Math－for－industry
Education \＆Research Hub

## Workshop on ＂Prohabilistic models with determinantal structure＂

Editor：Tomoyuki Shirai

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九州大学マス•フォア•インダストリ研究所
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# Workshop on <br> "Probabilistic models with determinantal structure" 

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## About MI Lecture Note Series

The Math-for-Industry (MI) Lecture Note Series is the successor to the COE Lecture Notes, which were published for the 21st COE Program "Development of Dynamic Mathematics with High Functionality," sponsored by Japan’s Ministry of Education, Culture, Sports, Science and Technology (MEXT) from 2003 to 2007. The MI Lecture Note Series has published the notes of lectures organized under the following two programs: "Training Program for Ph.D. and New Master’s Degree in Mathematics as Required by Industry," adopted as a Support Program for Improving Graduate School Education by MEXT from 2007 to 2009; and "Education-and-Research Hub for Mathematics-for-Industry," adopted as a Global COE Program by MEXT from 2008 to 2012.

In accordance with the establishment of the Institute of Mathematics for Industry (IMI) in April 2011 and the authorization of IMI's Joint Research Center for Advanced and Fundamental Mathematics-for-Industry as a MEXT Joint Usage / Research Center in April 2013, hereafter the MI Lecture Notes Series will publish lecture notes and proceedings by worldwide researchers of MI to contribute to the development of MI.

October 2014
Yasuhide Fukumoto
Director
Institute of Mathematics for Industry

# Workshop on <br> "Probabilistic models with determinantal structure" 

MI Lecture Note Vol.63, Institute of Mathematics for Industry, Kyushu University ISSN 2188-1200

Editor: Tomoyuki Shirai
Date of issue: 20 August 2015
Publisher: Institute of Mathematics for Industry, Kyushu University Graduate School of Mathematics, Kyushu University Motooka 744, Nishi-ku, Fukuoka, 819-0395, JAPAN Tel +81-(0)92-802-4402, Fax +81-(0)92-802-4405
URL http://www.imi.kyushu-u.ac.jp/

Printed by
Kijima Printing, Inc.
Shirogane 2-9-6, Chuo-ku, Fukuoka, 810-0012, Japan
TEL +81-(0)92-531-7102 FAX +81-(0)92-524-4411

## Preface

The present volume of Math-for-Industry Lecture Note Series collects the manuscripts and slides of invited talks at the workshop on "Probabilistic models with determinantal structure" held at Institute of Mathematics for Industry (IMI), Ito-Campus, Kyushu University, Fukuoka, Japan, April 30th and May 1st, 2015. The workshop is held during the visit to IMI of Professor Evgeny Verbitskiy (Leiden and Groningen) and Professor Subhroshekhar Ghosh (Princeton).

The purpose of this workshop is to overview recent developments around several probabilistic models with determinantal structure such as abelian sandpile models and determinantal point processes from various points of view. Topics are ranging from but not limited to algebraic dynamical systems, random walk on random spanning trees, persistent homology and random topology, quantum Rabi model and representation theory, abelian sandpile models, forest-fire models, rigidity in point processes, and diffusions associated with Gaussian analytic functions.

The 42 participants had many fruitful discussions and exchanges that contributed to the success of the workshop. We are very much grateful to all the participants, especially the invited speakers for their contribution to preparing manuscripts and giving talks. We are also grateful to Ms. Tsubura Imabayashi for her help. Without her generous effort, the workshop would not have been so smoothly organized.

We also hope all the participants enjoyed this workshop and had a pleasant stay in Fukuoka.

This workshop is financially supported by Progress 100 (World Premier International Researcher Invitation Program), Kyushu University, and Grant-in-Aid for Scientific Research Kiban(B) 26287019 (PI: Tomoyuki Shirai) and Challenging Exploratory Research 26610026 (PI: Hiroyuki Ochiai).

July 2015

Organizer: Tomoyuki Shirai (IMI, Kyushu University)

# Workshop on uProbabilistic models with determinantal structure" April 30th and May 1st, 2015 

## Venue: <br> Seminar Room 1, Faculty of Mathematics Institute of Mathematics for Industry, Ito campus, Kyushu University

## Speakers:

| Evgeny Verbitskiy
Abysian Sandpiles and Algebraic Dynamics (I)\&(II)

## Subhroshekhar Ghosh

Priceton University
Rigidity Phenomena in Point Processes:
Perspectives and Recent Progress (I)\&(II)
Yasuaki Hiraoka
Tohoku University
Random Topology, Minimum Spanning Acycle, and Persistent Homology

## Makoto Katori

Abelian Sandpile Models in Statistical Mechanics

## Takashi Kumagai

Subsequential Scaling Limits of Simple Random
Walk on the Two-Dimensional Uniform Spanning Tree

## Tetsuya Mitsudo

Kyoto University Large Deviations for Frequencies in Forest-Fire Models, Diffusive Elements and Real Earthquake Data

Hirofumi Osada
Kyushu University
Diffusions Associated with Gaussian Analytic Functions
Masato Wakayama
Kyushu University
Spectrum of the Quantum Rabi Model and
Representation Theory

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# Workshop on <br> "Probabilistic models with determinantal structure" 

April 30th and May 1st, 2015 at IMI, Kyushu University

## Program

## April 30th (Thur.)

13:00-13:45 Evgeny Verbitskiy (Kyushu, Leiden)
Abelian Sandpiles and Algebraic Dynamics (I)
13:55-14:40 Evgeny Verbitskiy (Kyushu, Leiden)
Abelian Sandpiles and Algebraic Dynamics (II)
14:50-15:40 Takashi Kumagai (Kyoto)
Subsequential scaling limits of simple random walk on the two-dimensional uniform spanning tree

15:50-16:40 Yasuaki Hiraoka (Tohoku)
Random Topology, Minimum Spanning Acycle, and Persistent Homology
16:50-17:40 Masato Wakayama (Kyushu)
Spectrum of the quantum Rabi model and representation theory

## May 1st (Fri.)

10:00-10:50 Makoto Katori (Chuo)
Abelian sandpile models in statistical mechanics
11:00-11:50 Tetsuya Mitsudo (Kyoto)
Large deviations for frequencies in forest-fire models, diffusive elements and real earthquake data

## 13:00-13:45 Subhroshekhar Ghosh (Princeton)

Rigidity phenomena in point processes: perspectives and recent progress (I)
13:55-14:40 Subhroshekhar Ghosh (Princeton)
Rigidity phenomena in point processes: perspectives and recent progress (II)
15:00-15:50 Hirofumi Osada (Kyushu)
Diffusions associated with Gaussian analytic functions

This workshop is supported by Progress 100 (World Premier International Researcher Invitation Program), Kyushu University.
Grant-in-Aid for Scientific Reserach:
Kiban(B) 26287019 (PI: Tomoyuki Shirai)
Challenging Exploratory Research 26610026 (PI: Hiroyuki Ochiai)

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# Spanning Trees, Abelian Sandpiles, and Algebraic Dynamical Systems 

Evgeny Verbitskiy<br>Institute of Mathematics for Industry, Kyushu University, Japan<br>Mathematical Institute, Leiden University, The Netherlands


#### Abstract

This overview talk is based on joint works with K. Schmidt and T. Shirai.


## 1 Szegö's Theorem and Mahler Measure

This year marks the anniversary of the seminal result by G. Szegö - First Limit Theorem for Toeplitz determinants, published in Math.Ann. in 1915.

### 1.1 Toeplitz determinants

Let $\mathbb{T}=[0,1)$ and $\mathbb{S}=\{z \in \mathbb{C}:|z|=1\}=\left\{e^{2 \pi i \theta}: \theta \in \mathbb{T}\right\}$. Suppose $\phi: \mathbb{S} \rightarrow \mathbb{R}$, and let

$$
\hat{\phi}_{n}=\int_{\mathbb{T}} e^{-2 \pi i n \theta} \phi\left(e^{2 \pi i \theta}\right) d \theta, \quad n \in \mathbb{Z}
$$

be the $n$-th Fourier coefficient of $\phi$. Consider the following Toeplitz matrix

$$
T_{N}=\left(\hat{\phi}_{n-m}\right)_{n, m=1}^{N}
$$

Theorem 1.1 (Szegö's First Limit Theorem). If $\phi: \mathbb{S} \rightarrow \mathbb{R}$ is positive and $\log \phi$ is integrable, then

$$
\lim _{N \rightarrow \infty} \frac{1}{N} \log \operatorname{det} T_{N}=\int_{\mathbb{T}} \log \phi\left(e^{2 \pi i \theta}\right) d \theta
$$

Szegö's First Limit Theorem is a truly fundamental result, with applications in analysis, probability, combinatorics, and even signal processing.

Theorem 1.1 admits multidimensional generalizations:

Theorem 1.2. Suppose $\phi: \mathbb{S}^{d}=\left\{\mathbf{z}=\left(z_{1}, \ldots, z_{d}\right) \in \mathbb{C}^{d}:\left|z_{j}\right|=1\right.$ for all $\left.j\right\} \rightarrow \mathbb{R}$ is positive and such that $\log \phi$ is integrable. Let

$$
\hat{\phi}_{\boldsymbol{n}}=\int_{\mathbb{T}^{d}} e^{-2 \pi i\langle\boldsymbol{n}, \boldsymbol{\theta}\rangle} \phi\left(e^{2 \pi i \boldsymbol{\theta}}\right) d \boldsymbol{\theta}, \quad \boldsymbol{n} \in \mathbb{Z}^{d}
$$

Then determinant of the corresponding block Toeplitz matrix

$$
T_{N}=\left(\hat{\phi}_{\boldsymbol{n}-\boldsymbol{m}}\right)_{\boldsymbol{n}, \boldsymbol{m} \in \Lambda_{N}}, \quad \Lambda_{N}=\{1, \ldots, N\}^{d}
$$

satisfies

$$
\lim _{N \rightarrow \infty} \frac{1}{N^{d}} \log \operatorname{det} T_{N}=\int_{\mathbb{T}^{d}} \log \phi\left(e^{2 \pi i \boldsymbol{\theta}}\right) d \boldsymbol{\theta}
$$

### 1.2 Mahler measure

Laurent polynomial in $d$ variables $u_{1}, \ldots, u_{d}$ with integer coefficients is

$$
f=\sum_{\boldsymbol{n} \in \mathbb{Z}^{d}} f_{\boldsymbol{n}} u^{\boldsymbol{n}}=\sum_{\boldsymbol{n} \in \mathbb{Z}^{d}} f_{\boldsymbol{n}} u_{1}^{n_{1}} \ldots u_{d}^{n_{d}}
$$

where $f_{n} \in \mathbb{Z}$ for every $\boldsymbol{n} \in \mathbb{Z}^{d}$ and there are only finitely $\boldsymbol{n}$ with $f_{\boldsymbol{n}} \neq 0$. Two Laurent polynomial $f$ and $g$ can be added and multiplied in the usual fashion

$$
f+g=\sum_{n \in \mathbb{Z}^{d}}(f+g)_{n} u^{n}, \quad f \cdot g=\sum_{n \in \mathbb{Z}^{d}}(f \cdot g)_{n} u^{n},
$$

with

$$
(f+g)_{n}=f_{n}+g_{n}, \quad(f \cdot g)_{n}=\sum_{k \in \mathbb{Z}^{d}} f_{n-k} g_{k}
$$

for all $\boldsymbol{n} \in \mathbb{Z}^{d}$. The ring of Laurent polynomials in $d$-variables will be denoted by $R_{d}$. Equivalently, we may say that $R_{d}$ is group ring of $\mathbb{Z}^{d}$, denoted by $\mathbb{Z}\left[\mathbb{Z}^{d}\right]$.

It is an easy application of Szegö's First Limit Theorem to show the following result.
Theorem 1.3. Suppose $f \in R_{d}$ and $f \neq \mathbf{0}$, then

$$
\lim _{N \rightarrow \infty} \frac{1}{N^{d}} \log \left|\operatorname{det} T_{N}(f)\right|=\int_{\mathbb{T}^{d}} \log \left|f\left(e^{2 \pi i \boldsymbol{\theta}}\right)\right| d \boldsymbol{\theta}
$$

where now

$$
T_{N}(f)=\left(f_{\boldsymbol{n}-\boldsymbol{m}}\right)_{\boldsymbol{n}, \boldsymbol{m} \in \Lambda_{N}}, \quad \Lambda_{N}=\{1, \ldots, N\}^{d}
$$

Definition 1.4. The logarithmic Mahler measure of $f \in R_{d}$ is defined as

$$
m_{f}= \begin{cases}\int_{\mathbb{T}^{d}} \log \left|f\left(e^{2 \pi i \boldsymbol{\theta}}\right)\right| d \boldsymbol{\theta}, & f \neq \mathbf{0} \\ +\infty, & f=\mathbf{0}\end{cases}
$$

Example 1.5. Let $f(z)=4-z-\frac{1}{z}$, then

$$
T_{N}=\left(\begin{array}{ccccc}
4 & -1 & & & \\
-1 & 4 & -1 & & \\
& \ddots & \ddots & \ddots & \\
& & \ddots & \ddots & -1 \\
& & & -1 & 4
\end{array}\right)
$$

and $a_{N}=\operatorname{det} T_{N}$ satisfies $a_{N}=4 a_{N-1}-a_{N-2}$, with $a_{0}=1, a_{1}=4$ :

| $N$ | 1 | 2 | 3 | 4 | 5 |
| :--- | :---: | :---: | :---: | :---: | :---: |
| $a_{N}$ | 4 | 15 | 56 | 209 | 780 |

The roots of characteristic polynomial of $f$ are $2 \pm \sqrt{3}$, and using standard techniques one easily gets that

$$
a_{N}=\frac{(2+\sqrt{3})^{N+1}-(2-\sqrt{3})^{N+1}}{2 \sqrt{3}}
$$

and hence

$$
\frac{1}{N} \log a_{N} \rightarrow \log (2+\sqrt{3})=\int_{0}^{1} \log (4-2 \cos (2 \pi \theta)) d \theta
$$

In fact, the logarithmic Mahler measure of $f \in R_{1}$ is easily computable: if

$$
f=a_{m} z^{m}+\ldots+a_{1} z+a_{0}, \quad a_{m} \neq 0
$$

factorizes over $\mathbb{C}$ as

$$
f=a_{m}\left(z-z_{1}\right) \ldots\left(z-z_{m}\right), \quad z_{j} \in \mathbb{C}
$$

then the so-called Jensen's formula gives that

$$
m_{f}=\log \left|a_{m}\right|+\sum_{j:\left|z_{j}\right|>1} \log \left|z_{j}\right| .
$$

Values of Mahler measures of multivariate polynomials are much more interesting and are subject of active studies. For example,

$$
m_{1+u_{1}+u_{2}}=\frac{3 \sqrt{3}}{4 \pi} L\left(\chi_{-3}, 2\right), \quad \text { where } L\left(\chi_{-3}, s\right)=\sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^{s}}
$$

is the Dirichlet $L$-series of the character

$$
\chi_{-3}(n)= \begin{cases}1, & n \equiv 1 \quad \bmod 3 \\ -1, & n \equiv-1 \quad \bmod 3 \\ 0, & n \equiv 0 \quad \bmod 3\end{cases}
$$

Few other interesting values:

$$
m\left(1+u_{1}+u_{2}+u_{3}\right)=\frac{7}{2 \pi^{2}} \zeta(3) .
$$

One particular family of polynomials (which related to dissipative sandpiles) has been studied rather extensively:

$$
f_{k}=k-\left(u_{1}+u_{1}^{-1}+u_{2}+u_{2}^{-1}\right), \quad k \in \mathbb{Z},
$$

Boyd [2] verified numerically (to a very high degree of accuracy) that for $1 \leq k \leq 100$, $k \neq 4$, one has

$$
\begin{equation*}
m_{f_{k}}=r_{k} L^{\prime}\left(E_{k}, 0\right), \tag{1.6}
\end{equation*}
$$

where $r_{k} \in \mathbb{Q}, E_{k}$ is the elliptic curve corresponding to the null set $\left\{f_{k}=0\right\}$, and $L$ is the corresponding $L$-function. Deninger [3] also related the logarithmic Mahler measure $m_{f_{k}}$ to Eisenstein-Kronecker series. His result, together with the Bloch-Beilinson conjectures, implies (1.6). Rodriguez-Villegas [19] developed alternative approaches to the evaluation of $m_{f_{k}}$. For $k=4$, i.e.,

$$
f=4-u_{1}-\frac{1}{u_{1}}-u_{2}-\frac{1}{u_{2}},
$$

the Mahler measure can be computed exactly:

$$
m_{f}=\frac{4}{\pi} G, \quad \text { where } \quad G=\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2 n+1)^{2}}=0.915965 \ldots
$$

is the so-called Catalan constant, which is again a value of number theoretic function, the Dirichlet $\beta$-function

$$
\beta(s)=\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2 n+1)^{s}} .
$$

## 2 Uniform Spanning Trees

### 2.1 Enumeration of Spanning Trees of Finite Graphs

Suppose $G=(V, E)$ is a finite connected undirected graph, possibly with multiple edges, but without loops.

Definition 2.1. A spanning tree $T$ of $G$ is a subgraph that includes all of the vertices of $G$ that is a tree, i.e., has no cycles.

Enumeration of spanning tress is a classical problem in combinatorics. The number of spanning trees is given by the Kirkhoff Matrix Tree Theorem.

Theorem 2.2. Suppose $G=(V, E)$ is a finite connected undirected multigraph without loops. Denote by $A_{v_{1}, v_{2}}$ the number of edges connecting $v_{1}$ and $v_{2}$ in $G$. Note that

$$
\operatorname{deg}\left(v_{1}\right)=\sum_{v_{2} \in V} A_{v_{1}, v_{2}}
$$

The graph Laplacian of $G$ is a matrix of size $|V| \times|V|$ given by

$$
\Delta_{G}\left(v_{1}, v_{2}\right)= \begin{cases}\operatorname{deg}\left(v_{1}\right), & \text { if } v_{1}=v_{2} \\ -A_{v_{1}, v_{2}}, & \text { if } v_{1} \neq v_{2}\end{cases}
$$

Then $\mathrm{t}(G)$ - the number of distinct spanning trees $\mathrm{t}(G)$ of $G$, is given by

$$
\mathrm{t}(G)=\frac{1}{|V|} \prod_{i=1}^{|V|-1} \lambda_{i}
$$

where $\lambda_{1}, \ldots, \lambda_{|G|-1}$ are the non-zero eigenvalues of $\Delta_{G}$. Equivalently, the number of distinct spanning trees of $G$ is equal to the absolute value of any principal minor of $\Delta_{G}$, i.e., determinant of a matrix obtained by deleting the column and the row corresponding to some vertex $v \in V$.

Often the graph Laplacian is represented as

$$
\Delta_{G}=D_{G}-A_{G}
$$

where $D_{G}$ is the diagonal degree matrix, and $A_{G}$ is the adjacency matrix of $G$. Note that determinant of $\Delta_{G}$ is 0 since all row sums are all equal to 0 , and hence $\Delta_{G} \mathbf{1}=\mathbf{0}$.

Let us now define the incidence matrix $M$ for a graph $G=(V, E)$ with $n=|V|$ vertices and $k=|E|$ edges as an $n \times k$ indicating which edges are incident to which vertices. More specifically, let us assume that vertices are indexed by $i, i=1, \ldots, n$, and edges are indexed by $j=1, \ldots, k$. Then $M=\left(m_{i, j}\right)_{i=1, j=1}^{n, k}$ is defined as follows: if $j$-th edge $e_{j}$ connects vertices $v_{i_{1}}$ and $v_{i_{2}}, i_{1}<i_{2}$, then

$$
m_{i, j}= \begin{cases}-1, & \text { if } i=i_{1} \\ 1, & \text { if } i=i_{2} \\ 0, & \text { otherwise }\end{cases}
$$

in other words, edges are directed from lower numbered vertices to higher numbered vertices.

Proposition 2.3. The following equality holds $\Delta_{G}=M M^{T}$.

Similar equality holds for the reduced Laplacians. For example, if $\widetilde{\Delta}_{G}$ is obtained by removing first column and the first row of $\Delta_{G}$, then

$$
\widetilde{\Delta}_{G}=\widetilde{M} \widetilde{M}^{T}
$$

where $\widetilde{M}$ is obtained by removing first row from $M$.
By the Cauchy-Binet formula

$$
\operatorname{det}\left(\widetilde{M} \widetilde{M}^{T}\right)=\sum_{I \in\binom{[k]}{n-1}} \operatorname{det}\left(\widetilde{M}_{I}\right) \operatorname{det}\left(\widetilde{M}_{I}^{T}\right)=\sum_{I \in\binom{[k]}{n-1}} \operatorname{det}\left(\widetilde{M}_{I}\right)^{2},
$$

where sum is taken over all $(n-1)$-subsets of $[k]=\{1, \ldots, k\}$. It turns out that for every collection of edges $I \subset\{1, \ldots, k\}$ of cardinality $n-1$, one has

$$
\operatorname{det}\left(\widetilde{M}_{I}\right)^{2}=\left\{\begin{array}{lc}
1, & \text { if } I \text { gives a spanning tree } \\
0, & \text { otherwise }
\end{array}\right.
$$

If $G=(V, E)$ is a multigraph and $e \in E^{\prime}$ is an arbitrary edge, then the number $\mathrm{t}(G)$ of spanning trees satisfies the deletion-contraction recurrence

$$
\mathrm{t}(G)=\mathrm{t}(G-e)+\mathrm{t}(G / e)
$$

where $G-e$ is the multigraph obtained by deleting $e$, and $G / e$ is the edge contraction of $G$ by $e$. Here, $\mathrm{t}(G-e)$ counts the spanning trees of $G$ that do not use edge $e$, and the term $\mathrm{t}(G / e)$ counts the spanning trees of $G$ that use $e$.

