}=\frac{\alpha_{k}^{2}}{\beta_{l}^{2}}
$$

To see that the cross-ratio is negative we can compute

$$
\begin{aligned}
& \left(1-c r\left(F_{k, l}, F_{k+1, l}, F_{k+1, l+1}, F_{k, l+1}\right)\right)^{2}=\frac{\left\langle\hat{F}, \hat{F}_{12}\right\rangle}{\left\langle\hat{F}_{12}, \hat{F}_{2}\right\rangle} \frac{\left\langle\hat{F}_{2}, \hat{F}_{1}\right\rangle}{\left\langle\hat{F}_{1}, \hat{F}\right\rangle} \\
= & \frac{\frac{(\alpha+\beta)^{2}}{\left\langle\hat{F}_{1}, \hat{F}_{2}\right\rangle}\left\langle\hat{F}_{1}, \hat{F}_{2}\right\rangle}{\beta^{2}}=\left(1+\frac{\alpha}{\beta}\right)^{2}=\left(1+\left|\operatorname{cr}\left(F_{k, l}, F_{k+1, l}, F_{k+1, l+1}, F_{k, l+1}\right)\right|\right)^{2} .
\end{aligned}
$$

So the cross-ratio needs to be negative and $F$ is discrete isothermic.
If on the other hand $F$ is discrete isothermic map in $\mathbb{R}^{n}$ with crossratio $-\frac{\alpha_{k}}{\beta_{l}}$ we can scale its lift $\hat{F} \in \mathbb{R}^{n+2}$ point-wise in such a way that $\left\langle\hat{F}_{k, l}, \hat{F}_{k+1, l}\right\rangle=\alpha_{k}$ and $\left\langle\hat{F}_{k, l}, \hat{F}_{k, l+1}\right\rangle=\beta_{l}$. Now given $\hat{F}, \hat{F}_{1}, \hat{F}_{12}$, and $\hat{F}_{2}$ with $\left\langle\hat{F}, \hat{F}_{1}\right\rangle=\left\langle\hat{F}_{2}, \hat{F}_{12}\right\rangle=\alpha$ and $\left\langle\hat{F}, \hat{F}_{2}\right\rangle=\left\langle\hat{F}_{1}, \hat{F}_{12}\right\rangle=\beta$ we know that they are linear dependent (since the corresponding $F$ 's are concircular) and

$$
\mu \hat{F}_{12}=\hat{F}+\nu \hat{F}_{1}+\eta \hat{F}_{2}
$$

must hold for some $\mu, \nu$, and $\eta$.

$$
\begin{aligned}
0 & =\mu^{2}\left\langle\hat{F}_{12}, \hat{F}_{12}\right\rangle
\end{aligned}=2\left(\nu \alpha+\eta \beta+\nu \eta\left\langle\hat{F}_{1}, \hat{F}_{2}\right\rangle\right)
$$

So together one concludes

$$
\eta=-\mu \nu \text { and } \nu=-\mu \eta \Rightarrow \mu= \pm 1 \text { and } \nu=\mp \eta
$$

and $\hat{F}_{12}-\hat{F}=\lambda\left(\hat{F}_{1}-\hat{F}_{2}\right)$ or $\hat{F}_{12}+\hat{F}=\lambda\left(\hat{F}_{1}+\hat{F}_{2}\right)$. But we already saw that the second solution corresponds to a negative cross-ratio of $F$ (and thus the first would give a positive cross-ratio). Thus we finally conclude that the scaled $\hat{F}$ solves the Moutard equation.
Remark. The scaling factor that makes $\hat{F}$ a solution to the Moutard equation can be interpreted as a discrete version of the metric factor $e^{-u}$.

## 4.3 s-isothermic surfaces

We will now briefly discuss solutions to the discrete Moutard equation in the space-like unit sphere. This sphere is sometimes called deSitter space.