### 2.2 Uniform Random Spanning Trees on Finite Graphs

Let $\boldsymbol{T}$ be a uniformly distributed random spanning tree of $G$, i.e., $\boldsymbol{T}$ assumes every possible (among $\mathrm{t}(G)$ possibilities) value with equal probability.

The definition of a uniform random spanning tree does not allow to compute local characteristics of the random tree. For example, the probabilities

$$
\mathbb{P}\left(e_{1} \in \boldsymbol{T}\right)=\frac{\mathrm{t}\left(G /\left\{e_{1}\right\}\right)}{\mathrm{t}(G)}, \quad \mathbb{P}\left(e_{1}, e_{2} \in \boldsymbol{T}\right)=\frac{\mathrm{t}\left(G /\left\{e_{1}, e_{2}\right\}\right)}{\mathrm{t}(G)},
$$

of the events like that "the edge $e_{1}$ belongs to $\boldsymbol{T}$ ", or "the pair of edges $e_{1}, e_{2}$ belong to $T^{\prime \prime}$, at first glance, cannot be computed without the need to enumerate all spanning trees.

The following beautiful result is due to Burton and Pemantle [1].

Theorem 2.4 (Transfer-Impedance Theorem). Let $G=(V, E)$ be any finite connected graph. There is a symmetric function $H(e, f)$ on pairs of edges such that for any $e_{1}, \ldots, e_{r} \in$ E,

$$
\mathbb{P}\left(e_{1}, \ldots, e_{r} \in \boldsymbol{T}\right)=\operatorname{det}\left(H\left(e_{i}, e_{j}\right)\right)_{i, j=1}^{r}
$$

where $H(e, f)$ with $e=(x, y)$ is the expected signed number of transits of $f$ by a random walk started at $x$ and stopped when it hits $y$.

Remark 2.5. Recall that $X=\left\{X_{1}, \ldots, X_{m}\right\} \in\{0,1\}^{m}$ or $X=\left\{X_{\boldsymbol{n}} \mid \boldsymbol{n} \in \mathbb{Z}^{d}\right\}$ with $X_{n} \in\{0,1\}$ is called a determinantal process if for all $\boldsymbol{n}_{1}, \ldots, \boldsymbol{n}_{k}$

$$
\mathbb{P}\left[X_{\boldsymbol{n}_{1}}=\ldots=X_{\boldsymbol{n}_{k}}=1\right]=\operatorname{det}\left(H\left(\boldsymbol{n}_{1}, \boldsymbol{n}_{2}\right)\right) .
$$

### 2.3 Uniform Spanning Forests

We will consider infinite graphs like $\mathbb{Z}^{d}, d \geq 1$, or the ladder graphs $\mathbb{Z} \times G, G$ is a finite graph with

$$
\left(n_{1}, v_{1}\right) \sim\left(n_{2}, v_{2}\right) \text { if and only if } n_{1}=n_{2} \text { and } v_{1} \sim_{G} v_{2}, \text { or }, n_{1}-n_{2}= \pm 1 \text { and } v_{1}=v_{2}
$$

For these graphs, the uniform spanning forest (USF) is the weak limit of uniform spanning trees in larger and larger finite boxes. Pemantle [17] showed for $\mathbb{Z}^{d}$ that the limit exists, that it does not depend on the sequence of boxes, and that every connected component of the USF is an infinite tree. Moreover, the limits with respect to two extremal boundary conditions, free and wired are the same.

Let us consider the wired boundary conditions. Fix $N \in \mathbb{N}$ and let $\Lambda_{N}=\{-N, \ldots, N\}^{d}$. Let $\Gamma_{N}$ be the finite graph obtained by contracting all vertices outside of $\Lambda_{N}$. The graph $\Gamma_{N}$ is a finite graph with $(2 N+1)^{d}+1$ vertices. The vertex set of $\Gamma_{N}$ is $\Lambda_{N} \cup s$. Vertices in $\Lambda_{N}$ are connected if they are nearest-neighbours, and the $\boldsymbol{s}$ is connected to $\boldsymbol{n} \in \Lambda_{N}$ by

$$
2 d-\left|\left\{\boldsymbol{k} \in \Lambda_{N}:\|\boldsymbol{k}-\boldsymbol{n}\|_{1}=1\right\}\right|
$$

edges. Therefore, $s$ is connected only to vertices on the boundary of $\Lambda_{N}$.
Let $\boldsymbol{T}_{N}$ be a uniform spanning tree on $\Gamma_{N}$. Weak convergence established by Pemantle [17] means that for fixed edges $e_{1}, \ldots, e_{r}$ connecting some vertices in $\mathbb{Z}^{d}$, probability

$$
\mathbb{P}_{N}\left(e_{1}, \ldots, e_{r} \in \boldsymbol{T}_{N}\right)
$$

converges as $N \rightarrow \infty$, which allows us to define a random variable $\boldsymbol{T}$ - the uniform spanning forest on $\mathbb{Z}^{d}$, by

$$
\mathbb{P}_{\mathbb{Z}^{d}}\left(e_{1}, \ldots, e_{r} \in \boldsymbol{T}\right)=\lim _{N \rightarrow \infty} \mathbb{P}_{N}\left(e_{1}, \ldots, e_{r} \in \boldsymbol{T}_{N}\right)
$$

The limiting process, $\boldsymbol{T}$ is almost surely a single tree if and only if $d \leq 4$.

The determinantal structure is preserved in the limit, and the Transfer-Impedance Theorem remains true for infinite graphs like $\mathbb{Z}^{d}$ or $\mathbb{Z} \times G$. In fact, expressions of the kernel function become more tractable in the limit.

### 2.4 Computation of Entropy

The constructed measure $\mathbb{P}_{\mathbb{Z}^{d}}$ on $\{0,1\}^{E\left(\mathbb{Z}^{d}\right)}$ is translation invariant. Therefore, we have a measure preserving dynamical system $\left(\Omega_{d}, S_{d}, \mathbb{P}_{\mathbb{Z}^{d}}\right)$, where

$$
\Omega_{d} \subset\{0,1\}^{E\left(\mathbb{Z}^{d}\right)}
$$

is the set of all infinite edge configurations without loops (spanning trees), and $S_{d}$ is the action of $\mathbb{Z}^{d}$ by shifts.

Burton \& Pemantle [1] have shown that the law of the USF is the unique measure of maximal entropy on essential spanning trees (spanning forests where every component is infinite), and computed the entropy

$$
h\left(S_{d}, \mathbb{P}_{\mathbb{Z}^{d}}\right)=h_{\text {top }}\left(S_{d}, \Omega_{d}\right)=\lim _{N \rightarrow \infty} \frac{1}{\left|\Lambda_{N}\right|} \log \mathrm{t}\left(\Gamma_{N}^{d}\right)
$$

The number $\mathrm{t}\left(\Gamma_{N}\right)$ is the determinant of any principal minor, e.g., the minor obtained by removing the row and the column corresponding to the special vertex $\boldsymbol{s}$. The corresponding matrix is of size $\left|\Lambda_{N}\right| \times\left|\Lambda_{N}\right|=(2 N+1)^{d} \times(2 N+1)^{d}$

$$
\Delta_{N}^{\prime}=\left(\delta_{\boldsymbol{n}, \boldsymbol{k}}\right)_{\boldsymbol{n}, \boldsymbol{k} \in \Lambda_{N}}
$$

with

$$
\delta_{\boldsymbol{n}, \boldsymbol{k}}= \begin{cases}2 d, & \text { if } \boldsymbol{n}=\boldsymbol{k} \\ -1, & \text { if }\|\boldsymbol{n}-\boldsymbol{k}\|_{1}=1 \\ 0, & \text { otherwise }\end{cases}
$$

Therefore, $\delta_{\boldsymbol{n}, \boldsymbol{k}}=f_{\boldsymbol{n}-\boldsymbol{k}}$, where $f \in R_{d}=\mathbb{Z}\left[\mathbb{Z}^{d}\right]$ is given by

$$
\begin{equation*}
f^{(d)}=2 d-\sum_{j=1}^{d}\left(u_{j}+u_{j}^{-1}\right) \tag{2.6}
\end{equation*}
$$

Applying Szegö's limit theorem, one concludes that

$$
h\left(S_{d}, \mathbb{P}_{\mathbb{Z}^{d}}\right)=m_{f^{(d)}}=\int_{\mathbb{T}^{d}} \log \left(2 d-2 \sum_{j=1}^{d} \cos \left(2 \pi \theta_{j}\right)\right) d \boldsymbol{\theta}
$$

and in particular, for $d=2$,

$$
h\left(S_{2}, \mathbb{P}_{\mathbb{Z}^{2}}\right)=\frac{4 G}{\pi}
$$

Burton \& Pemantle also observed that the entropy of the USF processes coincide with entropies of algebraic dynamical systems associated to $f^{(d)}$. They wrote:

We are at a loss to explain this apparent coincidence.

General formula for entropies of algebraic dynamical systems was obtained earlier by Lind, Schmidt, and Ward in [12].

Remark 2.7. Uniform Spanning Forest on the lattice $\mathbb{Z}^{d}$ can be seen as a solvable model in the sense of Statistical Mechanics. Rather remarkably for many solvable models, the free energy - obtained as a thermodynamic limit (i.e., the limit as the number of particles tends to infinity)

$$
F_{\beta}=\lim _{\Lambda \backslash \mathbb{L}}-\frac{1}{\beta|\Lambda|} \log Z_{\Lambda}(\beta)=\lim _{\Lambda \nmid \mathbb{L}}-\frac{1}{\beta|\Lambda|} \log \sum_{\sigma_{\Lambda}} \exp \left(-\beta H\left(\sigma_{\Lambda}\right)\right)
$$

coincides with the logarithmic Mahler measure of a certain polynomial. Below we list some of the models, their resulting free energies $F$, and the corresponding polynomials $f$. For simplicity we take $\beta=1$.

Dimer model $[9] \quad F=-\frac{1}{4} m_{f}, \quad f=4-\left(u_{1}+u_{1}^{-1}+u_{2}+u_{2}^{-1}\right)$
2D Ising model [16] $\quad F=-\frac{1}{2} m_{f}, \quad f=4\left(\frac{1+T^{2}}{1-T^{2}}\right)^{2}-\frac{4 T}{1-T^{2}}\left(u_{1}+u_{1}^{-1}+u_{2}+u_{2}^{-1}\right)$, where $T=\tanh (J)$

Conjugate model [5] $\quad F=-\frac{1}{2} m_{f}, \quad f=a-b\left(u_{1} u_{2}^{-1}+u_{1}^{-1} u_{2}\right)-c\left(u_{1} u_{2}+u_{1}^{-1} u_{2}^{-1}\right)$
Free-fermion model [5] $\quad F=-\frac{1}{2} m_{f}, \quad f=a-b\left(u_{1}+u_{1}^{-1}\right)-c\left(u_{2}+u_{2}^{-1}\right)$
$-d\left(u_{1} u_{2}^{-1}+u_{1}^{-1} u_{2}\right)-e\left(u_{1} u_{2}+u_{1}^{-1} u_{2}^{-1}\right)$

## 3 Algebraic Dynamical Systems

Let $d \geq 1$. We define the shift-action $\alpha$ of $\mathbb{Z}^{d}$ on $\mathbb{T}^{\mathbb{Z}^{d}}$ by

$$
\begin{equation*}
\left(\alpha^{\boldsymbol{m}} x\right)_{\boldsymbol{n}}=x_{\boldsymbol{m}+\boldsymbol{n}} \tag{3.1}
\end{equation*}
$$

for every $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{Z}^{d}$ and $x=\left(x_{\boldsymbol{n}}\right) \in \mathbb{T}^{\mathbb{Z}^{d}}$ and consider, for every $f \in R_{d}$, the group homomorphism

$$
\begin{equation*}
f(\alpha)=\sum_{m \in \mathbb{Z}^{d}} f_{m} \alpha^{m}: \mathbb{T}^{\mathbb{Z}^{d}} \longrightarrow \mathbb{T}^{\mathbb{Z}^{d}} \tag{3.2}
\end{equation*}
$$

Since $R_{d}$ is an integral domain, Pontryagin duality implies that $f(\alpha)$ is surjective for every nonzero $f \in R_{d}$ (it is dual to the injective homomorphism from $R_{d} \cong \widehat{\mathbb{T}^{\mathbb{Z}^{d}}}$ to itself consisting of multiplication by $f$ ).

Then

$$
X_{f}=\operatorname{ker} f(\alpha) \subset \mathbb{T}^{\mathbb{Z}^{d}}
$$

is a compact translation invariant group. More specifically,

$$
X_{f}=\left\{x=\left(x_{\boldsymbol{n}}\right) \in \mathbb{T}^{\mathbb{Z}^{d}}: \sum_{\boldsymbol{m} \in \mathbb{Z}^{d}} f_{\boldsymbol{m}} x_{\boldsymbol{n}+\boldsymbol{m}}=0_{\mathbb{T}} \text { for every } \boldsymbol{n} \in \mathbb{Z}^{d}\right\}
$$

For example, if $f^{(d)} \in R_{d}$ be given by (2.6), then

$$
\begin{array}{r}
X_{f^{(d)}}=\operatorname{ker} f^{(d)}(\alpha)=\left\{x=\left(x_{\boldsymbol{n}}\right) \in \mathbb{T}^{\mathbb{Z}^{d}}: 2 d x_{\boldsymbol{n}}-\sum_{j=1}^{d}\left(x_{\boldsymbol{n}+\mathbf{e}^{(j)}}+x_{\boldsymbol{n}-\mathbf{e}^{(j)}}\right)=0\right.  \tag{3.3}\\
\text { for every } \left.\boldsymbol{n} \in \mathbb{Z}^{d}\right\} .
\end{array}
$$

Denote by $\alpha_{f}$ the restriction of $\alpha$ to $X_{f}$. Since every $\alpha_{f}^{\boldsymbol{m}}, \boldsymbol{m} \in \mathbb{Z}^{d}$, is a continuous automorphism of $X_{f}$, the $\mathbb{Z}^{d}$-action $\alpha_{f}$ preserves the normalized Haar measure $\lambda_{X_{f}}$ of $X_{f}$. The algebraic $\mathbb{Z}^{d}$-action $\alpha_{f}$ on $X_{f}$ is completely determined by $f$, and one can express its dynamical properties in terms of the Laurent polynomial $f$ as follows:
(a) $X_{f}$ is infinite if and only if $f$ is not a unit in $R_{d}$, i.e., if and only if $f$ is not of the form $\pm u^{\boldsymbol{n}}$ for some $\boldsymbol{n} \in \mathbb{Z}^{d}$;
(b) $X_{f}$ is connected if and only if $f$ is primitive, i.e., not divisible by an integer $m>1$;
(c) $\alpha_{f}$ is mixing (with respect to the normalized Haar measure $\lambda_{X_{f}}$ of $X_{f}$ ) if and only if $f$ is not divisible by a polynomial of the form $c\left(u^{\boldsymbol{n}}\right)$, where $\mathbf{0} \neq \boldsymbol{n} \in \mathbb{Z}^{d}$ and $c \in R_{1}$ is a cyclotomic polynomial;
(d) If $\alpha_{f}$ is mixing it has positive entropy and is isomorphic to a Bernoulli shift;
(e) $\alpha_{f}$ is expansive if and only if

$$
\begin{equation*}
\mathrm{U}(f)=\left\{\mathbf{c}=\left(c_{1}, \ldots, c_{d}\right) \in(\mathbb{C} \backslash\{0\})^{d}: f(\mathbf{c})=0\right\} \cap \mathbb{S}^{d}=\varnothing \tag{3.4}
\end{equation*}
$$

where $\mathbb{S}=\{z \in \mathbb{C}:|z|=1\}$.
The Laurent polynomial $f^{(d)}$ can be viewed as a Laplacian on $\mathbb{Z}^{d}$, and every $x=\left(x_{n}\right) \in$ $X_{f^{(d)}}$ is harmonic $(\bmod 1)$ in the sense that, for every $\boldsymbol{n} \in \mathbb{Z}^{d}, 2 d \cdot x_{\boldsymbol{n}}$ is the sum of its $2 d$ neighbouring coordinates $(\bmod 1)$. This is the reason for calling $\left(X_{f^{(d)}}, \alpha_{f^{(d)}}\right)$ the $d$-dimensional harmonic model.

The Kolmogorov-Sinai entropy of $\alpha_{f^{(d)}}$ with respect to $\lambda_{X_{f^{(d)}}}$ coincides with the topological entropy of $\alpha_{f^{(d)}}$ and is given by

$$
\begin{equation*}
h_{\lambda_{X_{f^{(d)}}}}\left(\alpha_{f^{(d)}}\right)=h_{\text {top }}\left(\alpha_{f^{(d)}}\right)=m_{f^{(d)}} . \tag{3.5}
\end{equation*}
$$

Moreover, since $m_{f^{(d)}}>0$, the dynamical system $\left(X_{f^{(d)}}, \alpha_{f^{(d)}}, \lambda_{X_{f^{(d)}}}\right)$ is Bernoulli. The same is true for $\left(\Omega_{d}, S, \mathbb{P}_{d}\right)$. Therefore,

$$
\left(X_{f^{(d)}}, \alpha_{f^{(d)}}, \lambda_{X_{f^{(d)}}}\right) \text { and }\left(\Omega_{d}, S_{d}, \mathbb{P}_{d}\right)
$$

are measure-theoretically isomorphic (as two Bernoulli systems with equal entropy): thus there exist sets

$$
\Omega_{d}^{\prime} \subset \Omega_{d}, \mathbb{P}_{\mathbb{Z}^{d}}\left(\Omega_{d}^{\prime}\right)=1, \quad \text { and } \quad X_{f^{(d)}}^{\prime} \subset X_{f^{(d)}}, \lambda_{f^{(d)}}\left(X_{f^{(d)}}^{\prime}\right)=1,
$$

and a measure-preserving bijection $\xi: \Omega_{d}^{\prime} \mapsto X_{f^{(d)}}^{\prime}$ such that

$$
\begin{array}{clll}
\Omega_{d}^{\prime} & \xrightarrow{S^{n}} & \Omega_{d}^{\prime} & \\
\xi \downarrow & & \downarrow^{\xi} & \forall \boldsymbol{n} \in \mathbb{Z}^{d} . \\
X_{f^{(d)}}^{\prime} & \xrightarrow[\alpha_{f(d)}^{n}]{ } & X_{f^{(d)}}^{\prime} &
\end{array}
$$

### 3.1 Symbolic covers of algebraic dynamical systems

Suppose we are given an algebraic dynamical system: a group $X_{f} \subset \mathbb{T}^{\mathbb{Z}^{d}}, f \in R_{d}$, and a symbolic system: subshift $Y \subset V^{\mathbb{Z}^{d}}$, finite set $V \subset \mathbb{Z}$, e.g., $V=\{1, \ldots, M\}, M \in \mathbb{N}$. Suppose furthermore, they have equal entropies

$$
\mathrm{h}_{\mathrm{top}}(Y)=\mathrm{h}_{\mathrm{top}}\left(X_{f}\right)=m_{f}
$$

Then $Y$ is called an equal entropy symbolic cover of $X_{f}$, if there exist a continuous surjective equivariant map $\xi: Y \mapsto X_{f}$.

The theory of symbolic covers of algebraic dynamical systems has been started by A. Vershik [26], and continued by R. Kenyon [10], N. Sidorov [23], K. Schmidt [20], M. Einsiedler [4], E. Lindenstrauss [14], and several other. In the paper with K. Schmidt [22], we conjectured that: Solvable models are symbolic covers of their natural algebraic counterparts. There is one class of solvable models - the so-called Abelian Sandpile Models (ASM), which were proven to form symbolic covers of their algebraic counterparts.

## 4 Abelian Sandpiles

For simplicity we will only consider abelian sandpile model on $\mathbb{Z}^{2}$, generalisation to $\mathbb{Z}^{d}$ is straightforward.

### 4.1 Sandpile configurations on finite volumes

Let $\Lambda \subset \mathbb{Z}^{2}$ and $|\Lambda|<\infty$, notation $\Lambda \Subset \mathbb{Z}^{2}$. Configurations on $\Lambda$ are elements of $\mathbb{N}^{\Lambda}$ :

$$
y=\left(y_{\boldsymbol{n}}\right)_{\boldsymbol{n} \in \Lambda}, \quad \boldsymbol{n}=\left(n_{1}, n_{2}\right), \quad y_{\boldsymbol{n}} \in \mathbb{N}=\{1,2, \ldots\} .
$$

Configuration $y \in \mathbb{N}^{\Lambda}$ is called stable if

$$
y_{\boldsymbol{n}} \leq 4 \quad \forall \boldsymbol{n} \in \Lambda
$$

Suppose $y \in \mathbb{N}^{\Lambda}$ is unstable: there exists an $\boldsymbol{n} \in \Lambda$ such that $y_{\boldsymbol{n}}>4$. Then we perform a toppling at site $\boldsymbol{n}: y \mapsto y^{\prime}=T_{\boldsymbol{n}}(y)$, where

$$
\begin{array}{r}
y_{n}^{\prime}=y_{n}-4 \\
y_{\boldsymbol{m}}^{\prime}=y_{\boldsymbol{m}}+1 \quad \text { for all } \boldsymbol{m} \in \Lambda:\|\boldsymbol{m}-\boldsymbol{n}\|_{1}=1 .
\end{array}
$$

If $y \in \mathbb{N}^{\Lambda}$ is unstable, keep toppling unstable sites (i.e., apply appropriate $T_{n}$ 's) until you are left with a stable configurations. Denote the resulting configuration by $\mathcal{T}(y)$.

## Remarks:

- Dissipativity property: if a site $\boldsymbol{n}$ on the boundary of $\Lambda$ is toppled, then some grains of sand are lost. Hence, only a finite number of topplings is possible.
- Abelian property: The order of topplings is not important, hence $\mathcal{T}$ is well defined.

For $\Lambda \Subset \mathbb{Z}^{2}$, define the Laplacian $\Delta=\Delta_{\Lambda}=\left(\Delta_{n, m}\right)_{n, m \in \Lambda}$ as a matrix of size $|\Lambda| \times|\Lambda|$ with

$$
\Delta_{n, \boldsymbol{m}}= \begin{cases}4, & \text { if } \boldsymbol{n}=\boldsymbol{m} \\ -1, & \text { if }\|\boldsymbol{n}-\boldsymbol{m}\|=1 \\ 0, & \text { otherwise }\end{cases}
$$

If $y \in \mathbb{N}^{\Lambda}$ (viewed as column vector) and $y_{n}>4$, then

$$
T_{n}(y)=y-\Delta \delta_{n}
$$

where $\delta_{\boldsymbol{n}}=\left(\delta_{\boldsymbol{n}}(\boldsymbol{m})\right)_{\boldsymbol{m} \in \Lambda}$ a column vector of 0 's and one 1 .
Corollary 4.1. If $y \in \mathbb{N}^{\Lambda}$, then

$$
\mathcal{T}(y)=y-\Delta q
$$

where $q=\left(q_{\boldsymbol{n}}\right)$ where $q_{\boldsymbol{n}}$ is the number of times the site $\boldsymbol{n}$ toppled.

### 4.2 Recurrent configurations

A stable configuration $y \in \mathbb{N}^{\Lambda}$ is called recurrent if there exists another element $v \in \mathbb{N}^{\Lambda}$ such that

$$
\mathcal{T}(y+v)=y .
$$

Denote by $\mathcal{R}_{\Lambda}$ the set of all stable recurrent configurations on $\Lambda \Subset \mathbb{Z}^{2}$. Then

- $\mathcal{R}_{\Lambda}$ is an additive group: for $y, z \in \mathcal{R}_{\Lambda}$ let

$$
y \oplus z=\mathcal{T}(y+z)
$$

i.e., add two configurations and then topple the resulting configuration.

- $\mathcal{R}_{\Lambda}$ is called a sandpile or a critical group for $\Lambda$
- $\mathcal{R}_{\Lambda} \cong \mathbb{Z}^{\Lambda} / \Delta_{\Lambda} \mathbb{Z}^{\Lambda}$
- $\left|\mathcal{R}_{\Lambda}\right|=\operatorname{det}\left(\Delta_{\Lambda}\right)$

Note that if $\Lambda=\{-N, \ldots, N\}^{2}$, then the reduced laplacian of $\Gamma_{N}^{2}$ is precisely the Laplacian $\Delta_{\Lambda}$, and hence,

$$
\mathcal{R}_{\Lambda_{N}}=\mathrm{t}\left(\Gamma_{N}^{2}\right),
$$

i.e., numbers of recurrent configurations and of spanning trees coincide. Dhar found explicit bijection between these sets, known as the the burning algorithm [7,18]. Burning algorithm can also be used as a simple test to decide whether a stable configuration $y$ on $\Lambda$ is recurrent, i.e., $y \in \mathcal{R}_{\Lambda}$.

In [21], using the so-called summable homoclinic points, we constructed an equivariant surjective map $\xi: \mathcal{R}_{\mathbb{Z}^{2}} \rightarrow X_{f^{(2)}}$. The question whether the USF on $\mathbb{Z}^{2}$ is also a symbolic cover of $X_{f^{(2)}}$ remains open.

## 5 Ladder Sandpiles

In a joint work with T. Shirai [25], we consider the simplest infinite graph, namely,

$$
\Gamma=\mathbb{Z} \times\{1,2\}
$$

with non-trivial USF and abelian sandpile models. The USF on $\Gamma$ has been studied by Häggström [6], and the abelian sandpile model on $\Gamma$ has been studied by Járai \& Lyons [8]. These papers provide very detailed description of the corresponding models.

We start by defining the algebraic harmonic model on $\Gamma$. By analogy with the lattice sandpiles, the corresponding algebraic model should be

$$
X=\left\{\omega=\left(\omega_{n}\right)_{n \in \mathbb{Z}}, \omega_{n}=\binom{\omega_{n}^{(1)}}{\omega_{n}^{(2)}} \in \mathbb{T}^{2}, \begin{array}{l}
3 \omega_{n}^{(1)}-\omega_{n-1}^{(1)}-\omega_{n+1}^{(1)}-\omega_{n}^{(2)}=0_{\mathbb{T}} \\
3 \omega_{n}^{(2)}-\omega_{n-1}^{(2)}-\omega_{n+1}^{(2)}-\omega_{n}^{(1)}=0_{\mathbb{T}}
\end{array} \text { for all } n \in \mathbb{Z} .\right\}
$$

Clearly, $X$ is compact translation invariant subgroup of $\left(\mathbb{T}^{2}\right)^{\mathbb{Z}}$. Denote by $\alpha$ the $\mathbb{Z}$-action by left-shifts on $X$. Similarly, to algebraic systems defined by one Laurent polynomial, one can show that the Pontryagin dual of $X$ is given by the module

$$
\widehat{X}=R^{2} / M R^{2},
$$

where $R$ is the ring of Laurent polynomials in one variable, $R=\mathbb{Z}\left[z^{ \pm 1}\right]$, and $M$ is a $2 \times 2$ matrix with coefficients in $R$,

$$
M=\left[\begin{array}{cc}
3-z-\frac{1}{z} & -1 \\
-1 & 3-z-\frac{1}{z}
\end{array}\right], \quad R^{2}=\left\{\binom{f_{1}}{f_{2}}: f_{i} \in R\right\}, \quad M R^{2} \subsetneq R^{2}
$$

Note that

$$
\operatorname{det}(M)=\left(3-z-\frac{1}{z}\right)^{2}-1=\left(4-z-\frac{1}{z}\right)\left(2-z-\frac{1}{z}\right)=: f \cdot g
$$

By the result of [12], the topological entropy of shift action $\alpha$ on $X$ is

$$
h_{\text {top }}(X, \alpha)=\int_{0}^{1} \log \left|\operatorname{det}(M)\left(e^{2 \pi i \theta}\right)\right| d \theta=m_{f}+m_{g}=m_{f}=\log (2+\sqrt{3}),
$$

coincides with the entropies of sandpile and spanning trees on $\Gamma$. Therefore, the relevant part of the algebraic system is given by $X_{f}, f=4-z-z^{-1}$, and we will discuss symbolic covers of $X_{f}$. Note that $f=z^{2}-4 z+1$, has two roots $2+\sqrt{3}>1>2-\sqrt{3}$, the largest root thus a Pisot number (unit). Moreover, the shift action $\alpha_{f}: X_{f} \mapsto X_{f}$ is topologically conjugated to the total automorphism $T_{A}: \mathbb{T}^{2} \mapsto \mathbb{T}^{2}$, where $A$ is the companion matrix of $z^{2}-4 z+1$ :

$$
A=\left(\begin{array}{cc}
0 & 1 \\
-1 & 4
\end{array}\right)
$$

Symbolic covers of toral automorphisms with largest root being a Pisot number $\beta$ have been studied extensively: it is known that the so-called $\beta$-shifts - particular sofic shifts arising in number theory, form almost everywhere one-to-one symbolic covers to ( $\left.\mathbb{T}^{2}, T_{A}\right) \cong\left(X_{f}, \alpha_{f}\right)$.

The sets of spanning trees, the left-burnable (respectively, right-burnable) recurrent configurations on $\Gamma$ are also sofic $[6,8]$, and in fact they form symbolic covers of $X_{f}$.
Theorem 5.1 ([25]). There exist equivariant surjective maps from the sets of spanning trees and the left-burnable (respectively, right-burnable) recurrent configurations on $\Gamma$ onto $X_{f}$.

This is the first result where it is shown that USF is a symbolic cover of its algebraic counterpart.

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# Rigidity Phenomena in random point sets 

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## 1 Introduction

Rigidity phenomena in random point sets have attracted a fair amount of attention in the recent years. At its core, this involves singularity behaviour that can be observed under spatial conditioning in many natural point processes. While the phenomenon is interesting, and at first, counterintuitive in its own right, several applications of rigidity phenomena have been found in natural probabilistic questions concerning point processes, particularly those of a stochastic geometric flavour. In this article, we will try to give a brief overview of these results.

The most canonical example of point processes is the Poisson point process on Euclidean spaces (and other Riemannian manifolds). A key characteristic of the Poisson process is the property that the points in mutually disjoint domains are statistically independent. For the Poisson process, therefore, spatial conditioning is a triviality, and as a result, it is not interesting from the point of view of rigidity phenomena. However, there is a wide range of naturally occuring point processes such that do exhibit remarkable behaviour with regard to spatially conditioned measures. These include key examples arising from random matrices and random polynomials.

The basic question in studying rigidity phenomena is the following. Suppose we have a point process $\Pi$ on a space $\Xi$, which in general we will think of as a locally Euclidean metric space. Let $\mathcal{S}$ be the Polish space of locally finite point configurations on $\Xi$, which means that the point process $\Pi$ can be thought of as a probability measure on $\mathcal{S}$. Let $\mathcal{D} \subset \Xi$ be a bounded open set. The partitioning $\Xi=\mathcal{D} \cup \mathcal{D}^{c}$ induces a decomposition $\mathcal{S}=\mathcal{S}_{\text {in }} \times \mathcal{S}_{\text {out }}$, where $\mathcal{S}_{\text {in }}$ and $\mathcal{S}_{\text {out }}$ are respectively the spaces of finite point configurations on $\mathcal{D}$ and locally finite point configurations on $\mathcal{D}^{c}$. This immediately leads to the natural decomposition $\Upsilon=\left(\Upsilon_{\text {in }}, \Upsilon_{\text {out }}\right)$ for any $\Upsilon \in \mathcal{S}$, and consequently a decomposition of the point process $\Pi$ as $\Pi=\left(\Pi_{\mathrm{in}}, \Pi_{\text {out }}\right)$.

By abstract nonsense, we can define the conditional law (that is, the regular conditional distribution) of $\Pi_{\mathrm{in}}$ given $\Pi_{\text {out }}$. We will denote this conditional measure by $\rho_{\omega}$, where $\omega$ is the value of $\Pi_{\text {out }}$. In the case of the Poisson process, $\rho_{\omega}$ does not
depend on $\omega$, and is itself a Poisson point process on $\mathcal{D}$. In the case of naturally occurring point processes with non-trivial spatial correlation, one would expect that $\rho_{\omega}$ would still show some regularity, e.g. being absolutely continuous with respect to the Poisson process on $\mathcal{D}$. However, this turns out to be far from the case. E.g., for the Ginibre ensemble on $\mathbb{R}^{2}$, which comes from the canonical non-Hermitian Gaussian random matrix ensemble, the points outside any bounded open set $\mathcal{D}$ determine a.s. the number of points inside $\mathcal{D}$.

This is an opportune moment to introduce the formal definition of rigidity:
Definition 1. A measurable function $f_{\mathrm{in}}: \mathcal{S}_{\mathrm{in}} \rightarrow \mathbb{C}$ is said to be rigid with respect to the point process $X$ on $\mathcal{S}$ if there is a measurable function $f_{\text {out }}: \mathcal{S}_{\text {out }} \rightarrow \mathbb{C}$ such that a.s. we have $f_{\text {in }}\left(X_{\text {in }}\right)=f_{\text {out }}\left(X_{\text {out }}\right)$.

## 2 Rigidity Phenomena in point processes

In [GP], the authors undertook a systematic study of rigidity phenomena, with particular reference to the Ginibre ensemble and the Gaussian zero process on the plane. We briefly discuss these point processes here; for a detailed account we refer the reader to [HKPV].

The Ginibre ensemble was introduced in the physics literature by Ginibre [Gin] as a model based on non-Hermitian random matrices. For a positive integer $n$, consider the eigenvalues of a $n \times n$ random matrix whose entries are i.i.d. complex Gaussians. The Ginibre ensemble is the weak limit of these (finite-dimensional) eigenvalue processes. It is a determinantal point process with the determinantal kernel $K(z, w)=\sum_{j=0}^{\infty} \frac{(z \bar{w})^{j}}{j!}$ and the background measure $e^{-|z|^{2}} d \mathcal{L}(z)$, where $\mathcal{L}$ denotes the Lebesgue measure on $\mathbb{C}$. The standard planar Gaussian Analytic Function (abbreviated henceforth as GAF) is the random entire function defined by the series development

$$
f(z)=\sum_{k=0}^{\infty} \frac{\xi_{k}}{\sqrt{k!}} z^{k}
$$

where $\xi_{k}$-s are i.i.d. standard complex Gaussians. The zero set of this random analytic function is the GAF zero process. In comparison to the Ginibre ensemble, it can be realized as the weak limit of the zero processes of the random polynomials

$$
f_{n}(z)=\sum_{k=0}^{n} \frac{\xi_{k}}{\sqrt{k!}} z^{k}
$$

Both the Ginibre and the GAF zero ensembles are translation-invariant and are ergodic under the action of the rigid motions of the plane.

In what follows, we will show that for the Ginibre ensemble, the points outside any bounded open set $\mathcal{D}$ determine a.s. the number of points inside $\mathcal{D}$, and in the GAF zero process, the points outside $\mathcal{D}$ determine the number and the sum of the points inside. If we think of the points as a particle system, then this can be described by saying that in the Ginibre ensemble, there is local conservation of mass, while in the standard planar GAF zero process, the mass as well as the centre of mass are locally conserved. Moreover, they showed that these are the "only" conservation laws in these ensembles, in a natural sense. To be precise, we quote the relevant theorems from [GP].

In Theorems 2.1-2.4 we denote the Ginibre ensemble by $\mathcal{G}$ and the GAF zero ensemble by $\mathcal{Z}$. As before, $\mathcal{D}$ is a bounded open set in $\mathbb{C}$.

In the case of the Ginibre ensemble, we prove that a.s. the points outside $\mathcal{D}$ determine the number of points inside $\mathcal{D}$, and "nothing more".

Theorem 2.1. For the Ginibre ensemble, there is a measurable function $N: \mathcal{S}_{\text {out }} \rightarrow$ $\mathbb{N} \cup\{0\}$ such that a.s.

$$
\text { Number of points in } \mathcal{G}_{\text {in }}=N\left(\mathcal{G}_{\text {out }}\right) .
$$

Since a.s. the length of (the vector of) inside points $\underline{\zeta}$ equals $N\left(\mathcal{G}_{\text {out }}\right)$, we can assume that each measure $\rho\left(\Upsilon_{\text {out }}, \cdot\right)$ is supported on $\mathcal{D}^{N\left(\Upsilon_{\text {out }}\right)}$.

Theorem 2.2. For the Ginibre ensemble, a.s. the measure $\rho\left(\mathcal{G}_{\text {out }}, \cdot\right)$ and the Lebesgue measure $\mathcal{L}$ on $\mathcal{D}^{N\left(\mathcal{G}_{\text {out }}\right)}$ are mutually absolutely continuous.

In the case of the GAF zero process, we prove that the points outside $\mathcal{D}$ determine the number as well as the centre of mass (or equivalently, the sum) of the points inside $\mathcal{D}$, and "nothing more".

Theorem 2.3. For the GAF zero ensemble,
(i) There is a measurable function $N: \mathcal{S}_{\text {out }} \rightarrow \mathbb{N} \cup\{0\}$ such that a.s.

Number of points in $\mathcal{Z}_{\text {in }}=N\left(\mathcal{Z}_{\text {out }}\right)$.
(ii)There is a measurable function $S: \mathcal{S}_{\text {out }} \rightarrow \mathbb{C}$ such that a.s.

$$
\text { Sum of the points in } \mathcal{Z}_{\text {in }}=S\left(\mathcal{Z}_{\text {out }}\right)
$$

For a possible value $\Upsilon_{\text {out }}$ of $\mathcal{Z}_{\text {out }}$, define the set of admissible vectors of inside points (obtained by considering all possible orderings of such inside point configurations)

$$
\Sigma_{S\left(\Upsilon_{\text {out }}\right)}:=\left\{\underline{\zeta} \in \mathcal{D}^{N\left(\Upsilon_{\text {out }}\right)}: \sum_{j=1}^{N\left(\Upsilon_{\text {out }}\right)} \zeta_{j}=S\left(\Upsilon_{\text {out }}\right)\right\}
$$

where $\underline{\zeta}=\left(\zeta_{1}, \cdots, \zeta_{N\left(\Upsilon_{\text {out }}\right)}\right)$.
Since a.s. the length of (the vector of) inside points $\underline{\zeta}$ equals $N\left(\Upsilon_{\text {out }}\right)$, we can assume that each measure $\rho\left(\Upsilon_{\text {out }}, \cdot\right)$ gives us the distribution of a random vector in $\mathcal{D}^{N\left(\Upsilon_{\text {out }}\right)}$ supported on $\Sigma_{S\left(\Upsilon_{\text {out }}\right)}$.

Theorem 2.4. For the GAF zero ensemble, a.s. the measure $\rho\left(\mathcal{Z}_{\text {out }}, \cdot\right)$ and the Lebesgue measure $\mathcal{L}_{\Sigma}$ on $\Sigma_{S\left(\mathcal{Z}_{\text {out }}\right)}$ are mutually absolutely continuous.

In [G-I], the rigidity (of the number of points) in the famous sine kernel process (on the real line) was established, see Theorem 4.2 therein (which establishes such behaviour for a more general class of ensembles). On a related note, similar results were also established for a wide range of translation invariant determinantal point processes on $\boldsymbol{Z}$ (introduced by Lyons and Steif [LySt]) that correspond to function spaces characterised by vanishing Fourier transform outside a given set. To be more precise, let $f: \mathbb{T} \rightarrow[0,1]$ be a measurable function. Then it is not difficult to check that one can define a determinantal point process on $\boldsymbol{Z}$ with $K(i, j):=\hat{f}(i-j)$ and counting measure as the background measure. In Theorem 1.5, [G-I], it has been shown that whenever $f$ is the indicator function of an interval, the corresponding determinantal process exhibits rigidity of the number of points in a domain. In particular, this settles in the negative a conjecture in [LySt] to the effect that essentially all such processes are insertion and deletion tolerant.

In [OsSh], Osada and Shirai showed that for the Ginibre ensemble, the Palm measures with respect to different point sets are mutually absolutely continuous if the conditioning set of points have the same cardinality, and are mutually absolutely continuous otherwise. Such dichotomy is similar in spirit to the rigidity phenomena under our consideration.

## 3 Rigidity Hierarchies

Theorems 2.2 and 2.4 lead to a natural definition of "tolerance" for point processes. Heuristically, this corresponds to the regularity of the (spatially) conditional measures, modulo local rigidity constraints. Before making a formal definition, let us recall the definition of linear statistics:

Definition 2. Let $\Pi$ be a point proces on $\Xi$ and $\varphi: \Xi \rightarrow \mathbb{C}$ be a measurable function. Then the linear statistics $\Lambda(\varphi)$ of $\Pi$ is defined to be the random variable

$$
\Lambda(\varphi)[\Pi]:=\int_{\Xi} \varphi(z) d[\Pi](z) .
$$

In the above and in what follows, $d[\Pi]$ denotes the (random) counting measure naturally associated with the point process $\Pi$.