Definition 4.15 A map $\hat{F}: \mathbb{Z}^{2} \rightarrow \mathbb{R}^{n+2}$ with $\langle\hat{F}, \hat{F}\rangle=1$ is called a (discrete) s-isothermic map iff $\hat{F}$ solves the discrete Moutard equation.

Lemma 4.16 s-isothermic maps are Möbius invariant.
Proof. By definition.
s-isothermic maps are build from spheres (which are represented by points in deSitter space that have for each quadrilateral a common orthogonal circle or a common pair of points: The four spheres of a quadrilateral are collinear, so they span a 3 -space. The orthogonal compliment of that 3 -space is a 2 dimensional subspace, that - if space-like - describes a circle (the 1-dim set of space-like unit vectors in the 2-space give all the spheres that contain the circle), but if time-like contains exactly two light-like directions that give two points contained in all four spheres (remember: a point lies on a sphere iff their vectors in Minkowski $\mathbb{R}^{5}$ are perpendicular). In the limiting case of the 2 -space being light-like the touching light-like direction should be counted as a double point.

In case that the four spheres do touch, the orthogonal circle is inscribed in the quadrilateral formed by the centres of the four spheres. Remark.

- Four cyclically touching spheres in $\mathbb{R}^{3}$ always have a circle through the four touching points.
- Four cyclically touching circles in $\mathbb{R}^{3}$, that do not lie on a common sphere have a unique orthogonal sphere.

Definition 4.17 (and Lemma) Let $F$ be s-isothermic with centres $c$ and radii $r . F^{*}$ given by

$$
\begin{gathered}
c_{1}^{*}-c^{*}=\frac{c_{1}-c}{r_{1} r}, \quad c_{2}^{*}-c^{*}=-\frac{c_{2}-c}{r_{2} r} \\
r^{*}=\frac{1}{r}
\end{gathered}
$$

is s-isothermic again and called a dual surface of $F$.

Proof. we have to show, that $F^{*}$ is well defined and s-isothermic.
If $F$ is s-isothermic then its lift $\hat{F}$ solves a Moutard equation and we have

$$
\frac{1}{r}+\frac{1}{r_{12}}=\lambda\left(\frac{1}{r_{1}}+\frac{1}{r_{2}}\right), \quad \frac{c}{r}+\frac{c_{12}}{r_{12}}=\lambda\left(\frac{c_{1}}{r_{1}}+\frac{c_{2}}{r_{2}}\right)
$$

and $|c|^{2}+r^{2}+\left|c_{12}\right|^{2}+r_{12}^{2}=\lambda\left(\left|c_{1}\right|^{2}+r_{1}^{2}+\left|c_{2}\right|^{2}+r_{2}^{2}\right)$ which fixes $\lambda$. Now

$$
\begin{aligned}
& 0=\left(c_{1}^{*}-c^{*}\right)+\left(c_{12}^{*}-c_{1}^{*}\right)-\left(c_{12}^{*}-c_{2}^{*}\right)-\left(c_{2}^{*}-c^{*}\right) \\
\Leftrightarrow & 0=\frac{c_{1}-c}{r_{1} r}-\frac{c_{12}-c_{1}}{r_{12} r_{1}}+\frac{c_{12}-c_{2}}{r_{12} r_{2}}-\frac{c_{2}-c}{r_{2} r} \\
\Leftrightarrow & \left(\frac{1}{r}+\frac{1}{r_{12}}\right) \frac{c_{1}}{r_{1}}-\frac{1}{r_{1}}\left(\frac{c}{r}+\frac{c_{12}}{r_{12}}\right)=-\left(\frac{1}{r_{12}}+\frac{1}{r}\right) \frac{c_{2}}{r_{2}}+\frac{1}{r_{2}}\left(\frac{c_{12}}{r_{2}}+\frac{c}{r}\right) \\
\Leftrightarrow & \left(\left(\frac{1}{r}+\frac{1}{r_{12}}\right)-\lambda\left(\frac{1}{r_{1}}+\frac{1}{r_{2}}\right)\right) \frac{c_{1}}{r_{1}}=-\left(\left(\frac{1}{r_{12}}+\frac{1}{r}\right)-\lambda\left(\frac{1}{r_{1}}+\frac{1}{r_{2}}\right)\right) \frac{c_{2}}{r_{2}} \\
\Leftrightarrow & (0) \frac{c_{1}}{r_{1}}=-(0) \frac{c_{2}}{r_{2}}
\end{aligned}
$$