Definition 3. Let $\Pi$ be a point process on a Riemannian manifold $\Xi$ with volume measure $\mu$. Let $\mathcal{D} \subset \Xi$ be a bounded open set, and let $\Lambda\left(\Phi_{0}\right), \Lambda\left(\Phi_{1}\right), \cdots, \Lambda\left(\Phi_{t}\right)$ be rigid linear statistics of the point process $\Pi_{i n}$ on $\mathcal{D}$, with $\Phi_{0} \equiv 1$ and $\Phi_{1}, \cdots, \Phi_{t}: \mathcal{D} \rightarrow \mathbb{C}$ smooth functions.

For an integer $m \geq 0$ and $\underline{s}:=\left(s_{1}, \cdots, s_{t}\right) \subset \mathbb{C}^{t}$, consider the submanifold of $\mathcal{D}^{m}$

$$
\Sigma_{m, \underline{s}}:=\left\{\underline{\zeta}=\left(\zeta_{1}, \cdots, \zeta_{m}\right) \in \mathcal{D}^{m}: \Lambda\left(\Phi_{j}\right)\left[\delta_{\underline{\zeta}}\right]=s_{j} ; 1 \leq j \leq t\right\}
$$

where $\delta_{\underline{\underline{\zeta}}}$ is the counting measure corresponding to the point set $\left\{\zeta_{i}\right\}_{i=1}^{m}$.
Then $\Pi$ is said to be tolerant subject to $\Lambda\left(\Phi_{0}\right), \Lambda\left(\Phi_{1}\right), \cdots, \Lambda\left(\Phi_{t}\right)$ if the conditional distribution $\left(\Pi_{\mathrm{in}} \mid \Pi_{\text {out }}=\omega\right)$ is mutually absolutely continuous with the point process of $\Lambda\left(\Phi_{0}\right)=N(\omega)$ points sampled independently from the submanifold $\Sigma_{N(\omega), \underline{s}}$ (where $\left.s_{i}=\Lambda\left(\Phi_{i}\right)=S_{i}(\omega), 1 \leq i \leq t\right)$ equipped with the restriction of the volume measure $\mu^{\otimes N(\omega)}$.

Theorems 2.2 and 2.4 can be phrased in terms tolerance as defined above. E.g., Theorem 2.2 can be rephrased by saying that for any bounded open set $\mathcal{D}$, the Ginibre ensemble is tolerant subject to the number of points in $\mathcal{D}$.

In view of these results, it is natural to ask whether there are point processes which exhibit higher levels of rigidity, in the sense that higher moments of the points inside $\mathcal{D}$ are also conserved. In a technical sense, we can formulate this question as follows.

Definition 4. We say that a point process $\Pi$ on $\mathbb{C}$ is rigid at level $k$ if the following conditions hold for every bounded open set $\mathcal{D} \subset \mathbb{C}$ :

- The linear statistics of $\Pi_{i n}$ given by $\left\{\Lambda\left(z^{j}\right)\right\}_{j=0}^{k-1}$ are rigid.
- $\Pi$ is tolerant subject to $\left\{\Lambda\left(z^{j}\right)\right\}_{j=0}^{k-1}$.

In terms of the last definition, we can say that the Ginibre ensemble is rigid at level 1 and the standard planar GAF zero process is rigid at level 2 . In an upcoming paper [GK], the authors exhibit a family of Gaussian entire functions indexed by a paramter $\alpha$, such that the (random) zero set shows phase transitions in its level of rigidity as $\alpha$ varies, and further, any level of rigidity can be attained by appropriate choice of $\alpha$. To be more speficic, they show that

Theorem 3.1. For a real number $\alpha>0$, define the $\alpha-G A F$

$$
f_{\alpha}(z)=\sum_{k=0}^{\infty} \frac{\xi_{k}}{(k!)^{\alpha / 2}} z^{k}
$$

The $\alpha$-GAF zero process is rigid at level $\left(\left\lfloor\frac{1}{\alpha}\right\rfloor+1\right)$.

In the same work, they show that a necessary condition for a determinantal point process to be rigid is that its kernel is a reproducing kernel:

Theorem 3.2. Let $\Pi$ be a determinantal point process with kernel $K$ and background measure $\mu$. Then $\Pi$ exhibits rigidity in the number of points in a domain only if $K$ is a projection operator on $L^{2}(\mu)$.

## 4 Applications to stochastic geometry

An understanding of rigidity phenomena has been used in the recent past to answer various natural questions related to spatially correlated point processes, extending the state of the art from the case of the Poisson point process. Many of these questions have a stochastic geometric flavour. In [GKP], the authors used an understanding of the rigidity behaviour of the Ginibre ensemble and the standard planar GAF zero process obtained in [GP] in order to study continuum percolation on these models. In particular, they showed the existence of a non-trivial critical radius for percolation for GAF zeroes, and established the uniqueness of the infinite cluster in the supercritical regime for both the Ginibre and the GAF zero processes (the existence of a critical radius for the Ginibre ensemble was known in the literature).

To briefly describe the continuum percolation model (also known in the relevant literature as the Boolean model or the Gilbert disk model), we define a random geometric graph whose vertices are the points of a point process, and where vertices are connected by an edge if their mutual distance is less than a threshold $r$. It is easy to see that the number of infinite clusters in this graph is a non-decreasing function of $r$. We say that there is a non-trivial critical radius for percolation if there exists a $0<r_{c}<\infty$ such that a.s. there is no infinite connected component in the random geometric graph whenever $r<r_{c}$, and a.s. there is an infinite connected component if $r>r_{c}$. Further, the number of infinite clusters is a translation invariant random variable, and hence is a.s. a constant whenever the underlying point process is ergodic.

While it would be too much of a digression to give a detailed outline of the proofs, let us sketch some of the major features, with emphasis on the aspects related to rigidity phenomena. The existence of a non-trivial critical radius uses the standard Pierls type argument, and the key property that is required of the underlying point process is an exponential decay of the probability of having a long vacant (or overcrowded) circuit. For the GAF zero process, this was not known, and was established in [GKP] (Theorem 1.3 therein) exploiting an almost-independence phenomenon exhibited by the GAF and using a Cantor set type construction.

To establish the uniqueness of the infinite cluster, we would ideally like to appeal to the famous Burton and Keane type argument from the classical Bond percolation theory. In the setting of continuum percolation, the same argument can be used to deal with the Poisson process. The main theme of the Burton and Keane type argument
is as follows: we want to rule out the possibility that, with positive probability, there are multiple (but finitely many) infinite clusters (the case of infinitely many infinite clusters is ruled out because of the amenability of the ambient space $\mathbb{R}^{2}$, and can be dealt with in a unified manner for all ergodic point process). In the representative scenario where there are two infinite clusters with positive probability, we intersect the two infinite clusturs with a large disk $\mathcal{D}$. Then, fixing the Poisson process outside $\mathcal{D}$, we introduce $N$ new points uniformly inside $\mathcal{D}$. For $N$ large enough, these new points can be used to connect the two infinite clusters with positive probability. Thus, having two infinite clusters with positive probability implies that we can have one infinite cluster with positive probability. This contradicts the fact that the number of infinite clusters, being a translation-invariant random variable defined on an ergodic point process, is a.s. a constant.

The Burton and Keane argument crucially depends on the fact that in a Poisson point process, conditioned on the points outside a bounded domain $\mathcal{D}$, one can insert more points inside $\mathcal{D}$ with positive probability. In the mathematical physics literature, this property is sometimes abstracted as the "finite energy condition", under which assumption many results can be obtained. However, for the Ginibre ensemble and the GAF zero process, the rigidity of the number of points precludes the finite energy condition, and therefore renders a direct application of the Burton and Keane argument invalid.

In order to remedy this difficulty, we observe that the number of points in a disk $\mathcal{D}$ of (large) radius $R$ is $\Theta\left(R^{2}\right)$, whereas the distance to be spanned in order to connect the two infinite clusters (that $\mathcal{D}$ intersects) is $O(R)$. Heuristically, therefore, there are typically "many more points than necessary" in order to connect the two infinite clusters. This obviates the need to introduce new points, and the question is whether the points already present in $\mathcal{D}$ can be spatially manipulated in order to "connect" the infinite clusters, at the same time maintaining all the rigidity constraints relevant to the point process in question. Invoking Theorems 2.2 and 2.4, this approach can be pushed through rigorously for the Ginibre and the GAF zero processes, and we refer the interested reader to the proofs of Theorems 1.1 and 1.2 in [GKP] for the (fairly elaborate) technical issues involved.

In [Os], an understanding of the quasi-Gibbs property, which has a somewhat similar flavour to our discussion of rigidity phenomena, was exploited to define an infinite particle SDE for invariant dynamics on the Ginibre process. In a recent work (draft under preparation), Osada has established a sub-diffusivity behaviour of tagged particles in the Ginibre interacting brownian motion, which is another manifestation of the rigidity of the Ginibre ensemble (compared to Poisson). To execute a similar programme for invariant dynamics on the standard planar GAF zero process involves new challenges involving the higher level of rigidity in that process, and there is some hope of advances in this direction in an ongoing work by Osada, Shirai and the author.

## 5 Completeness problems

Another class of problems on which rigidity phenomena has been brought to bear in the recent past is completeness problems for random point sets. To describe this class of problems in a simple setting, consider the case of the real line. For any $\lambda \in \mathbb{R}$, define the exponential function $e_{\lambda} \in L^{2}(-\pi, \pi)$ by

$$
e_{\lambda}(x):=e^{i \lambda x} .
$$

For a point set $\Lambda \subset \mathbb{R}$, this defines a set of exponential functions

$$
\mathcal{E}_{\Lambda}:=\left\{e_{\lambda} \mid \lambda \in \Lambda\right\} .
$$

Clearly, $\overline{\operatorname{Span}\left(\mathcal{E}_{\Lambda}\right)}$ is a closed subspace of $L^{2}(-\pi, \pi)$; the question is whether they are equal. In other words, do the exponentials arising from the set $\Lambda$ span $L^{2}(-\pi, \pi)$; we say that $\Lambda$ is "complete" in $L^{2}(-\pi, \pi)$ if they do. This question has been of considerable interest in the classical harmonic analysis literature; the interested reader can look at the works of Levinson, Beurling, Malliavin and Redheffer, to provide a partial list. We refer the reader to the comprehensive survey by Redheffer [Re]. For a more recent discussion of the completeness problem and its generalization to the world of determinantal point processes, we refer the reader to Lyons' excellent survey [Ly-II]

The classical results are usually stated in terms of an asymptotic density of a point set $\Lambda$ called the Beurling-Malliavin density. For reasons to be shortly explained, the exact definition of this density is not particularly germane to our discussion; the interested reader can look at $[\mathrm{Re}] . ~ \Lambda$ is complete if this density $>1$, and is incomplete if this density $<1$. The density 1 case is critical, and simple examples can be given of two locally finite point sets (e.g. $\boldsymbol{Z}$ and $\boldsymbol{Z} \backslash\{0\}$ ) which have the same density 1, but one is complete in $L^{2}(-\pi, \pi)$ while the other is not.

It is natural to argue that the counterexample above is rather pathological, in the sense that it demands a very specific geometry of the point set $\Lambda$, and therefore, it is of interest to try and prove a theorem with regard to completeness for a "generic" point set. A canonical way to define a "generic" point set is to think of $\Lambda$ as a realization of an ergodic point process on $\mathbb{R}$. For many natural point processes on $\mathbb{R}$ (including the homogeneous Poisson process and the sine kernel process), the Beurling-Malliavin density turns out to be the same as the one-point intensity (see [Ly-I]), hence the question boils down to the completeness properties of an ergodic point process of intensity 1.

Curiously, the only case of this question where the answer is known in the literature is that of an i.i.d lattice perturbation, where considerable mileage can be derived from the fact that we are starting from $\boldsymbol{Z}$, which is an orthogonal basis of $L^{2}(-\pi, \pi)$. In other processes, which are bereft of a "skeleton" like $\boldsymbol{Z}$, the answer was not known, including that of the homogeneous Poisson process on $\mathbb{R}$ with unit intensity. A first
step in this direction was taken in [G-I], where it was proven that if $\Lambda$ is a realization of the sine kernel process (of intensity 1 ), then it is complete in $L^{2}(-\pi, \pi)$ a.s. In doing so, the rigidity of the number of points in the sine kernel process was exploited. In fact, the theorem for the sine kernel process was deduced as a special case of a spanning theorem for rigid determinant point processes, which we explain below.

Consider a determinantal point process with a kernel $K$ and background measure $\mu$ such that $K$ acts as a projection operator onto a subspace $\mathcal{H}$ of $L^{2}(\mu)$; in other words, $K$ is a reproducing kernel for $\mathcal{H} \subset L^{2}(\mu)$. If $\Lambda$ is a realization of this point process, then it is not hard to see that the (random) set of exponentials $\{K(\cdot, x): x \in \Lambda\} \subset \mathcal{H}$; and the question is whether there is equality. In the case where $\operatorname{dim}(\mathcal{H})<\infty$ or the ambient space is countable, the answer to this question is known to be positive, for details see [Ly-I]. In general, the answer to this question is not known, and it ties to many interesting questions, as explained in [Ly-II]. In particular, when the determinant process is the sine kernel process (of unit intensity) on $\mathbb{R}$, this question is equivalent to the completeness question on exponentials described above. In [G-I], this question was settled in the affirmative in the case where the determinantal process is, in addition, rigid :

Theorem 5.1. Let $\Pi$ be a determinantal point process with a kernel $K(\cdot, \cdot)$ on a second countable locally compact proper metric space $(E, d)$ and a background measure $\mu$ which is a non-negative regular Borel measure. Suppose $K(\cdot, \cdot)$, as an integral operator from $L^{2}(\mu)$ to itself, is the projection onto a closed subspace $\mathcal{H} \subset L^{2}(\mu)$

Let $\Pi$ be rigid, in the sense that for any open ball $B$ with a finite radius, the point configuration outside $B$ a.s. determines the number of points $N_{B}$ of $\Pi$ inside $B$. Then $\{K(\cdot, x): x \in \Pi\}$ is a.s. complete in $\mathcal{H}$, that is, a.s. this set of functions spans $\mathcal{H}$.

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## Subsequential scaling limits of simple random walk

 on the two-dimensional uniform spanning treeProbabilistic models with determinantal structure Kyushu University, 30 April, 2015

Takashi Kumagai (Kyoto University)
joint with
M.T. Barlow (UBC) and D.A. Croydon (Warwick)

Ann. Probab., to appear.

## UNIFORM SPANNING TREE IN TWO DIMENSIONS

Let $\wedge_{n}:=[-n, n]^{2} \cap \mathbb{Z}^{2}$.
A subgraph of the lattice is a spanning tree of $\Lambda_{n}$ if it connects all vertices, no cycles.

Let $\mathcal{U}^{(n)}$ be a spanning tree of $\Lambda_{n}$ selected uniformly at random from all possibilities.

The UST on $\mathbb{Z}^{2}, \mathcal{U}$, is then the local limit of $\mathcal{U}^{(n)}$.
NB. Wired/free boundary conditions unimportant.
Almost-surely, $\mathcal{U}$ is a spanning tree of $\mathbb{Z}^{2}$.
[Aldous, Benjamini, Broder, Häggström, Kirchoff, Lyons, Pemantle, Peres, Schramm...]

## WILSON'S ALGORITHM ON $\mathbb{Z}^{2}$

Let $x_{0}=0, x_{1}, x_{2}, \ldots$ be an enumeration of $\mathbb{Z}^{2}$.
Let $\mathcal{U}(0)$ be the graph tree consisting of the single vertex $x_{0}$.
Given $\mathcal{U}(k-1)$ for some $k \geq 1$, define $\mathcal{U}(k)$ to be the union of $\mathcal{U}(k-1)$ and the loop-erased random walk (LERW) path run from $x_{k}$ to $\mathcal{U}(k-1)$.

The UST $\mathcal{U}$ is then the local limit of $\mathcal{U}(k)$.


## LERW SCALING IN $\mathbb{Z}^{d}$

Consider LERW as a process $\left(L_{n}\right)_{n \geq 0}$.

In $\mathbb{Z}^{d}, d \geq 5, L$ rescales diffusively to BM [Lawler 1980].
In $\mathbb{Z}^{4}$, with logarithmic corrections rescales to BM [Lawler].
In $\mathbb{Z}^{3},\left\{L_{n}: n \in[0, \tau]\right\}$ has a scaling limit [Kozma 2007].

In $\mathbb{Z}^{2},\left\{L_{n}: n \in[0, \tau]\right\}$ has $\operatorname{SLE}(2)$ scaling limit, UST peano curve has SLE(8) scaling limit [Lawler/Schramm/Werner 2004]. Growth exponent is 5/4 [Kenyon, Masson, Lawler].

Let $M_{n}=\left|\operatorname{LERW}\left(0, B_{E}(0, n)\right)\right|$ be the length of a LERW run from 0 to $B_{E}(0, n)^{c}$.

Theorem. $(d=2)$
[Kenyon 2000] $\lim _{n \rightarrow \infty} \frac{\log E^{0} M_{n}}{\log n}=5 / 4$
[Lawler 2014] $c_{1} n^{5 / 4} \leq E^{0} M_{n} \leq c_{2} n^{5 / 4}$

Now consider random walk on the UST.

RW on random graphs: General theory.
Let $\mathcal{G}(\omega)$ be a random graph on $(\Omega, \mathbb{P})$. Assume $\exists 0 \in \mathcal{G}(\omega)$.

Let $D \geq 1$. For $\lambda \geq 1$, we sat that $B(0, R)$ in $\mathcal{G}(\omega)$ is $\lambda$-good if

$$
\begin{array}{r}
\lambda^{-1} R^{D} \leq|B(0, R)| \leq \lambda R^{D} \\
\lambda^{-1} R \leq R_{\operatorname{eff}}\left(0, B(0, R)^{c}\right) \leq R+1
\end{array}
$$

$\lambda$-good is a nice control of the volume and resistance for $B(0, R)$.
Theorem. [Barlow/Jarai/K/Slade 2008, K/Misumi 2008]

Suppose $\exists p>0$ such that

$$
\mathbb{P}(\{\omega: B(0, R) \text { is } \lambda \text {-good. }\}) \geq 1-\lambda^{-p} \quad \forall R \geq R_{0}, \forall \lambda \geq \lambda_{0}
$$

Then $\exists \alpha_{1}, \alpha_{2}>0$ and $N(\omega), R(\omega) \in \mathbb{N}$ s.t. the following holds for $\mathbb{P}$-a.e. $\omega$ :

$$
\begin{array}{ll}
(\log n)^{-\alpha_{1}} n^{-\frac{D}{D+1}} \leq p_{2 n}^{\omega}(0,0) \leq(\log n)^{\alpha_{1}} n^{-\frac{D}{D+1}}, & \forall n \geq N(\omega) \\
(\log R)^{-\alpha_{2}} R^{D+1} \leq E_{\omega}^{0} \tau_{B(0, R)} \leq(\log R)^{\alpha_{2}} R^{D+1}, & \forall R \geq R(\omega)
\end{array}
$$

In particular,

$$
d_{s}(G):=\lim _{n \rightarrow \infty} \frac{\log p_{2 n}^{\omega}(0,0)}{\log n}=\frac{2 D}{D+1}
$$

## Examples. (See K 2014: LNM (St. Flour Lect. Notes))

$D=2$ and $d_{s}=4 / 3$

- Critical percolation on regular trees conditioned to survive forever. (Barlow/K '06)
- Infinite incipient cluster (IIC) for spread out oriented percolation for $d \geq 6$ (Barlow/Jarai/K/Slade '08)
- Invasion percolation on a regular tree. (Angel/Goodman/den Hollander/Slade '08)
- IIC for percolation on $\mathbb{Z}^{d}, d \geq 19$ (Kozma/Nachmias '09)

More general

- $\alpha$-stable Galton-Watson trees conditioned to survive forever
(Croydon/K '08) $d_{s}=2 \alpha /(2 \alpha-1)$


## VOLUME AND RESISTANCE ESTIMATES

 [BARLOW/MASSON 2010,2011]

With high probability,
$B_{E}\left(x, \lambda^{-1} R\right) \subseteq B_{\mathcal{U}}\left(x, R^{5 / 4}\right) \subseteq B_{E}(x, \lambda R)$, as $R \rightarrow \infty$ then $\lambda \rightarrow \infty$.

It follows that with high probability,

$$
\mu_{\mathcal{U}}\left(B_{\mathcal{U}}(x, R)\right) \asymp R^{8 / 5} .
$$

Also with high probability,
Resistance $\left(x, B_{\mathcal{U}}(x, R)^{c}\right) \asymp R$.
$\Rightarrow$ Exit time for intrinsic ball radius $R$ is $R^{13 / 5}$,
HK bounds $p_{2 n}^{U}(0,0) \asymp n^{-8 / 13} .\left(D=8 / 5, d_{s}=16 / 13\right)$
(Q) How about scaling limit for UST?

Barlow/Masson obtained further detailed properties.

Theorem.[Barlow/Masson 2010]

$$
\begin{aligned}
\mathbb{P}\left(M_{n}>\lambda E M_{n}\right) & \leq 2 e^{-c_{1} \lambda} \\
\mathbb{P}\left(M_{n}<\lambda^{-1} E M_{n}\right) & \leq 2 e^{-c_{2} \lambda^{c_{3}}}
\end{aligned}
$$

Theorem.[Barlow/Masson 2011]

$$
\begin{aligned}
\mathbb{P}\left(B_{\mathcal{U}}\left(0, R^{5 / 4} / \lambda\right) \not \subset B_{E}(0, R)\right) & \leq c_{4} e^{-\lambda^{2 / 3}} \\
\mathbb{P}\left(B_{E}(0, R) \not \subset B_{\mathcal{U}}\left(0, \lambda R^{5 / 4}\right)\right) & \leq c_{\epsilon} \lambda^{-4 / 15-\epsilon}
\end{aligned}
$$

While for most points $x \in \mathbb{Z}^{2}$, the balls $B_{E}(0, R)$ and $B_{\mathcal{U}}\left(0, R^{5 / 4}\right)$ will be comparable, there are neighboring points in $\mathbb{Z}^{2}$ which are far in $\mathcal{U}$.

Lemma. [Benjamini et. al. 2001]
The box $[-n, n]^{2}$ contains with probability 1 neighbouring points $x, y \in \mathbb{Z}^{2}$ with $d_{\mathcal{U}}(x, y) \geq n$.

Proof. Consider the path (in $\mathbb{Z}^{2}$ ) of length $8 n$ around the box $[-n, n]^{2}$ : If each neiboring pair were connected by a path in $\mathcal{U}$ of length less than $n$, then this path would not contain 0 . So we would obtain a loop around $0-$ which is impossible since $\mathcal{U}$ is a tree.

## UST SCALING [SCHRAMM 2000]

Consider $\mathcal{U}$ as an ensemble of paths:

$$
\mathfrak{U}=\left\{\left(a, b, \pi_{a b}\right): a, b \in \mathbb{Z}^{2}\right\}
$$

where $\pi_{a b}$ is the unique arc connecting $a$ and $b$ in $\mathcal{U}$. cf. [Aizenman/Burchard/Newman/Wilson 1999].


Scaling limit $\mathfrak{T}$ a.s. satisfies:

- each pair $a, b \in \dot{\mathbb{R}}^{2}$ connected by a path;
- if $a \neq b$, then this path is simple;
- if $a=b$, then this path is a point or a simple loop;
- the trunk, $\cup_{\mathfrak{T}} \pi_{a b} \backslash\{a, b\}$, is a dense topological tree with degree at most 3.

ISSUE : This topology does not carry information about intrinsic distance, volume, or resistance.

## GENERALISED GROMOV-HAUSDORFF TOPOLOGY

(cf. [GROMOV, LE GALL/DUQUESNE])
Define $\mathbb{T}$ to be the collection of measured, rooted, spatial trees, i.e.

$$
\left(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}}\right)
$$

where:

- $\left(\mathcal{T}, d_{\mathcal{T}}\right)$ is a locally compact real tree;
- $\mu_{\mathcal{T}}$ is a Borel measure on $\left(\mathcal{T}, d_{\mathcal{T}}\right)$;
- $\phi_{\mathcal{T}}$ is a cont. map from $\left(\mathcal{T}, d_{\mathcal{T}}\right)$ into $\mathbb{R}^{2}$;
- $\rho_{\mathcal{T}}$ is a distinguished vertex in $\mathcal{T}$.

On $\mathbb{T}_{c}$ (compact trees only), define a distance $\Delta_{c}$ by
$\inf _{\substack{Z, \psi, \psi^{\prime}, \mathcal{C}: \\\left(\rho_{\mathcal{T}}, \rho_{\mathcal{T}}^{\prime}\right) \in \mathcal{C}}}\left\{d_{P}^{Z}\left(\mu_{\mathcal{T}} \circ \psi^{-1}, \mu_{\mathcal{T}}^{\prime} \circ \psi^{\prime-1}\right)+\sup _{\left(x, x^{\prime}\right) \in \mathcal{C}}\left(d_{Z}\left(\psi(x), \psi^{\prime}\left(x^{\prime}\right)\right)+\left|\phi_{\mathcal{T}}(x)-\phi_{\mathcal{T}}^{\prime}\left(x^{\prime}\right)\right|\right)\right\}$.
Can be extended to locally compact case.

## TIGHTNESS OF UST

Theorem. If $\mathbf{P}_{\delta}$ is the law of the measured, rooted spatial tree

$$
\left(\mathcal{U}, \delta^{5 / 4} d_{\mathcal{U}}, \delta^{2} \mu_{\mathcal{U}}(\cdot), \delta \phi_{\mathcal{U}}, 0\right)
$$

under $\mathbf{P}$, then the collection $\left(\mathbf{P}_{\delta}\right)_{\delta \in(0,1)}$ is tight in $\mathcal{M}_{1}(\mathbb{T})$.
Proof involves:

- strengthening estimates of [Barlow/Masson],
- comparison of Euclidean and intrinsic distance along paths.



## UST LIMIT PROPERTIES

If $\tilde{\mathbf{P}}$ is a subsequential limit of $\left(\mathbf{P}_{\delta}\right)_{\delta \in(0,1)}$, then for $\tilde{\mathbf{P}}$-a.e.
( $\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}}$ ) it holds that:
(i) $\mu_{\mathcal{T}}$ is non-atomic, supported on the leaves of $\mathcal{T}$,
i.e. $\mu_{\mathcal{T}}\left(\mathcal{T}^{o}\right)=0$, where $\mathcal{T}^{o}:=\mathcal{T} \backslash\left\{x \in \mathcal{T}: \operatorname{deg}_{\mathcal{T}}(x)=1\right\}$;
(ii) for any $R>0$,

$$
\liminf _{r \rightarrow 0} \frac{\inf _{x \in B_{\mathcal{T}}\left(\rho_{\mathcal{T}}, R\right)} \mu_{\mathcal{T}}\left(B_{\mathcal{T}}(x, r)\right)}{r^{8 / 5}\left(\log r^{-1}\right)^{-c}}>0,
$$

(iii) $\phi_{\mathcal{T}}$ is a homeo. between $\mathcal{T}^{o}$ and $\phi_{\mathcal{T}}\left(\mathcal{T}^{o}\right)$ (dense in $\mathbb{R}^{2}$ );
(iv) $\max _{x \in \mathcal{T}} \operatorname{deg}_{\mathcal{T}}(x)=3$;
(v) $\mu_{\mathcal{T}}=\mathcal{L} \circ \phi_{\mathcal{T}}$.

To prove this, we need the following 'uniform control':

$$
\begin{aligned}
& \lim _{\eta \rightarrow 0} \liminf _{\delta \rightarrow 0} \mathbf{P}\left(\sup _{\substack{x, y \in B \mathcal{U}\left(0, c_{1} \delta^{-5 / 4} r\right): \\
d_{\mathcal{U}}(x, y) \leq c_{2} \delta^{-5 / 4} \eta}} d_{\mathcal{U}}^{S}(x, y)>\delta^{-1} \varepsilon\right)=0, \\
& \lim _{\varepsilon \rightarrow 0} \limsup _{\delta \rightarrow 0} \mathbf{P}\left(\inf _{\substack{x, y \in B_{\mathcal{U}}\left(0, \delta^{-5 / 4} r\right): \\
d_{\mathcal{U}}(x, y) \geq \delta^{-5 / 4} \eta}} d_{\mathcal{U}}^{S}(x, y)<\delta^{-1} \varepsilon\right)=0,
\end{aligned}
$$

where $d_{\mathcal{U}}^{S}=\operatorname{diam}(\gamma(x, y)$ ) (Euclidean diameter of the LERW between $x$ and $y$; Schramm's distance).
$\Rightarrow$ This involves uniform control and requires more detailed estimates than those of Barlow/Masson.

Given such generalized G-H convergence of trees, we can prove convergence of the process on the trees (generalization of the theory due to Crodon (2008)).

On the (limiting) real treee $\left(\mathcal{T}, d_{\mathcal{T}}, \mu^{\mathcal{T}}\right)$ s.t. $\mu^{\mathcal{T}}$ has full support, one can define a 'Brownian motion' $X^{\mathcal{T}}=\left(X_{t}^{\mathcal{T}}\right)_{t \geq 0}$.

- For $x, y, z \in \mathcal{T}$,

$$
P_{z}^{\mathcal{T}}\left(\tau_{x}<\tau_{y}\right)=\frac{d_{\mathcal{T}}(b(x, y, z), y)}{d_{\mathcal{T}}(x, y)} .
$$



- Mean occupation density when started at $x$ and killed at $y$,

$$
2 d_{\mathcal{T}}(b(x, y, z), y) \mu^{\mathcal{T}}(d z) .
$$

## Requirement :

$$
\liminf _{r \rightarrow 0} \frac{\inf _{x \in \mathcal{T}} \mu_{\mathcal{T}}\left(B_{\mathcal{T}}(x, r)\right)}{r^{\kappa}}>0 . \quad \exists \kappa>0
$$

## LIMITING PROCESS FOR SRW ON UST

Suppose $\left(\mathbf{P}_{\delta_{i}}\right)_{i \geq 1}$, the laws of

$$
\left(\mathcal{U}, \delta_{i}^{5 / 4} d_{\mathcal{U}}, \delta_{i}^{2} \mu_{\mathcal{U}}, \delta_{i} \phi_{\mathcal{U}}, 0\right),
$$

form a convergent sequence with limit $\tilde{\mathbf{P}}$.
Let $\left(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}}\right) \sim \tilde{\mathbf{P}}$.
It is then the case that $\mathbb{P}_{\delta_{i}}$, the annealed laws of

$$
\left(\delta_{i} X_{\delta_{i}^{-13 / 4} t}^{\mathcal{U}}\right)_{t \geq 0}
$$

converge to $\widetilde{\mathbb{P}}$, the annealed law of

$$
\left(\phi_{\mathcal{T}}\left(X_{t}^{\mathcal{T}}\right)\right)_{t \geq 0},
$$

as probability measures on $C\left(\mathbb{R}_{+}, \mathbb{R}^{2}\right)$.

## HEAT KERNEL ESTIMATES FOR SRW LIMIT

Let $R>0$. For $\tilde{P}$-a.e. realisation of $\left(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}}\right)$, there exist random constants $c_{1}, c_{2}, c_{3}, c_{4}, t_{0} \in(0, \infty)$ and deterministic constants $\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4} \in(0, \infty)$ such that the heat kernel associated with the process $X^{\mathcal{T}}$ satisfies:
$p_{t}^{\mathcal{T}}(x, y) \leq c_{1} t^{-8 / 13} \ell\left(t^{-1}\right)^{\theta_{1}} \exp \left\{-c_{2}\left(\frac{d_{\mathcal{T}}(x, y)^{13 / 5}}{t}\right)^{5 / 8} \ell\left(d_{\mathcal{T}}(x, y) / t\right)^{-\theta_{2}}\right\}$,
$p_{t}^{\mathcal{T}}(x, y) \geq c_{3} t^{-8 / 13} \ell\left(t^{-1}\right)^{-\theta_{3}} \exp \left\{-c_{4}\left(\frac{d_{\mathcal{T}}(x, y)^{13 / 5}}{t}\right)^{5 / 8} \ell\left(d_{\mathcal{T}}(x, y) / t\right)^{\theta_{4}}\right\}$,
for all $x, y \in B_{\mathcal{T}}\left(\rho_{\mathcal{T}}, R\right), t \in\left(0, t_{0}\right)$, where $\ell(x):=1 \vee \log x$.

# Random Topology, <br> Minimum Spanning Acycle, and Persistent Homology 

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## Content

1. Motivation: Materials Science
2. From Random Graph to Random Topology
3. Persistent Homology
4. Main Result: Generalization of Frieze's zeta(3)-Limit Theorem


- Atomic configurations of crystal, glass, and liquid states of SiO2
- Geometric properties of glass are not well-understood
- Industrially important (solar energy panel, DVD, BD, etc)
- Can we distinguish between crystal and glass?


## Persistence Diagram (PD)





- 1st Persistence Diagrams (PDs) distinguish three states (info. of persistent ring)
- Inverse problem from PDs to atomic configurations clarifies new geometric characterizations of glass
What are the topological properties in liquid (random) state?
Brief Sketch of PDs for points data

blow up balls and detect appearance and disappearance of topological features



## Content

## 1. Motivation: Materials Science

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## Erdös-Rényi Random Graph

- $K_{n}=V_{n} \sqcup E_{n}$ : complete graph with $n$ vertices

$$
V_{n}=\{1, \ldots, n\}, E_{n}=\{|i j|: i<j\}
$$

- $t_{e} \in[0,1]$ : i.i.d. uniform random variable for $e \in E_{n}$

Erdös-Rényi Random Graph (Process)

$$
K_{n}(t)=V_{n} \sqcup\left\{e \in E_{n}: t_{e} \leq t\right\}
$$


2. From Random Graph to Random Topology

## Erdös-Rényi Random Graph

Thm(Erdös-Rényi): For $t=(\log n+\omega(n)) / n$, $\omega(n) \rightarrow \infty \Longrightarrow K_{n}(t)$ is a.s. connected as $n \rightarrow \infty$
$\omega(n) \rightarrow-\infty \Longrightarrow K_{n}(t)$ is a.s. disconnected as $n \rightarrow \infty$

- rewriting by reduced homology

$$
\begin{aligned}
& \omega(n) \rightarrow \infty \Longrightarrow \tilde{H}_{0}\left(K_{n}(t)\right)=0 \\
& \omega(n) \rightarrow-\infty \Longrightarrow \tilde{H}_{0}\left(K_{n}(t)\right) \neq 0
\end{aligned}
$$

Random Graph $G \longrightarrow H_{0}(G), H_{1}(G)$ (connectivity, cycle)

In this talk, we study
Random Simplicial Complex X
$H_{k}(X)$ (higher topological features)
2. From Random Graph to Random Topology

## Frieze's zeta(3)-Limit Theorem

- $K_{n}(t)$ : ER random graph $\bullet K_{n}=K_{n}(1)$ : complete graph
- $T \subset K_{n}$ is a spanning tree if $T$ is a tree containing all vertices
- $\mathcal{S}^{(1)}$ : the set of spanning trees in $K_{n}$


Cayley's formula: $\left|\mathcal{S}^{(1)}\right|=n^{n-2}$

- $T \in \mathcal{S}^{(1)}$ is the minimum spanning tree if the weight $\mathrm{wt}(T)=\sum_{e \in T} t_{e}$ is minimum in $\mathcal{S}^{(1)}$


Frieze's zeta(3)-limit thm:

$$
\mathbb{E}\left[\min _{T \in \mathcal{S}^{(1)}} \operatorname{wt}(T)\right] \rightarrow \zeta(3)=1.202 \cdots \quad \text { as } n \rightarrow \infty
$$

2. From Random Graph to Random Topology

## Persistence and Frieze's Thm

Key observation: $\min _{T \in \mathcal{S}^{(1)}} \mathrm{wt}(T)=\int_{0}^{1} \beta_{0}(t) d t \star$


$\int_{0}^{1} \beta_{0}(t) d t=$ lifetime sum
Generalization of Frieze's Thm

- random graph $\longrightarrow$ random simplicial complex
- spanning tree $\longrightarrow$ spanning acycle
- $\quad \longrightarrow$ identity using persistent homology

2. From Random Graph to Random Topology

## Simplicial Complex

- Simplicial Complex $X$ on the vertices $\{1, \ldots, n\}$ :
- a collection of nonempty subsets
$-\{i\} \in X, \quad \forall i \in\{1, \ldots, n\}$
- $\sigma \in X, \tau \subset \sigma \Longrightarrow \tau \in X$
- $\sigma=\left\{i_{0}, \ldots, i_{k}\right\}$ : $k$-simplex, $\operatorname{dim} \sigma=k$
- $\operatorname{dim} X=\max _{\sigma \in X} \operatorname{dim} \sigma$
- $X_{k}=\{\sigma \in X: \operatorname{dim} \sigma=k\}, X^{(k)}=\sqcup_{i=0}^{k} X_{i}$
- graph $=1$ dim simplicial complex



## Random Simplicial Complex 1

- Random Clique Complex CL $(G)$


## $\mathrm{CL}(G)$ : clique of a graph $G$

$$
\mathrm{CL}(G) \ni\left\{i_{1}, \ldots, i_{k}\right\} \Longleftrightarrow\left\{i_{s} i_{t}\right\} \in G, \forall s, t \in\{1, \ldots, k\}
$$



Take ER random graph $G=K_{n}(t)$, then
Kahle (2009): For random clique complex $\mathrm{CL}\left(K_{n}(t)\right)$,

$$
\begin{gathered}
t=\left(\frac{(2 k+1) \log n+\omega(n)}{n}\right)^{1 /(2 k+1)} \Longrightarrow H_{i}\left(\mathrm{CL}\left(K_{n}(t)\right)\right)=0 \\
\omega(n) \rightarrow \infty
\end{gathered}
$$

$k=0$ recovers one-side of the ER theorem

## Random Simplicial Complex 2

- $\Delta_{n-1}:(n-1)$-dim maximal simpl cplex on $\{1, \ldots, n\}$

- $t_{\sigma} \in[0,1]$ : i.i.d. uniform random variable for $\sigma \in\left(\Delta_{n-1}\right)_{d}$

Linial-Meshulam Process

$$
\mathcal{K}^{(d)}(t)=\Delta_{n-1}^{(d-1)} \sqcup\left\{\sigma \in\left(\Delta_{n-1}\right)_{d}: t_{\sigma} \leq t\right\}
$$

- $d=1$ gives ER random graph
higher dim generalization of ER graph process


$$
d=2, n=5
$$



## Spanning Acycle

- $X$ : simplicial complex
we use
integer coefficient reduced homology
- For a set of k-splexs $S \subset X_{k}$, define $X_{S}=S \sqcup X^{(k-1)}$
$S$ is called a $k$-spanning acycle if
(a) $H_{k}\left(X_{S}\right)=0$
(b) $\left|H_{k-1}\left(X_{S}\right)\right|<\infty$
$\mathcal{S}^{(k)}$ : the set of $k$-spanning acycles
- For $k=1$, the graph $X_{S}$ has (a) no cycles and is (b) connected, meaning a spanning tree.
- This definition is originally introduced by Kalai. (also relating to simplicial spanning tree, k-bases, etc)
- Set $\gamma_{k}(X)=\left|X_{k}^{(k)}\right|-\beta_{k}\left(X^{(k)}\right)+\beta_{k-1}\left(X^{(k)}\right)$

Any two of (a), (b), and (c) $|S|=\gamma_{k}(X)$ imply the third.
2. From Random Graph to Random Topology

## Minimum Spanning Acycle

- $\{X(t)\}_{t}$ : random filtration of simpl. cplx. (e.g., clique ER process, LM process)
- $t_{\sigma}$ : birth time of the simplex, i.e.,

$$
t_{\sigma}=\min \{t: \sigma \in X(t)\}
$$

- $S \in \mathcal{S}^{(k)}$ is a minimum spanning acycle if the weight $\operatorname{wt}(S)=\sum_{\sigma \in S} t_{\sigma}$ is minimum in $\mathcal{S}^{(k)}$


## Content

## 1. Motivation: Materials Science

2. From Random Graph to Random Topology
3. Persistent Homology
4. Main Result: Generalization of Frieze's zeta(3)-Limit Theorem
5. Persistent Homology

## (Persistent) Homology 1

Quick review of simplicial homology

- $X$ : simplicial complex
- chain complex in $\mathbb{Z}$-coefficient

$$
\begin{array}{r}
\cdots \longrightarrow C_{k+1}(X) \xrightarrow{\partial_{k+1}} C_{k}(X) \xrightarrow{\partial_{k}} C_{k-1}(X) \longrightarrow \cdots \\
\partial_{k}\left\langle v_{0} \cdots v_{k}\right\rangle=\sum_{i}(-1)^{i}\left\langle v_{0} \cdots \widehat{v_{i}} \cdots v_{k}\right\rangle: \\
\left(\partial_{k} \circ \partial_{k+1}=0\right)
\end{array}
$$

- $Z_{k}(X)=\operatorname{ker} \partial_{k}$ : k-cycle, $B_{k}(X)=\operatorname{im} \partial_{k+1}$ : k-boundary betti number
- $H_{k}(X)=Z_{k}(X) / B_{k}(X) \simeq \mathbb{Z}^{\beta_{k}} \oplus T_{k} \quad$ (Z्Z-module)

Homology to Persistent Homology
Replace $\mathbb{Z}$-module with graded $K\left[\mathbb{R}_{\geq 0}\right]$-module using filtration