So $F^{*}$ is well defined. Since $F$ and $F^{*}$ are dual to each other the condition that the edges of $F$ sum to 0 is equivalent to $F^{*}$ being a solution to the moutard equation.

Example 4.3 Following the method in example 4.2 we can produce discrete $s$-isothermic minimal surfaces by means of a discrete Weierstraß method. The discrete holomorphic map is replaced here by a circle pattern that can be thought of as a s-isothermic surface in the plane. This kind of discrete holomorphic map can be generated by means of a variational principle from given combinatorics [BS04]. Fig. 15 shows a s-isothermic Enneper surface and Fig. 16 shows a s-isothermic Catenoid. A full treatment of s-isothermic minimal surfaces can be found in [BHS06].

## 4.4 curvatures

There are many possible approaches towards notions of curvature for discrete surfaces. The one we will follow here emerged in the last years and it turns out, that it covers astonishingly many cases of special surfaces that have already been discretized. It uses the Steiner formula and is covered in detail in [BPW].

### 4.4.1 the Steiner formula

Definition 4.18 Let $f$ be a smooth immersion with unit normal field $N$. A parallel surface $f^{t}$ is given by $f^{t}=f+t N$. For sufficiently small $t$, $f^{t}$ is a smooth immersion again and $N^{t}=N$.


Definition 4.19 Let $f$ be a smooth immersion with normal field $N$. The area $A(f)$ of $f$ is given by

$$
A(f)=\int \operatorname{det}\left(f_{x}, f_{y}, N\right) d x d y=\int \omega_{f}
$$

where $\omega_{f}$ is the "volume form" of $f$.
Lemma 4.20 Let $f$ be a smooth immersion with normal field $N$, mean curvature $H$ and Gauß curvature $K$. If $f^{t}$ is a smooth parallel surface for $f$ then

$$
A\left(f^{t}\right)=\int\left(1+2 t H+t^{2} K\right) \operatorname{det}\left(f_{x}, f_{y}, N\right) d x d y=A(f)+2 t H(f)+t^{2} K(f)
$$

where $H(f)$ and $K(f)$ are the integrals over mean and Gauß curvature of $f$.
Proof. Using the coefficients $a, b, c, d$ of the Weingarten operator ${ }^{20}$ as read from $N_{x}=a f_{x}+c f_{y}$ and $N_{y}=b f_{x}+d f_{y}$, we calculate the derivatives of $f^{t}$ to be:

$$
\begin{aligned}
\left(f^{t}\right)_{x} & =f_{x}+t N_{x}=(1+a t) f_{x}+c t f_{y} \\
\left(f^{t}\right)_{y} & =f_{y}+t N_{y}=b t f_{x}+(1+d t) f_{y}
\end{aligned}
$$

Now

$$
\begin{aligned}
A\left(f^{t}\right) & =\int \operatorname{det}\left(\left(f^{t}\right)_{x},\left(f^{t}\right)_{y}, N\right) d x d y \\
& =\int \operatorname{det}\left((1+a t) f_{x}+c t f_{y}, b t f_{x}+(1+d t) f_{y}, N\right) d x d y \\
& =\int(1+a t)(1+d t) \operatorname{det}\left(f_{x}, f_{y}, N\right)+c b t^{2} \operatorname{det}\left(f_{y}, f_{x}, N\right) d x d y \\
& =\int\left(1+2 t H+t^{2} K\right) \operatorname{det}\left(f_{x}, f_{y}, N\right) .
\end{aligned}
$$