## 3. Persistent Homology

## Persistent Homology 2

- $\mathcal{X}=\left\{X(t): t \in \mathbb{R}_{\geq 0}\right\}$ : (right cont.) filtration of a simpl. cplex $X$ $\left(X(t) \subset X(s) \subset X, t \leq s, \quad X(t)=\bigcap_{t<s} X(s)\right)$
Assume there exists a saturation time $T$ s.t. $X(T)=X$
- $t_{\sigma}$ : birth time of the simplex $\sigma$, i.e., $t_{\sigma}=\min \{t: \sigma \in X(t)\}$
- $K$ : field with $\operatorname{char}(K)=0$
$K\left[\mathbb{R}_{\geq 0}\right]$ : monoid ring, i.e., the set of formal polynomials with

$$
a z^{t} \cdot b z^{s}=a b z^{t+s}, \quad a, b \in K, t, s \in \mathbb{R}_{\geq 0}
$$

- $C_{k}(X(t))$ : $K$-vector space spanned by $k$-simplices in $X(t)$
- graded $K\left[\mathbb{R}_{\geq 0}\right]$-module

$$
\begin{gathered}
C_{k}(\mathcal{X})=\bigoplus_{t \in \mathbb{R}_{\geq 0}} C_{k}(X(t))=\left\{\left(c_{t}\right): c_{t} \in C_{k}(X(t)), t \in \mathbb{R}_{\geq 0}\right\} \\
z^{s} \cdot\left(c_{t}\right)=\left(c_{t}^{\prime}\right), \quad c_{t}^{\prime}=\left\{\begin{array}{cl}
c_{t-s}, & t \geq s \\
0, & t<s
\end{array}\right.
\end{gathered}
$$

3. Persistent Homology

## Persistent Homology 3

- For oriented simplex $\langle\sigma\rangle$, define $\langle\langle\sigma\rangle\rangle=\left(c_{t}\right), \quad c_{t}=\left\{\begin{array}{cc}\langle\sigma\rangle, & t=t_{\sigma} \\ 0, & t \neq t_{\sigma}\end{array}\right.$
$\longrightarrow \Xi_{k}=\left\{\langle\langle\sigma\rangle\rangle: \sigma \in X_{k}\right\}$ forms a basis of $C_{k}(\mathcal{X})$
- boundary map: $\delta_{k}\langle\langle\sigma\rangle\rangle=\sum_{j=0}^{k}(-1)^{j} z^{t_{\sigma}-t_{\sigma_{j}}}\left\langle\left\langle\sigma_{j}\right\rangle\right\rangle$

$$
\langle\sigma\rangle=\left\langle v_{0} \cdots v_{k}\right\rangle, \quad \sigma_{j}=\sigma \backslash\left\{v_{j}\right\}
$$

$$
B_{k}(\mathcal{X})=\operatorname{im} \delta_{k+1} \subset Z_{k}(\mathcal{X})=\operatorname{ker} \delta_{k}
$$

- persistent homology: $H_{k}(\mathcal{X})=Z_{k}(\mathcal{X}) / B_{k}(\mathcal{X}) \quad$ (graded $K[\mathbb{R} \geq 0]$-module)
- structure thm of PID module (with saturation) implies

$$
H_{k}(\mathcal{X}) \simeq \bigoplus_{i=1}^{p} I\left[b_{i}, d_{i}\right]
$$

$$
\begin{aligned}
& I\left[b_{i}, d_{i}\right]=\left(z^{b_{i}}\right) /\left(z^{d_{i}}\right): \text { interval representation } \\
& \left(z^{a}\right)=\left\{z^{a} f(z): f(z) \in K\left[\mathbb{R}_{\geq 0}\right]\right\}: \text { ideal generated by } z^{a}
\end{aligned}
$$

3. Persistent Homology

## Persistence Diagram 1

Interval decomp: $H_{k}(\mathcal{X}) \simeq \bigoplus_{i=1}^{p} I\left[b_{i}, d_{i}\right]$

- $I[b, d]$ represents appearance and disappearance of birth time death time a topological feature at $t=b, d$ in $\mathcal{X}=\{X(t)\}_{t}$

- $D_{k}(\mathcal{X})=\left\{\left(b_{i}, d_{i}\right) \in \mathbb{R}_{\geq 0}^{2}: i=1, \ldots, p\right\}:$ persistence diagram


3. Persistent Homology

## Persistence Diagram 2

- persistence diagram (multiset):

$$
D_{k}(\mathcal{X})=\left\{\left(b_{i}, d_{i}\right) \in \mathbb{R}_{\geq 0}^{2}: i=1, \ldots, p\right\}
$$

- persistence diagram (counting measure):

$$
\xi_{k}=\sum_{0 \leq x<y<\infty} m_{(x, y)} \delta_{(x, y)}
$$

where $\delta_{(x, y)}$ is the delta measure and


$$
m_{(x, y)}=\left|\left\{1 \leq i \leq p \mid\left(b_{i}, d_{i}\right)=(x, y)\right\}\right|
$$

- $\beta_{k}(t)=\beta_{k}(X(t))=\xi_{k}([0, t] \times[t, \infty))$

3. Persistent Homology

## Persistence Diagram 2

- Let $L_{k}=\sum_{i}\left(d_{i}-b_{i}\right)$ be the lifetime sum

$$
\longrightarrow L_{k}=\int_{[0, \infty)} \beta_{k}(t) d t
$$

pf) By Fubini,


$$
\begin{aligned}
L_{k} & =\int_{\Delta}(y-x) \xi_{k}(d x d y) \\
& =\int_{\Delta} \xi_{k}(d x d y) \int_{[0, \infty]} I(0 \leq x \leq t \leq y \leq \infty) d t \\
& =\int_{[0, \infty]} d t \int_{\Delta} I_{[0, t]}(x) I_{[t, \infty]}(y) \xi_{k}(d x d y) \\
& =\int_{[0, \infty]} \beta_{k}(t) d t .
\end{aligned}
$$



## Content

## 1. Motivation: Materials Science <br> 2. From Random Graph to Random Topology <br> 3. Persistent Homology <br> 4. Main Result: Generalization of Frieze's zeta(3)-Limit Theorem

4. Generalization of Frieze's zeta(3)-Limit Theorem

## Algebraic Formula of $\mathrm{L}_{\mathrm{d}-1}$

key observation of ER graph process:

$$
L_{0}=\min _{T \in \mathcal{S}^{(1)}} \mathrm{wt}(T)=\int_{0}^{\infty} \beta_{0}(t) d t \star
$$




Theorem: Let $X$ be a simpl. cplx $(1 \leq d \leq \operatorname{dim} X)$ with

$$
\beta_{d-1}\left(X^{(d)}\right)=\beta_{d-2}\left(X^{(d-1)}\right)=0
$$

and $\mathcal{X}=\left\{X(t): t \in \mathbb{R}_{\geq 0}\right\}$ be a filtration of $X$. Then,

$$
L_{d-1}=\min _{T \in \mathcal{S}^{(d)}} \operatorname{wt}(T)-\max _{S \in \mathcal{S}^{(d-1)}} \operatorname{wt}\left(X_{d-1} \backslash S\right)=\int_{0}^{\infty} \beta_{d-1}(t) d t
$$

Remark: $\mathrm{d}=1$ recovers $\star$
4. Generalization of Frieze's zeta(3)-Limit Theorem

## Sketch of Proof: Algebraic Formula of $L_{d-1}$

$M$ : matrix form of the $\mathbf{d}$-boundary map of P.H. under standard bases (entries are $\pm z^{t}$ )
$D=\left.M\right|_{z=1}$ : matrix form of the d-boundary map of $\mathbf{H}$
For $K \subset X_{d-1}, S \subset X_{d}, M_{K}, D_{K S}$ (etc) mean the restriction to $K, S$
 where $\tau(S)=\mathrm{wt}(S)-\min _{S \in \mathcal{S}^{(d)}} \mathrm{wt}(S), \quad e(K)=\min _{S \in \mathcal{S}^{(d)}} \mathrm{wt}(S)-\mathrm{wt}(K)$

## Pf) Binet-Cauchy.

Prop: For the elementary divisors $d_{1}=z^{e_{1}}, \ldots, d_{r}=z^{e_{r}}$ of $M$,

$$
\min _{K \in \mathcal{S}_{c}^{(d-1)}} e(K)=e_{1}+\cdots+e_{r}
$$

where $\mathcal{S}_{c}^{(d-1)}=\left\{X_{d-1} \backslash L: L \in \mathcal{S}^{(d-1)}\right\}$
Pf) Use $d_{1} \cdots d_{r}=\Delta_{r}(M)$ (deteminantal divisor) and

$$
D_{K S} \neq 0 \Longleftrightarrow S \in \mathcal{S}^{(d)}, K \in \mathcal{S}_{c}^{(d-1)}
$$

$L_{d-1}=e_{1}+\cdots+e_{r}$ leads to
$L_{d-1}=\min _{T \in \mathcal{S}^{(d)}} \mathrm{wt}(T)-\max _{S \in \mathcal{S}^{(d-1)}} \mathrm{wt}\left(X_{d-1} \backslash S\right)$
4. Generalization of Frieze's zeta(3)-Limit Theorem

## Main Result

Frieze's zeta(3)-limit thm:

$$
\mathbb{E}\left[L_{0}\right]=\mathbb{E}\left[\min _{T \in \mathcal{S}^{(1)}} \mathrm{wt}(T)\right] \rightarrow \zeta(3)=1.202 \cdots \text { as } n \rightarrow \infty
$$

d-Linial-Meshulam Process: $\mathcal{K}^{(d)}(t)=\Delta_{n-1}^{(d-1)} \sqcup\left\{\sigma \in\left(\Delta_{n-1}\right)_{d}: t_{\sigma} \leq t\right\}$


Theorem: For d-LM process, $\mathbb{E}\left[L_{d-1}\right]=O\left(n^{d-1}\right)$, as $n \rightarrow \infty$
Clique Complex Process: $\Delta_{n-1}^{(0)}=\mathcal{C}(0) \subset \mathcal{C}(t) \subset \mathcal{C}(1)=\Delta_{n-1}$
(clique of ER process) $\quad$ where $\mathcal{C}(t)=\mathrm{Cl}\left(\mathcal{K}^{(1)}(t)\right), \quad 0 \leq t \leq 1$
Theorem: For clique complex process,

$$
\begin{array}{ll}
c n^{d-1} \leq \mathbb{E}\left[L_{d-1}\right] \leq C n^{d-1} \log n & (d=1,2) \\
c n^{\frac{(d+2)(d-1)}{2 d}} \leq \mathbb{E}\left[L_{d-1}\right] \leq C n^{d-1} & (d \geq 3) \\
\hline
\end{array}
$$

4. Generalization of Frieze's zeta(3)-Limit Theorem

## Sketch of Proof: d-LM process

$$
\begin{gathered}
\text { Prove: } \mathbb{E}\left[L_{d-1}\right]=O\left(n^{d-1}\right), \text { as } n \rightarrow \infty \\
L_{d-1}=\min _{T \in \mathcal{S}^{(d)}} \mathrm{wt}(T)-\max _{S \in \mathcal{S}^{(d-1)}} \mathrm{wt}\left(X_{d-1} \backslash S\right)=\int_{0}^{1} \beta_{d-1}(t) d t
\end{gathered}
$$

Let $Y_{t}=\mathcal{K}^{(d)}(t)$.

- For a lower bound, use $\square$ (with complete (d-1)-skeleton)

$$
L_{d-1}=\mathrm{wt}(T) \geq \sum_{i=1}^{|T|} u_{i} \quad N=\binom{n}{d+1}
$$

$T$ : minimum spanning acycle

$$
u_{1} \leq u_{2} \leq \cdots \leq u_{N}: \text { rearrangement of }\left\{t_{\sigma}: \sigma \in\left(\Delta_{n-1}\right)_{d}\right\}
$$

$$
\longrightarrow \mathbb{E}\left[L_{d-1}\right] \geq \sum_{i=1}^{|T|} \mathbb{E}\left[u_{i}\right]=\sum_{i=1}^{|T|} \frac{i}{N+1} \sim \frac{d+1}{2 d!} n^{d-1}
$$

- For an upper bound, use

$$
\mathbb{E}\left[L_{d-1}\right]=\int_{0}^{1} \mathbb{E}\left[\beta_{d-1}(t)\right] d t \leq \frac{d+1}{n} \int_{0}^{1} \mathbb{E}\left|\mathcal{R}_{d}\left(Y_{t}\right)\right| d t \leq 8 \frac{d+1}{n}\binom{n}{d} \sim \frac{8(d+1)}{d!} n^{d-1}
$$

4. Generalization of Frieze's zeta(3)-Limit Theorem

## Sketch of Proof: d-LM process

## - For an upper bound, use $\square$

$$
\begin{aligned}
& \mathbb{E}\left[L_{d-1}\right]=\int_{0}^{1} \mathbb{E}\left[\beta_{d-1}(t)\right] d t \leq \frac{d+1}{n} \int_{0}^{1} \mathbb{E}\left|\mathcal{R}_{d}\left(Y_{t}\right)\right| d t \leq 8 \frac{d+1}{n}\binom{n}{d} \sim \frac{8(d+1)}{d!} n^{d-1} \\
& \leq \quad \mathcal{R}_{d}(Y)=\left\{\sigma \in\left(\Delta_{n-1}\right)_{d}: \beta_{d-1}(Y \cup \sigma)=\beta_{d-1}(Y)-1\right\} \\
& \mathcal{S}_{d}(Y)=\left\{\sigma \in\left(\Delta_{n-1}\right)_{d}: \beta_{d-1}(Y \cup \sigma)=\beta_{d-1}(Y)\right\}
\end{aligned}
$$

Lemma: $\beta_{d-1}(Y) \leq \frac{d+1}{n}\left|\mathcal{R}_{d}(Y)\right|$

$$
\text { pf) } \begin{aligned}
& \bar{Y}:=Y \cup \mathcal{S}_{d}(Y) \text {. Then, } \\
& \beta_{d-1}(Y)=\beta_{d-1}(\bar{Y})=\binom{n-1}{d}-\operatorname{rank} \partial_{\bar{Y}, d} \leq\binom{ n-1}{d}-\frac{d+1}{n}\left|\bar{Y}_{d}\right|
\end{aligned}
$$

$\leq \quad \mathcal{C}_{n}^{(d)}$ : the set of $d$-dim simpl. cplx on $n$ vertices with
$(d-1)$-complete skeleton
$\mathcal{C}_{n, m}^{(d)}=\left\{Y \in \mathcal{C}_{n}^{(d)}:\left|Y_{d}\right|=m\right\} \quad Y^{(d)}(n, m)$ : uniform dist on $\mathcal{C}_{n, m}^{(d)}$
First, we show $\int_{0}^{1} \mathbb{E}\left|\mathcal{R}_{d}\left(Y_{t}\right)\right| d t \leq \frac{m}{1-\rho_{n, m}}, 1 \leq \forall m \leq N$ where $\rho_{n, m}=\mathbb{P}\left(\sigma \in \mathcal{R}_{d}(Z)\right), \quad Z \sim Y^{(d)}(n, m)$
4. Generalization of Frieze's zeta(3)-Limit Theorem

## Sketch of Proof: d-LM process

Let us set $m_{c}(n)=\min \left\{m \leq N: \rho_{n, m} \leq c\right\}$.

By setting $c=1 / 2$

Hoffman-Kahle-Paquette $m_{1 / 2}(n) \leq 4\binom{n}{d}$

$$
\int_{0}^{1} \mathbb{E}\left|\mathcal{R}_{d}\left(Y_{t}\right)\right| d t \leq \frac{m}{1-\rho_{n, m}} \leq 2 m_{1 / 2}(n) \leq 8\binom{n}{d} .
$$

Hence, we have
$\mathbb{E}\left[L_{d-1}\right]=\int_{0}^{1} \mathbb{E}\left[\beta_{d-1}(t)\right] d t \leq \frac{d+1}{n} \int_{0}^{1} \mathbb{E}\left|\mathcal{R}_{d}\left(Y_{t}\right)\right| d t \leq 8 \frac{d+1}{n}\binom{n}{d} \sim \frac{8(d+1)}{d!} n^{d-1}$

## 4. Generalization of Frieze's zeta(3)-Limit Theorem

## Sketch of Proof: Clique complex process

Prove: $c n^{d-1} \leq \mathbb{E}\left[L_{d-1}\right] \leq C n^{d-1} \log n \quad(d=1,2)$

$$
c n^{\frac{(d+2)(d-1)}{2 d}} \leq \mathbb{E}\left[L_{d-1}\right] \leq C n^{d-1} \quad(d \geq 3)
$$

$$
L_{d-1}=\min _{T \in \mathcal{S}^{(d)}} \mathrm{wt}(T)-\max _{S \in \mathcal{S}^{(d-1)}} \mathrm{wt}\left(X_{d-1} \backslash S\right)=\int_{0}^{1} \beta_{d-1}(t) d t
$$

For both upper \& lower bounds, use $\square$ with the Morse inequality

$$
-f_{d-2}(t)+f_{d-1}(t)-f_{d}(t) \leq \beta_{d-1}(t) \leq f_{d-1}(t)
$$

where $\beta_{i}(t)=\beta_{i}(\mathcal{C}(t))$ and $f_{i}(t)=\left|\mathcal{C}(t)_{i}\right|$

- Lower bound: $\mathbb{E}\left[f_{j}(t)\right]=\binom{n}{j+1} t^{\binom{j+1}{2}}$ and straightforward cal.
- Upper bound: Discrete Morse Theory, i.e., reduce $f_{d-1}(t)$ by critical cells defined by a lex-order Morse function


## Further Discussions

- Limiting constant of d-LM process

$$
I_{d-1}=\lim _{n \rightarrow \infty} \frac{1}{n^{d-1}} \mathbb{E}\left[L_{d-1}\right]
$$

- Central limit theorem of $L_{d-1}$ in d-LM process
- Limit theorem of persistence diagram
- Order in the clique complex process
- Asymptotics of $\ell^{2}$-norm and persistence landscape
- Wilson's algorithm for "uniform" spanning acycles

> Thank you very much

# Spectrum of the quantum Rabi model and representation theory 

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30 April, 2015

## Non-commutative harmonic oscillators I

The story has begun by the study of non-commutative harmonic oscillators ( NcHO ):
The normal form the Hamiltonian $Q_{(\alpha, \beta)}(x, D)$ of NcHO is given by

$$
Q_{(\alpha, \beta)}(x, D)=A\left(-\frac{1}{2} \frac{d^{2}}{d x^{2}}+\frac{1}{2} x^{2}\right)+J\left(x \frac{d}{d x}+\frac{1}{2}\right)
$$

where $A=\left[\begin{array}{ll}\alpha & 0 \\ 0 & \beta\end{array}\right], J=\left[\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right]$.
A. Parmeggiani and M. Wakayama:

Oscillator representations and systems of ordinary differential equations, Proc. Natl. Acad. Sci. USA 98 (2001), 26-30.
Non-commutative harmonic oscillators-I, II, Corrigenda and remarks to I, Forum. Math. 14 (2002), 539-604, 669-690, ibid 15 (2003), 955-963.

## Non-commutative harmonic oscillators II

- Development of the study of NcHO including number theoretic investigations can be found in the book and its references:
A. Parmeggiani, Spectral Theory of Non-commutative Harmonic Oscillators: An Introduction. LNM. 1992, Springer, 2010.
- There is a second degree element $\mathcal{R}$ of $U\left(\mathfrak{s l}_{2}\right)$ such that the image of $\mathcal{R}$ under the oscillator representation of the Lie algebra $\mathfrak{s l}_{2}$ gives the NcHO :
H. Ochiai, Non-commutative harmonic oscillators and Fuchsian ordinary differential operators, Comm. Math. Phys. 217 (2001), 357-373.


## Purposes of the talk (1)

- Draw the following pictures:

1) the image of $\mathcal{R}$ under the non-unitary principal series representation of $\mathfrak{s l}_{2}$ gives a Heun ODE. Moreover, this Heun ODE provides the Heun picture of the quantum Rabi model under suitable (including a parameter of the representation) confluent procedure [2].
2) there exists another second degree element $\mathcal{K}$ of $U\left(\mathfrak{s l}_{2}\right)$ such that the image of $\mathcal{K}$ under non-unitary principal series representation of $\mathfrak{s l}_{2}$ gives the quantum Rabi model [1].

## Purposes of the talk (II)

- Describe the representation theoretic explanation (finite dimensional representations of $\mathfrak{s l}_{2}$ ) of the degenerate spectrum of the quantum Rabi model, which was described by Küs:
M. Kus: On the spectrum of a two-level system, J. Math. Phys., 26 (1985) 2792-2795.
- Provide a conjectural statement about the non-degenerate exceptional spectrum (relating the discrete series representation of $\mathfrak{s l}_{2}$ ). Numerical evidence is found in
A.J. Maciejewski, M. Przybylska and T. Stachowiak: Full spectrum of the Rabi model, Phys. Letter A 378, (2014), 16-20.


## the quantum Rabi model

The quantum Rabi model is defined by the Hamiltonian

$$
H_{\mathrm{Rabi}} / \hbar=\omega \psi^{\dagger} \psi+\Delta \sigma_{z}+g \sigma_{x}\left(\psi^{\dagger}+\psi\right)
$$

Here $\psi=\left(x+\partial_{x}\right) / \sqrt{2}\left(\right.$ resp. $\left.\psi^{\dagger}=\left(x-\partial_{x}\right) / \sqrt{2}\right)$ is the annihilation (resp. creation) operator for a bosonic mode of frequency $\omega$, $\sigma_{x}=\left[\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right], \sigma_{y}=\left[\begin{array}{cc}0 & -i \\ i & 0\end{array}\right], \sigma_{z}=\left[\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right]$ are the Pauli matrices for the two-level system, $2 \Delta$ is the energy difference between the two levels, and $g$ denotes the coupling strength between the two-level system and the bosonic mode. For simplicity and without loss of generality we may set $\hbar=1$ and $\omega=1$.


Figure: Courtesy of APS/Alan Stonebraker in E. Solano, Viewpoint: The dialogue between quantum light and matter, Physics 4, 68 (2011).

The Rabi model describes the simplest interaction between quantum light and matter. The model considers a two-level atom coupled to a quantized, single-mode harmonic oscillator.

## Spectrum of the quantum Rabi model

Spectrum of $H_{\text {Rabi }}$ is classified as

$$
\operatorname{Spec}\left(H_{\text {Rabi }}\right)=\{\text { Regular eigen. }\} \sqcup\{\text { Exceptional eigen. }\}
$$

Exceptional eigenvalues $\lambda$ are of the form $\lambda=N-g^{2}(N \in \mathbb{Z})$. Regular eigenvalues are the ones not of the form. Moreover,
\{Exceptional eigen.\}
$=\{$ non - degenerate Exceptional eigen. $\} \sqcup\{$ degenerate Exceptional eigen. $\}$
Degenerate exceptional eigenvalues $\left(\lambda=N-g^{2}\right.$ for some $\left.N \in \mathbb{N}\right)$ are described by Küs (1985).
The regular spectrum was described by D. Braak for the first time in about 70 years after the proposition of the quantum Rabi model:
D. Braak, On the Integrability of the Rabi Model, Phys. Rev. Lett. 107 (2011), 100401-100404.

## The element $\mathcal{R} \in U\left(\mathfrak{s l}_{2}\right)$

For the triplet $(\kappa, \varepsilon, \nu) \in \mathbb{R}_{>0}^{3}$, define a second order element $\mathcal{R}$ of the universal enveloping algebra $\mathcal{U}\left(\mathfrak{s l}_{2}\right)$ of $\mathfrak{s l}_{2}$ by

$$
\mathcal{R}:=\frac{2}{\sinh 2 \kappa}\left\{[(\sinh 2 \kappa)(E-F)-(\cosh 2 \kappa) H+\nu](H-\nu)+(\varepsilon \nu)^{2}\right\} .
$$

Here $H, E$ and $F$ be the standard generators of the Lie algebra $\mathfrak{s l}_{2}$ defined by

$$
H=\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right], \quad E=\left[\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right], \quad F=\left[\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right] .
$$

They satisfy the commutation relations

$$
[H, E]=2 E,[H, F]=-2 F,[E, F]=H .
$$

## NcHO and $\mathcal{R}$

Suppose that $\alpha \neq \beta$. Determine the triplet $(\kappa, \varepsilon, \nu) \in \mathbb{R}_{>0}^{3}$ by the formulas

$$
\cosh \kappa=\sqrt{\frac{\alpha \beta}{\alpha \beta-1}}, \quad \sinh \kappa=\frac{1}{\sqrt{\alpha \beta-1}}, \quad \varepsilon=\left|\frac{\alpha-\beta}{\alpha+\beta}\right|, \quad \nu=\frac{\alpha+\beta}{2 \sqrt{\alpha \beta(\alpha \beta-1)}} \lambda .
$$

Then the eigenvalue problem $Q \varphi=\lambda \varphi\left(\varphi \in L^{2}\left(\mathbb{R}, \mathbb{C}^{2}\right)\right)$ is equivalent to the equation $\pi^{\prime}(\mathcal{R}) u=0(u \in \overline{\mathbb{C}}[y])$. Here $\pi^{\prime}$ is the oscillator representation of $\mathfrak{s l}_{2}$ defined on the space $\mathbb{C}[y]$ by

$$
\pi^{\prime}(H)=y \partial_{y}+1 / 2, \pi^{\prime}(E)=y^{2} / 2, \pi^{\prime}(F)=-\partial_{y}^{2} / 2
$$

## The element $\mathcal{K} \in U\left(\mathfrak{s L}_{2}\right)$

$$
\begin{aligned}
\mathcal{K}:= & {\left[\frac{1}{2} H-E+1-\frac{\lambda+g^{2}}{2}\right]\left[F+4 g^{2}\right]+\left(\frac{1}{2}-\frac{\lambda+g^{2}}{2}\right)\left[H-\frac{1}{2}\right] . } \\
& \Lambda_{a}:=4 g^{2}\left(\frac{1}{2} a+1-\frac{\lambda+g^{2}}{2}\right)+\left(\frac{1}{2}-\frac{\lambda+g^{2}}{2}\right)\left(a-\frac{1}{2}\right) .
\end{aligned}
$$

Here $a$ is a parameter of non-unitary principal series $\varpi_{a}$ of the $\mathfrak{s l}_{2}$. (See [1].) Under the representation $\varpi_{a}$ of $\mathfrak{s l}_{2}$ we have the confluent Heun picture of the quantum Rabi model as ([1])

$$
\mathcal{H}_{1}^{\text {Rabi }}(\lambda)=\{x(x-1)\}^{-1} x^{-\frac{1}{2}\left(a-\frac{1}{2}\right)}\left(\varpi_{a}(\mathcal{K})-\Lambda_{a}\right) x^{\frac{1}{2}\left(a-\frac{1}{2}\right)} \quad\left(a:=-\left(\lambda+g^{2}\right)\right),
$$

where

$$
\mathcal{H}_{1}^{\text {Rabi }}(\lambda):=\frac{d^{2}}{d x^{2}}+\left\{-4 g^{2}+\frac{1-\left(\lambda+g^{2}\right)}{x}+\frac{1-\left(\lambda+g^{2}+1\right)}{x-1}\right\} \frac{d}{d x}+\frac{4 g^{2}\left(\lambda+g^{2}\right) x+\mu}{x(x-1)}
$$

with the accessory parameter

$$
\mu:=\left(\lambda+g^{2}\right)^{2}-4 g^{2}\left(\lambda+g^{2}\right)-\Delta^{2} .
$$

## References

The talk presented in the workshop is based on the results in the following works:
[1] M. Wakayama and T. Yamasaki, The quantum Rabi model and Lie algebra representations of $\mathfrak{s l}_{2}$, J. Phys. A: Math. Theor. 47 (2014), 335203 (17pp).
[2] M. Wakayama, Equivalence between the eigenvalue problem of non-commutative harmonic oscillators and existence of holomorphic solutions of Heun differential equations, eigenstates degeneration and the Rabi model, Int. Math. Res. Not., doi:10.1093/imrn/RNV145 (May 25, 2015) (36pp).

# Abelian Sandpile Models in Statistical Mechanics - Dissipative Abelian Sandpile Models - 

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27 May 2015


#### Abstract

We introduce a family of abelian sandpile models with two parameters $n, m \in \mathbb{N}$ defined on finite lattices on $d$-dimensional torus. Sites with $2 d n+m$ or more grains of sand are unstable and topple, and in each toppling $m$ grains dissipate from the system. Because of dissipation in bulk, the models are well-defined on the shift-invariant lattices and the infinitevolume limit of systems can be taken. From the determinantal expressions, we obtain the asymptotic forms of the avalanche propagators and the height- $(0,0)$ correlations of sandpiles for large distances in the infinite-volume limit in any dimensions $d \geq 2$. We show that both of them decay exponentially with the correlation length


$$
\xi(d, a)=\left(\sqrt{d} \sinh ^{-1} \sqrt{a(a+2)}\right)^{-1},
$$

if the dissipation rate $a=\frac{m}{2 d n}$ is positive. Considering a series of models with increasing $n$, we discuss the limit $a \downarrow 0$ and the critical exponent defined by $\nu_{a}=-\lim _{a \downarrow 0} \frac{\log \xi(d, a)}{\log a}$ is determined as

$$
\nu_{a}=\frac{1}{2}
$$

for all $d \geq 2$. Comparison with the $q \downarrow 0$ limit of $q$-state Potts model in external magnetic field is discussed.

Key words. Abelian sandpile models, Dissipation, Avalanches, Height correlations, Determinantal expressions, Correlation length exponent.

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## 1 Introduction

Let $d \in\{2,3, \ldots\}$ and $L \in \mathbb{N} \equiv\{1,2,3, \ldots\}$. Consider a box in the $d$-dimensional hypercubic lattice $B_{L}=\{-L,-L+1, \ldots, L\}^{d} \subset \mathbb{Z}^{d}$, where $\mathbb{Z}$ denotes the collection of all integers. We impose periodic boundary conditions for all $d$ directions and obtain a lattice on a torus (toroidal), which is denoted by $\Lambda_{L}$. The number of sites in $\Lambda_{L}$ is given by $\left|\Lambda_{L}\right|=(2 L+1)^{d}$. In the present paper we study a family of Markov processes on $\Lambda_{L}, h_{t}=\left\{h_{t}(\mathbf{z})\right\}_{\mathbf{z} \in \Lambda_{L}}$, with discrete-time $t \in \mathbb{N}_{0} \equiv\{0\} \cup \mathbb{N}$.

Assume $n, m \in \mathbb{N}$ and let

$$
a=\frac{m}{2 d n} \quad \text { and } \quad h_{\mathrm{c}}=2 d(1+a)
$$

Define a real symmetric matrix with size $(2 L+1)^{d}$,

$$
\Delta_{L}(\mathbf{x}, \mathbf{y})=\left\{\begin{align*}
h_{\mathrm{c}}, & \text { if } \mathbf{x}=\mathbf{y}  \tag{1.1}\\
-1, & \text { if }|\mathbf{x}-\mathbf{y}|=1 \\
0, & \text { otherwise }
\end{align*}\right.
$$

where $\mathbf{x}=\left(x_{1}, \ldots, x_{d}\right), \mathbf{y}=\left(y_{1}, \ldots, y_{d}\right) \in \Lambda_{L}$ and $|\mathbf{x}-\mathbf{y}|=\sqrt{\sum_{i=1}^{d}\left(x_{i}-y_{i}\right)^{2}}$. Let $\mathbf{1}(\omega)$ be the indicator function of an event $\omega ; \mathbf{1}(\omega)=1$, if $\omega$ occurs and $\mathbf{1}(\omega)=0$, otherwise. The configuration space is

$$
\mathcal{S}_{L}=\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, h_{\mathrm{c}}-\frac{1}{n}\right\}^{\Lambda_{L}}
$$

Given a configuration $h_{t} \in \mathcal{S}_{L}, t \in \mathbb{N}_{0}, h_{t+1} \in \mathcal{S}_{L}$ is determined by the following algorithm.
(i) Choose one site in $\Lambda_{L}$ at random. Let $\mathbf{x}$ be the chosen site and define

$$
\eta_{(1)}^{\mathbf{x}}(\mathbf{z})=h_{t}(\mathbf{z})+\frac{1}{n} \mathbf{1}(\mathbf{z}=\mathbf{x}), \quad \mathbf{z} \in \Lambda_{L} .
$$

If $\eta_{(1)}^{\mathbf{x}}(\mathbf{x})<h_{\mathrm{c}}$, then $\eta_{(1)}^{\mathbf{x}} \equiv\left\{\eta_{(1)}^{\mathbf{x}}(\mathbf{z})\right\}_{\mathbf{z} \in \Lambda_{L}} \in \mathcal{S}_{L}$. In this case, we set $h_{t+1}=\eta_{(1)}^{\mathbf{x}}$.


Figure 1: A toppling for the DASM with the parameters $d=2, n=2$ and $m=1$. In this case $h_{\mathrm{c}}=2 d n+m=9$, and thus the site $\mathbf{x}$ with height $h(\mathbf{x})=10$ is unstable. In a toppling, $h_{\mathrm{c}}=9$ grains of sand drop from the site $\mathbf{x}$, in which $n=2$ grains land on each nearest-neighbor site, $m=1$ grain is dissipated from the system, while $h(\mathbf{x})-h_{\mathrm{c}}=1$ grain remains on the site $\mathbf{x}$.
(ii) If $\eta_{(1)}^{\mathrm{x}}(\mathbf{x})=h_{\mathrm{c}}$, then $\eta_{(1)}^{\mathrm{x}} \notin \mathcal{S}_{L}$. In this case, we consider a finite series of configurations $\left\{\eta_{(1)}^{\mathbf{x}}, \cdots, \eta_{(\tau)}^{\mathbf{x}}\right\}$ with $\exists \tau \in \mathbb{N}$ recursively as follows. Assume that $\eta_{(\ell)}^{\mathbf{x}} \notin \mathcal{S}_{L}$ with $\ell \geq 1$, then $A_{(\ell)}^{\mathrm{x}}\left(h_{t}\right) \equiv\left\{\mathbf{z} \in \Lambda_{L}: \eta_{(\ell)}^{\mathbf{x}}(\mathbf{z}) \geq h_{\mathrm{c}}\right\} \neq \emptyset$ and define

$$
\eta_{(\ell+1)}^{\mathbf{x}}(\mathbf{z})=\eta_{(\ell)}^{\mathbf{x}}(\mathbf{z})-\sum_{\mathbf{y}: \mathbf{y} \in A_{(\ell)}^{\mathbf{x}}\left(h_{t}\right)} \Delta_{L}(\mathbf{y}, \mathbf{z}), \quad \mathbf{z} \in \Lambda_{L} .
$$

If $\eta_{(\ell+1)}^{\mathbf{x}} \in \mathcal{S}_{L}$, then $\tau=\ell+1$ and $h_{t+1}=\eta_{(\tau)}^{\mathrm{x}}$. Remark that $\tau=\tau\left(\mathbf{x}, h_{t}\right)$ and $\tau<\infty$ by $\sum_{\mathbf{z}: \mathbf{z} \in \Lambda_{L}} \Delta_{L}(\mathbf{y}, \mathbf{z})>0, \forall \mathbf{y} \in \Lambda_{L}$ as explained below.

We think that $1 / n$ is a unit of grain of sand and $h_{t}(\mathbf{z}) n$ represents the height of sandpile at site $\mathbf{z}$ measured in this unit. The step (i) simulates a random deposit of a grain of sand. In the step (ii), for each $1 \leq \ell \leq \tau$, the sites $\mathbf{y} \in A_{(\ell)}^{\mathbf{x}}\left(h_{t}\right)$ are regarded as unstable sites and the process

$$
\left\{\eta_{(\ell)}^{\mathbf{x}}(\mathbf{z})\right\}_{\mathbf{z} \in \Lambda_{L}} \rightarrow\left\{\eta_{(\ell)}^{\mathbf{x}}(\mathbf{z})-\Delta_{L}(\mathbf{y}, \mathbf{z})\right\}_{\mathbf{z} \in \Lambda_{L}},
$$

is called a toppling of the site $\mathbf{y}$ such that

$$
\Delta_{L}(\mathbf{y}, \mathbf{y}) n=h_{\mathrm{c}} n=2 d n+m \text { grains of sand drop from the unstable site } \mathbf{y}
$$

and

$$
\left|\Delta_{L}(\mathbf{y}, \mathbf{z})\right| n=n \text { grains of sand land on each nearest-neighbor site } \mathbf{z},|\mathbf{x}-\mathbf{z}|=1 .
$$

Since there are $2 d$ nearest-neighbor sites of each site, $m$ grains are annihilated in a toppling. (See Fig.1.) The total number of grains on $\Lambda_{L}$ decreases in each toppling and it guarantees $\tau<\infty$. The configuration space $\mathcal{S}_{L}$ is a set of all stable configurations of sandpiles in which height of sandpile is less than the threshold value $h_{\mathrm{c}}$ at every site; $h(\mathbf{z})<h_{\mathrm{c}}, \forall \mathbf{z} \in \Lambda_{L}$. From a stable configuration $h_{t}$ to another stable configuration $h_{t+1}, \sum_{\ell=1}^{\tau-1}\left|A_{(\ell)}^{\mathrm{x}}\left(h_{t}\right)\right|$ topplings occur.

Such a series of toppling is called an avalanche. (Note that, if $\tau=1$, toppling does not occur. Even in such a case, we call the transition from $h_{t}$ to $h_{t+1}$ an avalanche, which is just a random deposit of a grain of sand.) Define

$$
\begin{equation*}
T(\mathbf{x}, \mathbf{y}, h)=\sum_{\ell=1}^{\tau(\mathbf{x}, h)-1} \mathbf{1}\left(\mathbf{y} \in A_{(\ell)}^{\mathbf{x}}(h)\right), \quad \mathbf{x}, \mathbf{y} \in \Lambda_{L}, \quad h \in \mathcal{S}_{L} . \tag{1.2}
\end{equation*}
$$

This is the number of topplings at site $\mathbf{y} \in \Lambda_{L}$ in an avalanche caused by a deposit of a grain of sand at a site $\mathbf{x} \in \Lambda_{L}$ in the configuration $h \in \mathcal{S}_{L}$.

We have assumed that $n, m \in \mathbb{N}$ in the above definition of processes. If we set $n=1, m=0$, however, we have $a=0$ and $\left.\Delta_{L}\right|_{a=0}$ gives the 'rule matrix' of the sandpile model introduced by Bak, Tang and Wiesenfeld (BTW) [2, 3]. The BTW model have been studied on finite lattices with open boundary conditions in order to make $\tau$ be finite. For example, the BTW model is considered on a box $B_{L}$. The boundary of box $B_{L}$ is given by $\partial B_{L}=\left\{\mathbf{y}=\left(y_{1}, \cdots, y_{d}\right) \in B_{L}\right.$ : $1 \leq \exists i \leq d$ s.t. $y_{i}=-L$ or $\left.L\right\}$. In the BTW model defined on $B_{L},\left.\sum_{\mathbf{z}: \mathbf{z} \in \Lambda_{L}} \Delta_{L}\right|_{a=0}(\mathbf{y}, \mathbf{z})=0$ if $\mathbf{y} \in B_{L} \backslash \partial B_{L}$; that is, the number of grains of sand is conserved in any toppling in the bulk of system. By imposing the open boundary condition, we have $\left.\sum_{\mathbf{z}: \mathbf{z} \in \Lambda_{L}} \Delta_{L}\right|_{a=0}(\mathbf{y}, \mathbf{z})>0$ for $\mathbf{y} \in \partial B_{L}$ and dissipation of grains of sand can occur in topplings at the boundary sites. In the present model, in every toppling at any site $\mathbf{y} \in \Lambda_{L}, \sum_{\mathbf{z}: \mathbf{z} \in \Lambda_{L}} \Delta_{L}(\mathbf{y}, \mathbf{z}) n=m$ grains of sand dissipate from the system and hence $\tau<\infty$ is guaranteed in the shift-invariant system. The quantity $a$ indicates the rate of dissipation in a toppling.

The present process belongs to the class of abelian sandpile models (ASM) studied by Dhar [6]. We define the operators $\{\mathrm{a}(\mathbf{x})\}_{\mathbf{x} \in \Lambda_{L}}$ following Dhar by

$$
h_{t+1}=\mathrm{a}(\mathbf{x}) h_{t}, \quad \mathbf{x} \in \Lambda_{L},
$$

where $h_{t}, h_{t+1} \in \mathcal{S}_{L}$ and the site $\mathbf{x}$ is the chosen site in the first step (i) of the algorithm at time $t$. That is, $a(\mathbf{x})$ represents an avalanche caused by a deposit of a grain of sand at $\mathbf{x}$. Then the above algorithm guarantees the abelian property of avalanches (see Lemma 2.1 in Section 2.1)

$$
\begin{equation*}
[\mathrm{a}(\mathbf{x}), \mathrm{a}(\mathbf{y})] \equiv \mathrm{a}(\mathbf{x}) \mathrm{a}(\mathbf{y})-\mathrm{a}(\mathbf{y}) \mathrm{a}(\mathbf{x})=0, \quad \forall \mathbf{x}, \mathbf{y} \in \Lambda_{L} . \tag{1.3}
\end{equation*}
$$

We call the present Markov process the $d$-dimensional dissipative abelian sandpile model (DASM for short). The two-dimensional case was studied numerically [10] and analytically $[30,28,18]$. In the present paper, we will discuss the models in general dimensions $d \geq 2$ in finite and infinite lattices. See also [29]. As shown in $[17,26,16]$ the DASM is useful to construct the infinite-volume limit of avalanche models. Importance of the abelian sandpile models in the extensive study of self-organized criticality in the statistical mechanics and related fields is discussed in [25].