[^14]Here is an alternative proof: Since $N^{t}=N$ we find that $K \omega=K^{t} \omega^{t}$ (note that the focal points are the same) so $\frac{1}{\left(k_{1}\right)^{t}}=\frac{1}{k_{1}}+t$ and $\frac{1}{\left(k_{2}\right)^{t}}=\frac{1}{k_{2}}+t$. Now

$$
\begin{aligned}
A\left(f^{t}\right) & =\int \omega^{t}=\int \frac{K}{K^{t}} \omega=\int k_{1} k_{2}\left(\frac{1}{k_{1}}+t\right)\left(\frac{1}{k_{2}}+t\right) \omega \\
& =\int\left(1+\left(k_{1}+k_{2}\right) t+\left(k_{1} k_{2}\right) t^{2}\right) \omega=\int\left(1+2 t H+t^{2} K\right) \omega
\end{aligned}
$$

Remark. one can use this to show for example

$$
\begin{aligned}
K^{t} & =\frac{K}{1+2 t H+t^{2} K} \\
H^{t} & =\frac{H-t K}{1+2 t H+t^{2} K}
\end{aligned}
$$

If the immersion $f$ is of constant positive Gauß curvature $K \equiv$ const and $t= \pm \frac{1}{\sqrt{K}}$ then

$$
H^{t}=\frac{H \mp \sqrt{K} K}{1+ \pm 2 \frac{H}{\sqrt{K}}+1}=-\frac{\sqrt{K}}{2} .
$$

So parallel surfaces in distance $t= \pm \frac{1}{\sqrt{K}}$ are of constant mean curvature.
Definition 4.21 A line congruence net is a map $F: \mathbb{Z}^{2} \rightarrow \mathbb{R}^{3}$ with planar faces together with a map l from $\mathbb{Z}^{2}$ into the set of straight lines in $\mathbb{R}^{3}$ such that $F_{i, j} \in l_{i, j}$ and $l_{i, j}$ intersects $l_{i+1, j}$ and $l_{i, j+1}$ for all $i, j \in \mathbb{Z}$.
Lemma 4.22 A line congruence net has a 1-parameter family of parallel line congruence nets $F_{t}$ that have parallel faces and share the lines $F_{i, j}^{t} \in l_{i, j}$.
Proof. Write $l_{0,0}=F_{0,0}+t N_{0,0}$. Fixing $t_{0}$ we set $F_{0,0}^{t_{0}}:=F_{0,0}+t_{0} N_{0,0}$. Now if $F_{i, j}^{t_{0}}$ is known, then there is a unique $F_{i+1, j}^{t_{0}} \in l_{i+1, j}$ such that $F_{i+1, j}^{t_{0}}-$ $F_{i, j}^{t_{0}} \| F_{i+1, j}-F_{i, j}$ and analogously for the other lattice direction. Since $F$ has planar faces the closing condition around quadrilaterals is automatically satisfied.
Remark. Note: If one thinks of the lines $l$ as normal lines the definition forces the net to be something like a discrete curvature line parameterization: $F_{1}^{t}-$ $F^{t} \| F_{1}-F$ and $F_{2}^{t}-F^{t} \| F_{2}-F$ implies, that the "Weingarten operator" is diagonal. However, to really have a discrete curvature line parameterization the normal directions should meet additional conditions.

To derive a discrete Steiner formula, we will need some results on planar quadrilaterals with parallel faces.

## Definition 4.23 (and lemma)

The oriented area of a triangle $\Delta=\left(p_{1}, p_{2}, p_{3}\right)$ is $A(\Delta)=\frac{1}{2}\left[p_{2}-p_{1}, p_{3}-p_{1}\right]$ with $\left[p_{i}, p_{j}\right]=\operatorname{det}\left(p_{i}, p_{j}, N\right)$ and $N$ the normal of the plane containing the triangle. The oriented area of an $n$-gon $g$ with vertices $\left(p_{0}, \ldots, p_{n-1}\right)$ is

$$
A(g)=\frac{1}{2} \sum_{i=0}^{n-1}\left[p_{i}, p_{i+1}\right]
$$

with the understanding that $p_{n}=p_{0}$.


Figure 15: A s-isothermic Enneper surface.


Figure 16: A s-isothermic Catenoid.

Proposition 4.24 For a given polygon the set of all polygons with parallel edges forms a vector space.

Proof. Given an $n$-gon $\left(p_{1}, \ldots, p_{n}\right)$ any other $n$-gon $\left(q_{1}, \ldots, q_{n}\right)$ with parallel edges can be described by $q_{1}$ and the scaling factors $\lambda_{i}$ from $q_{i+1}-q_{i}=$ $\lambda\left(p_{i+1}-p_{i}\right), i=i, \ldots, n-2$. Sums and scalar multiples for the polygons correspond to sums and scalar multiples for these coordinates. The following definition can be found in [PAHA07].
Definition 4.25 Let $P=\left(p_{1}, \ldots, p_{n}\right)$ and $Q=\left(q_{1}, \ldots, q_{n}\right)$ be polygons with parallel edges then

$$
A(P, Q):=\frac{1}{4} \sum_{i=1}^{n-1}\left(\left[p_{i}, q_{i+1}\right]+\left[q_{i}, p_{i+1}\right]\right)
$$

is called the mixed area of $P$ and $Q$.
Lemma 4.26 1. $A(P, Q)$ is a symmetric bilinear form on the vector space of polygons with parallel faces.
2. $A(P)$ is a quadratic form thereon
3.

$$
A(P+t Q)=A(P)+2 t A(P, Q)+t^{2} A(Q)
$$

Proof. Direct computation. $A(P, Q)=A(Q, P)$ by definition. Also by definition $A(P)=A(P, P)$. Finally:

$$
\begin{aligned}
A(P+t Q) & =\frac{1}{2} \sum\left[p_{i}+t q_{i}, p_{i+1}+t q_{i+1}\right] \\
& =\frac{1}{2} \sum\left(\left[p_{i}, p_{i+1}\right]+t\left[p_{i}, q_{i+1}\right]+t\left[q_{i}, p_{i+1}\right]+t^{2}\left[q_{i}, q_{i+1}\right]\right) \\
& =A(P)+2 t A(P, Q)+t^{2} A(Q)
\end{aligned}
$$

Now given a line congruence net $(F, l)$ we define a normal map (generically not of unit length) by choosing one normal $N_{i, j}$ such that $F_{i, j}+N_{i, j} \in l_{i, j}$ and determine the length of all other normals by the condition that $F+N$ should be a parallel mesh. But if $F+N$ is a parallel mesh, then $N$ has parallel quadrilaterals as well (see Fig. 17), so

$$
A(F+t N)=A(F)+2 t A(F, N)+t^{2} A(N)
$$



Figure 17: A line congruence net and its Gauß map.

Definition 4.27 Let $(F, l)$ be a line congruence net with normal map $N$. The mean curvature $H$ and the Gauß curvature $K$ are given by

$$
A(F+t N)=\left(1+2 t H+t^{2} K\right) A(F)
$$

for each quadrilateral.
Remark. $H$ and $K$ depend on the scaling of $N$. Still surfaces of constant curvatures are well defined.

Minimal surfaces: Since $H=\frac{A(F, N)}{A(F)} . F$ is minimal iff $A(F, N) \equiv 0$.
Definition 4.28 Two quadrilaterals $P$ and $Q$ with parallel edges are said to be dual to each other if

$$
A(P, Q)=0
$$

Proposition 4.29 Two quadrilaterals $P=\left(p_{1}, \ldots, p_{4}\right)$ and $Q=\left(q_{1}, \ldots, q_{4}\right)$ with parallel edges are dual to each other iff

$$
p_{1}-p_{3} \| q_{2}-q_{4}, \quad \text { and } \quad p_{2}-p_{4} \| q_{1}-q_{3} .
$$

Proof. Let $a, b, c, d$ and $a^{*}, b^{*}, c^{*}, d^{*}$ denote the edges of $P$ and $Q$ respectively. For $R=P+t Q$ we know

$$
\begin{aligned}
A(R) & =\frac{1}{2}\left[R_{2}-R_{1}, R_{3}-R_{2}\right]+\frac{1}{2}\left[R_{4}-R_{3}, R_{1}-R_{4}\right] \\
& =\frac{1}{2}\left(\left[a+t a^{*}, b+t b^{*}\right]+\left[c+t c^{*}, d+t d^{*}\right]\right)
\end{aligned}
$$

Now $A(P, Q)$ is $1 / 2$ the term linear in $t$ therein so

$$
\begin{aligned}
4 A(P, Q) & =\left[a^{*}, b\right]+\left[a, b^{*}\right]+\left[c^{*}, d\right]+\left[c, d^{*}\right] \\
& =\left[a^{*}, a+b\right]+\left[a+b, b^{*}\right]+\left[c^{*}, c+d\right]+\left[c+d, d^{*}\right] \\
& =\left[a+b, b^{*}-a^{*}+c^{*}-d^{*}\right]=2\left[a+b, b^{*}+c^{*}\right]
\end{aligned}
$$

So $A(P, Q)=0 \Leftrightarrow a+b \| b^{*}+c^{*}$. The same argument works the other diagonal as well.

Lemma 4.30 Each planar quadrilateral has a dual one. It is unique up to translation and scaling.

Proof. Let $m$ be the intersection of the diagonals of the quadrilateral $p_{1}, p_{2}, p_{3}, p_{4}$ and let $v=\frac{p_{1}-m}{\left\|p_{1}-m\right\|}$ and $w=\frac{p_{2}-m}{\left\|p_{2}-m\right\|}$. We can assume that $m=0$. Then

$$
\begin{aligned}
& p_{1}=\alpha v, \quad p_{3}=\gamma v \\
& p_{2}=\beta w, \quad p_{4}=\delta w
\end{aligned}
$$

Now we define

$$
\begin{aligned}
& q_{1}=-\frac{1}{\alpha} w, \quad q_{3}=-\frac{1}{\gamma} w \\
& q_{2}=-\frac{1}{\beta} v, \quad q_{4}=-\frac{1}{\delta} v
\end{aligned}
$$

and find

$$
\begin{aligned}
q_{2}-q_{1} & =-\frac{1}{\beta} v+\frac{1}{\alpha} w=\frac{1}{\alpha \beta}(\beta w-\alpha v)=\frac{1}{\alpha \beta}\left(p_{2}-p_{1}\right) \\
& \vdots
\end{aligned}
$$

So $p$ and $q$ have parallel edges.
To show uniqueness we assume that $\tilde{q}$ is an other dual quadrilateral. By translation and scaling we can achieve, that the intersections of the diagonals of $q$ and $\tilde{q}$ coincide and that $\tilde{q}_{1}=q_{1}$. Since $\tilde{q}_{2}-\tilde{q}_{1} \| q_{2}-q_{1}$ and $\tilde{q}_{2}$ and $q_{2}$ lie
on the same diagonal they must coincide as well. Iteratively it follows that $\tilde{q}_{3}=q_{3}$ and $\tilde{q}_{4}=q_{4}$.
Remark. Note that although every planar quadrilateral has a dual one this is in general not true for entire meshes. Meshes that are dualizable are called Koenigs nets [BS].

Lemma 4.31 The dual of a quadrilateral with cross-ratio $\operatorname{cr}(\square)=-\frac{\alpha}{\beta}$ is dual in the above sense.

Proof. exercise.
It follows that the discrete (isothermic) minimal surfaces are minimal in the $H \cong 0$ sense.

Theorem 4.32 the line congruence net $F$ with normal $N$ has constant mean curvature $H \neq 0$ iff there exists a dual surface in constant distance

$$
F^{*}=F+d N
$$

$F^{*}$ with normal $N$ has mean curvature $-H$ and the surface $F+\frac{d}{2} N$ has constant Gauß curvature $K=4 H^{2}$.

Proof. We find $A(F, N)=H A(F) \leftrightarrow \frac{1}{H} A(F, N)=A(F, F) \leftrightarrow A(F, F-$ $\left.\frac{1}{H} N\right)=0$. So $d=-\frac{1}{H}$ and $F+d N$ needs to be dual. Since $F=F^{*}-d N$ it follows that $F^{*}$ has mean curvature $-H$. Finally

$$
K\left(F+\frac{d}{2} N\right)=\frac{A(N)}{A\left(F+\frac{d}{2} N\right)}=\frac{A(N)}{A(F)+d A(F, N)+\frac{d^{2}}{4} A(N)}=4 H^{2}
$$

This notion of discrete cmc surfaces coincides with the one introduced in an algebraic way by Bobenko and Pinkall [BP99].

With this we end our introduction in discrete differential geometry. The interested reader may find more on most of the subjects in the cited literature and especially in the books mentioned in the introduction.

## References

[BHS06] A. Bobenko, T. Hoffmann, and B. Springborn. Minimal surfaces from circle patterns: Geometry from combinatorics. Ann. of Math., 164(1):231-264, 2006.
[Bos32] R. C. Bose. On the number of circles of curvature perfectly enclosing or perfectly enclosed by a closed oval. Math. Ann., 35:16-24, 1932.
[BP96] A. Bobenko and U. Pinkall. Discrete isothermic surfaces. J. reine angew. Math., 475:178-208, 1996.
[BP99] A. Bobenko and U. Pinkall. Discretization of surfaces and integrable systems. In Bobenko A. and Seiler R., editors, Discrete Integrable Geometry and Physics. Oxford University Press, 1999.
[BPW] A. I. Bobenko, H. Pottmann, and J. Wallner. A curvature theory for discrete surfaces based on mesh parallelity. arXiv:0901.4620 [math.DG].
[BS] A. I. Bobenko and Y. B. Suris. Discrete Koenigs nets and discrete isothermic surfaces. arXiv:0709.3408 [math.DG].
[BS04] A. I. Bobenko and B. A. Springborn. Variational principles for circle patterns and Koebe's theorem. Trans. Amer. Math. Soc., 356, 2004.
[BS08] A. I. Bobenko and Y. B. Suris. Discrete Differential Geometry: Integrable Structure, volume 98 of Graduate Studies in Mathematics. AMS, 2008.
[EHH ${ }^{+} 92$ ] H. D. Ebbinghaus, H. Hermes, F. Hirzebruch, M. Koecher, K. Mainzer, J. Neukirch, A. Prestel, and R Remmert. Zahlen. Springer, 1992.
[EJ07] J.-H. Eschenburg and J. Jost. Differentialgeometrie und Minimalflächen. Springer, 2007.
[FT86] L. D. Faddeev and L. A. Takhtajan. Hamiltonian methods in the theory of solitons. Springer, 1986.
[HJ03] U. Hertrich-Jeromin. Introduction to Möbius Differential Geometry. Cambridge University Press, 2003.
[HJHP99] U. Hertrich-Jeromin, T. Hoffmann, and U. Pinkall. A discrete version of the Darboux transformation for isothermic surfaces. In A. Bobenko and R. Seiler, editors, Discrete Integrable Geometry and Physics, pages 59-81. Oxford University Press, 1999.
[HK04] T. Hoffmann and N. Kutz. Discrete curves in $C P^{1}$ and the Toda lattice. Stud. Appl. Math., 113(1):31-55, 2004.
[Muk09] S Mukhopadhyaya. New methods in the geometry of a plane arc. Bull. Calcutta Math. Soc., 1:21-27, 1909.
[Mus04] O. R. Musin. Curvature extrema and four-vertex theorems for polygons and polyhedra. Journal of Mathematical Sciences, 119:268-277(10), January 2004.
[NS98] J. J. C. Nimmo and W. K. Schief. An integrable discretization of a $2+1$-dimensional sine-gordon equation. Stud. Appl. Math., 100:295-309, 1998.
[PAHA07] H. Pottmann, A. Asperl, M. Hofer, and Kilian A. Architectural Geometry. Bentley Institute Press, 2007.
[Sau70] R. Sauer. Differenzengeometrie. Springer, 1970.
[Tab00] Serge Tabachnikov. A four vertex theorem for polygons. The American Mathematical Monthly, 107(9):830-833, 2000.


[^0]:    ${ }^{1} \mathrm{~A}$ contradiction in the thing itself.

[^1]:    ${ }^{2}$ the intersection of an interval in $\mathbb{R}$ with $\mathbb{Z}$

[^2]:    ${ }^{3}$ A smooth curve is called regular at a given point, if its derivative at that point does not vanish.

[^3]:    ${ }^{5}$ The osculating circle in a given point of a smooth regular curve is the best approximating circle in that point.
    ${ }^{6}$ We will consider a straight line to be a (degenerate) circle as well.

[^4]:    ${ }^{7}$ In the smooth setup one can define the centre of curvature as the intersection point of infinitesimal neighbouring (arbitrarily close) normal lines.
    ${ }^{8}$ Reversing the argument we could introduce a second edge osculating circle by taking the circle that touches the (possibly extended) edge and has its centre at the intersection of the two adjacent vertex normal lines.

[^5]:    ${ }^{9}$ Note that by assumption full circles can only touch in 1,2 , or three points.

[^6]:    ${ }^{10}$ hyperspheres in general but we are concerned with plane geometry for now. We will however, think of straight lines (hyperplanes) as circles (spheres) as well

[^7]:    ${ }^{11}$ this is a discrete version of Keppler's 2nd law: the line joining a planet and its sun sweeps out equal areas in equal intervals of time - or for the orbit $p(t): \operatorname{det}\left(p(t), p^{\prime}(t)\right)=$ const. Discretizing the derivative as a difference gives our normalization.

[^8]:    ${ }^{12}$ The notion of arc-length parameterization is not a Möbius invariant one but if it happens that the curve is arc-length parameterized with one choice of $\infty$, then it will stay so under this flow and choice.

[^9]:    ${ }^{13}$ The two fix-points may coincide: A translation only fixes $\infty$. If a Möbius transformation has three fix-points it is already the identity.

[^10]:    ${ }^{14}$ The set of closed Darboux transforms is in fact a Riemann surface. The map form it into the complex $\mu$-plane is a branched double cover.

[^11]:    ${ }^{15}$ One can actually show that the smooth complex curvature is given by $\Psi=\kappa e^{i \int \tau}$ in terms of the curvature $\kappa$ and the torsion $\tau$.

[^12]:    ${ }^{16}$ An immersion is a smooth map whose differential is injective in every point.
    ${ }^{17}$ It is slightly confusing, that classically conformal coordinates are called isothermic, while a surface is called isothermic if it allows for isothermal coordinates.

[^13]:    ${ }^{18}$ The Enneper surface was introduced 1863 in an explicit parameterization by Alfred Enneper.
    ${ }^{19}$ The catenoid was found 1744 by Leonard Euler. It was the first minimal surface that was discovered.

[^14]:    ${ }^{20}$ The Weingarten operator is the linear operator that represents the second fundamental form of a surface with respect to the first.