## 2 Basic Properties of Dissipative Abelian Sandpile Model

### 2.1 Abelian property

First we prove the abelian property of avalanches (1.3).

Lemma 2.1 (Dhar [6]) Assume that the avalanche operators $\{\mathrm{a}(\mathrm{x})\}_{\mathbf{x} \in \Lambda_{L}}$ act on $\mathcal{S}_{L}$. Then

$$
[\mathrm{a}(\mathrm{x}), \mathrm{a}(\mathbf{y})]=0, \quad \forall \mathbf{x}, \mathbf{y} \in \Lambda_{L} .
$$

Proof. Let $\mathcal{X}_{L}=\mathbb{Z}^{\Lambda_{L}}$. Define three sets of maps from $\mathcal{X}_{L}$ to $\mathcal{X}_{L} ;\{\tilde{t}(\mathbf{x})\}_{\mathbf{x} \in \Lambda_{L}},\{\mathrm{t}(\mathbf{x})\}_{\mathbf{x} \in \Lambda_{L}}$ and $\{\mathrm{d}(\mathbf{x})\}_{\mathbf{x} \in \Lambda_{L}}$ as follows. For $\mathbf{x} \in \Lambda_{L}$ and $\eta=\{\eta(\mathbf{x})\}_{\mathbf{x} \in \Lambda_{L}} \in \mathcal{X}_{L}$ define

$$
\begin{aligned}
& \tilde{\mathrm{t}}(\mathbf{x}) \eta(\mathbf{z})=\eta(\mathbf{z})-\Delta_{L}(\mathbf{x}, \mathbf{z}), \\
& \mathrm{t}(\mathbf{x}) \eta(\mathbf{z})= \begin{cases}\eta(\mathbf{z})-\Delta_{L}(\mathbf{x}, \mathbf{z}), & \text { if } \eta(\mathbf{x}) \geq h_{\mathrm{c}}, \\
\eta(\mathbf{z}), & \text { otherwise, }\end{cases} \\
& \mathrm{d}(\mathbf{x}) \eta(\mathbf{z})=\eta(\mathbf{z})+\frac{1}{n} \mathbf{1}(\mathbf{z}=\mathbf{x}), \quad \mathbf{z} \in \Lambda_{L} .
\end{aligned}
$$

By definition of $\tilde{\mathfrak{t}}$,

$$
\tilde{\mathfrak{t}}(\mathbf{y}) \tilde{\mathrm{t}}(\mathbf{x}) \eta(\mathbf{z})=\eta(\mathbf{z})-\Delta_{L}(\mathbf{x}, \mathbf{z})-\Delta_{L}(\mathbf{y}, \mathbf{z}), \quad \mathbf{z} \in \Lambda_{L} .
$$

Similarly we have

$$
\tilde{\mathfrak{t}}(\mathbf{x}) \tilde{\mathrm{t}}(\mathbf{y}) \eta(\mathbf{z})=\eta(\mathbf{z})-\Delta_{L}(\mathbf{y}, \mathbf{z})-\Delta_{L}(\mathbf{x}, \mathbf{z}), \quad \mathbf{z} \in \Lambda_{L} .
$$

Therefore $\tilde{\mathfrak{t}}(\mathbf{y}) \tilde{\mathfrak{t}}(\mathbf{x}) \eta=\tilde{\mathfrak{t}}(\mathbf{x}) \tilde{\mathfrak{t}}(\mathbf{y}) \eta, \forall \eta \in \mathcal{X}_{L}$, that is

$$
\begin{equation*}
[\tilde{t}(\mathbf{x}), \tilde{\mathrm{t}}(\mathbf{y})]=0, \quad \forall \mathbf{x}, \mathbf{y} \in \Lambda_{L} . \tag{2.1}
\end{equation*}
$$

Assume that $\mathbf{y} \neq \mathbf{x}$. Then

$$
\tilde{\mathrm{t}}(\mathbf{y}) \eta(\mathbf{x})=\eta(\mathbf{x})-\Delta(\mathbf{y}, \mathbf{x})= \begin{cases}\eta(\mathbf{x})+1, & \text { if } \\ \eta(\mathbf{x}), & \text { if } \\ |\mathbf{x}|=1 \\ \mathbf{x} \mid>1\end{cases}
$$

It implies that if $\eta(\mathbf{x}) \geq h_{\mathrm{c}}$ then $\tilde{\mathrm{t}}(\mathbf{y}) \eta(\mathbf{x}) \geq h_{\mathrm{c}}, \forall \mathbf{y} \neq \mathbf{x}$, that is, any site cannot be stabilized by topplings which occur at other sites. Therefore, the definition of $t(x)$ and (2.1) give

$$
\begin{equation*}
[\mathrm{t}(\mathrm{x}), \mathrm{t}(\mathbf{y})]=0, \quad \forall \mathbf{x}, \mathbf{y} \in \Lambda_{L} \tag{2.2}
\end{equation*}
$$

It is obvious that

$$
\begin{equation*}
[\mathrm{t}(\mathrm{x}), \mathrm{d}(\mathbf{y})]=0, \quad \forall \mathbf{x}, \mathbf{y} \in \Lambda_{L} \tag{2.3}
\end{equation*}
$$

Consider the situation that $h \in \mathcal{S}_{L}$ and $A_{(\ell)}^{\mathrm{x}}(h) \neq \emptyset, 1 \leq \ell \leq \tau$. By $(2.2), \prod_{\mathbf{z}: \mathbf{z} \in A_{(\ell)}^{\mathrm{x}}(h)} \mathrm{t}(\mathbf{z})$ is independent of the order of the products of $t(\mathbf{z})$ 's. Then we can write

$$
\mathrm{a}(\mathbf{x}) h=\left[\prod_{\ell=1}^{\tau-1}\left(\prod_{\mathbf{z}: \mathbf{z} \in A_{(\ell)}^{\mathrm{x}}(h)} \mathrm{t}(\mathbf{z})\right)\right] \mathrm{d}(\mathbf{x}) h, \quad \mathbf{x} \in \Lambda_{L}, \quad h \in \mathcal{S}_{L} .
$$

By (2.2) and (2.3), the lemma is proved.


Figure 2: The set of recurrent configurations $\mathcal{R}_{L}$ is closed under avalanches.

### 2.2 Recurrent configurations

Consider a subset of $\mathcal{S}_{L}$ defined by

$$
\mathcal{R}_{L}=\left\{h \in \mathcal{S}_{L}: \forall \mathbf{x} \in \Lambda_{L}, \exists k(\mathbf{x}) \in \mathbb{N} \text {, s.t. }(\mathrm{a}(\mathbf{x}))^{k(\mathbf{x})} h=h\right\},
$$

which is called the set of recurrent configurations.
Lemma 2.2 (Dhar [6]) If $h \in \mathcal{R}_{L}$, then $\mathrm{a}(\mathrm{x}) h \in \mathcal{R}_{L}$ for any $\mathrm{x} \in \Lambda_{L}$. That is, $\mathcal{R}_{L}$ is closed under avalanches (see Fig.2).
Proof. By definition, if $h \in \mathcal{R}_{L}$, then for any $\mathbf{y} \in \Lambda_{L}, \exists k(\mathbf{y}) \in \mathbb{N}$, s.t. $(\mathrm{a}(\mathbf{y}))^{k(\mathbf{y})} h=h$. If we operate $\mathrm{a}(\mathbf{x}), \mathbf{x} \in \Lambda_{L}$ on the both sides of this equation, then we have $\mathrm{a}(\mathbf{x})(\mathrm{a}(\mathbf{y}))^{k(\mathbf{y})} h=\mathrm{a}(\mathbf{x}) h$. By Lemma 2.1, LHS $=(\mathrm{a}(\mathbf{y}))^{k(\mathbf{y})} \mathrm{a}(\mathbf{x}) h$. This equality implies that $\mathrm{a}(\mathbf{x}) h \in \mathcal{R}_{L}$. Since it is valid for any $\mathbf{x} \in \Lambda_{L}$, the proof is completed.

Consider a $(2 L+1)^{d}$-dimensional vector space $\mathcal{V}_{L}$, in which the orthonormal basis is given by $\{\mathbf{e}(\mathbf{z})\}_{\mathbf{z} \in \Lambda_{L}}$. For each configuration $\eta \in \mathcal{X}_{L}$, we assign a vector

$$
\begin{equation*}
\boldsymbol{\eta}=\sum_{\mathbf{z}: \mathbf{z} \in \Lambda_{L}} \eta(\mathbf{z}) \mathbf{e}(\mathbf{z})=\sum_{\mathbf{z}: \mathbf{z} \in \Lambda_{L}} n \eta(\mathbf{z}) \frac{\mathbf{e}(\mathbf{z})}{n}, \tag{2.4}
\end{equation*}
$$

where $1 / n$ denotes the unit of grain of sand. Assume that $h \in \mathcal{R}_{L}$; for each $\mathbf{x} \in \Lambda_{L}$, there is $k(\mathbf{x}) \in \mathbb{N}$ such that

$$
\begin{equation*}
(\mathrm{a}(\mathbf{x}))^{k(\mathbf{x})} h=h \tag{2.5}
\end{equation*}
$$

Consider the vector corresponding to the configuration $(\mathrm{d}(\mathbf{x}))^{k(\mathbf{x})} h$,

$$
\begin{equation*}
\boldsymbol{\eta}=\left(h(\mathbf{x})+\frac{k(\mathbf{x})}{n}\right) \mathbf{e}(\mathbf{x})+\sum_{\mathbf{z}: \mathbf{z} \neq \mathbf{x}} h(\mathbf{z}) \mathbf{e}(\mathbf{z}) \in \mathcal{V}_{L} . \tag{2.6}
\end{equation*}
$$

Then (2.5) claims that there exists a set $\left\{r(\mathbf{z}) \in \mathbb{N}: \mathbf{z} \in \Lambda_{L}\right\}$ such that

$$
\begin{equation*}
\mathbf{h}=\boldsymbol{\eta}+\sum_{\mathbf{z}: \mathbf{z} \in \Lambda_{L}}\left(\sum_{\mathbf{y}: \mathbf{y} \in \Lambda_{L}} r(\mathbf{y}) \Delta_{L}(\mathbf{y}, \mathbf{z})\right) \mathbf{e}(\mathbf{z}) \tag{2.7}
\end{equation*}
$$

Note that (2.7) is written as

$$
\mathbf{h}=\boldsymbol{\eta}+\sum_{\mathbf{y}: \mathbf{y} \in \Lambda_{L}} r(\mathbf{y}) \mathbf{v}(\mathbf{y})
$$

with

$$
\begin{equation*}
\mathbf{v}(\mathbf{x})=\sum_{\mathbf{z}: \mathbf{z} \in \Lambda_{L}} \Delta_{L}(\mathbf{x}, \mathbf{z}) \mathbf{e}(\mathbf{z}), \quad \mathbf{x} \in \Lambda_{L} \tag{2.8}
\end{equation*}
$$



Figure 3: Hypercubic lattice $\Omega$ with the basis $\{\mathbf{v}(\mathbf{x})\}_{\mathbf{x} \in \Lambda_{L}}$ in $\mathcal{V}_{L}$. Every avalanche from an unstable configuration $\boldsymbol{\eta}$ given by (2.6) to a recurrent configuration $h \in \mathcal{R}_{L}$ is represented by a lattice path $\boldsymbol{\eta} \leadsto \mathbf{h}$ on $\Omega$.

We can say that, given $h \in \mathcal{R}_{L}$, all points $\{\boldsymbol{\eta}\}$ given by (2.6) are identified with sites of a hypercubic lattice $\Omega$ with the basis $\{\mathbf{v}(\mathbf{x})\}_{\mathbf{x} \in \Lambda_{L}}$ in $\mathcal{V}_{L}$. (See Fig.3.) Consider a primitive cell (fundamental domain) of the lattice defined by

$$
\begin{equation*}
\mathcal{U}_{L}=\left\{\sum_{\mathbf{x}: \mathbf{x} \in \Lambda_{L}} \mathbf{c}(\mathbf{x}) \mathbf{v}(\mathbf{x}): 0 \leq \mathbf{c}(\mathbf{x})<1, \mathbf{x} \in \Lambda_{L}\right\} \subset \mathcal{V}_{L} \tag{2.9}
\end{equation*}
$$

By definition, the intersection of the lattice $\Omega$ and $\mathcal{U}_{L}$ is a singleton, say $\mathbf{p}$. We assume that the origin of this lattice is given by $\mathbf{p}$ and express the lattice by $\Omega^{\mathbf{p}}$. We consider a collection of all lattices with the same basis (2.8) having distinct origin in $\mathcal{U}_{L},\left\{\Omega^{\mathbf{p}}: \mathbf{p} \in \mathcal{U}_{L}\right\}$. Then there establishes a bijection between $\mathcal{R}_{L}=\{h\}$ and $\left\{\Omega^{\mathbf{p}}: \mathbf{p} \in \mathcal{U}_{L}\right\}$.

Lemma 2.3 (Dhar [6]) The number of recurrent configuration is given by

$$
\left|\mathcal{R}_{L}\right|=n^{(2 L+1)^{d}} \operatorname{det} \Delta_{L}
$$

Proof. The above bijection implies $\left|\mathcal{R}_{L}\right|=\left|\left\{\Omega^{\mathbf{p}}: \mathbf{p} \in \mathcal{U}_{L}\right\}\right|$. Since the unit of grain of sand is $1 / n$, the origins $\{\mathbf{p}\}$ of lattices $\left\{\Omega^{\mathbf{p}}\right\}$ should be in $(\mathbb{Z} / n)^{\Lambda_{L}}$, and hence (see Fig.4)

$$
\left|\left\{\Omega^{\mathbf{p}}: \mathbf{p} \in \mathcal{U}_{L}\right\}\right|=\left|\mathcal{U}_{L} \cap(\mathbb{Z} / n)^{\Lambda_{L}}\right|=n^{(2 L+1)^{d}} \times\left(\text { the volume of } \mathcal{U}_{L}\right)
$$

The volume of $\mathcal{U}_{L}$ given by (2.9) with (2.8) is det $\Delta_{L}$ and the proof is completed.


Figure 4: A primitive cell of $\Omega$ on the lattice $(\mathbb{Z} / n)^{\Lambda_{L}}$. Since the unit of grain of sand is $1 / n$, the origin $\mathbf{p}$ of lattice $\Omega$ should be at a site of $(\mathbb{Z} / n)^{\Lambda_{L}}$.

### 2.3 Stationary distribution

For $h \in \mathcal{R}_{L}$, let $\mathbb{P}_{L}^{h}$ be the probability law of the DASM starting from the configuration $h_{0}=h$.
Definition 2.4 If we restrict $\{\mathrm{a}(\mathrm{x})\}_{\mathbf{x} \in \Lambda_{L}}$ to $\mathcal{R}_{L}$, inverse of the avalanche operator can be defined by

$$
\mathrm{a}(\mathbf{x})^{-1}=\mathrm{a}(\mathbf{x})^{k(\mathbf{x})-1}, \quad \mathbf{x} \in \Lambda_{L} .
$$

Assume that $h \in \mathcal{R}_{L}$ is given. Define

$$
\begin{aligned}
& \mu_{t}(X)=\mathbb{P}^{h}\left(h_{t}=X\right), \\
& W(X \rightarrow Y)=\mathbb{P}^{h}\left(h_{t+1}=Y \mid h_{t}=X\right), \quad t \in \mathbb{N}_{0}, \quad X, Y \in \mathcal{R}_{L} .
\end{aligned}
$$

Consider the Master equation

$$
\mu_{t+1}(X)=\mu_{t}(X)-\sum_{Y: Y \in \mathcal{R}_{L}} \mu_{t}(X) W(X \rightarrow Y)+\sum_{Y: Y \in \mathcal{R}_{L}} \mu_{t}(Y) W(Y \rightarrow X),
$$

where we have used the assumption that $h_{0}=h \in \mathcal{R}_{L}$ and Lemma 2.2. By definition of the DASM, we can find that, for $X, Y \in \mathcal{R}_{L}$,

$$
\begin{aligned}
W(X \rightarrow Y) & =\sum_{\mathrm{x}: \mathbf{x} \in \Lambda_{L}} \operatorname{Prob}(\mathrm{x} \text { is chosen }) \mathbf{1}(\mathrm{a}(\mathrm{x}) X=Y) \\
& =\frac{1}{\left|\Lambda_{L}\right|} \sum_{\mathrm{x}: \mathbf{x} \in \Lambda_{L}} \mathbf{1}(\mathrm{a}(\mathbf{x}) X=Y) \\
& =\frac{1}{(2 L+1)^{d}} \sum_{\mathrm{x}: \mathbf{x} \in \Lambda_{L}} \mathbf{1}\left(X=\mathrm{a}^{-1}(\mathbf{x}) Y\right) .
\end{aligned}
$$

Then we have

$$
\mu_{t+1}(X)-\mu_{t}(X)=\frac{1}{(2 L+1)^{d}} \sum_{\mathrm{x}: \mathbf{x} \in \Lambda_{L}}\left\{\mu_{t}\left(\mathrm{a}(\mathbf{x})^{-1} X\right)-\mu_{t}(X)\right\}, \quad \forall X \in \mathcal{R}_{L}
$$

It implies that the uniform measure on $\mathcal{R}_{L}$,

$$
\mu(X)=\frac{1}{\left|\mathcal{R}_{L}\right|} \mathbf{l}\left(X \in \mathcal{R}_{L}\right)=\frac{1}{n^{(2 L+1)^{d}} \operatorname{det} \Delta_{L}} \mathbf{1}\left(X \in \mathcal{R}_{L}\right), \quad X \in \mathcal{X}_{L}
$$

is a stationary distribution of the process.

Lemma 2.5 The DASM on $\Lambda_{L}$ is irreducible on $\mathcal{R}_{L}$.
Proof. Consider the configuration $\bar{h} \in \mathcal{S}_{L}$, such that $\bar{h}(\mathbf{x})=h_{\mathrm{c}}-1 / n, \forall \mathbf{x} \in \Lambda_{L}$. Now we take two arbitrary configurations $X$ and $Y$ from $\mathcal{R}_{L}$. We have

$$
\begin{equation*}
\bar{h}=\prod_{\mathbf{x}: X(\mathbf{x})<h_{\mathrm{c}}-1 / n}(\mathrm{a}(\mathbf{x}))^{h_{\mathrm{c}}-1 / n-X(\mathbf{x})} X=\prod_{\mathbf{x}: Y(\mathbf{x})<h_{\mathrm{c}}-1 / n}(\mathrm{a}(\mathbf{x}))^{h_{\mathrm{c}}-1 / n-Y(\mathbf{x})} Y \tag{2.10}
\end{equation*}
$$

Since this means that the configuration $\bar{h}$ is reachable form $X$ and $Y$ by avalanches, Lemma 2.2 guarantees that $\bar{h} \in \mathcal{R}_{L}$. Since we have assumed that $Y \in \mathcal{R}_{L},(\mathrm{a}(\mathbf{x}))^{k(\mathbf{x})} Y=Y$ with some $k(\mathbf{x}) \in \mathbb{N}$ for any $\mathbf{x} \in \Lambda_{L}$. Therefore, the second equality of (2.10) gives (see Definition 2.4)

$$
\begin{equation*}
Y=\prod_{\mathbf{x}: Y(\mathbf{x})<h_{\mathrm{c}}-1 / n}(\mathrm{a}(\mathbf{x}))^{k(\mathbf{x})-\left(h_{\mathrm{c}}-1 / n-Y(\mathbf{x})\right)} \bar{h} \tag{2.11}
\end{equation*}
$$

Combining (2.10) and (2.11) gives

$$
Y=\prod_{\mathbf{x}: Y(\mathbf{x})<h_{\mathrm{c}}-1 / n}(\mathrm{a}(\mathbf{x}))^{k(\mathbf{x})-\left(h_{\mathrm{c}}-1 / n-Y(\mathbf{x})\right)} \prod_{\mathbf{y}: X(\mathbf{y})<h_{\mathrm{c}}-1 / n}(\mathrm{a}(\mathbf{y}))^{h_{\mathrm{c}}-1 / n-X(\mathbf{y})} X
$$

Let $\sigma=\sum_{\mathbf{x}: Y(\mathbf{x})<h_{\mathrm{c}}-1 / n}\left\{k(\mathbf{x})-\left(h_{\mathrm{c}}-1 / n-Y(\mathbf{x})\right)\right\}+\sum_{\mathbf{x}: X(\mathbf{x})<h_{\mathrm{c}}-1 / n}\left\{h_{\mathrm{c}}-1 / n-X(\mathbf{x})\right\}$. Then we see

$$
\mathbb{P}^{h_{0}}\left(h_{t+s}=Y \mid h_{t}=X\right) \geq\left(\frac{1}{\left|\Lambda_{L}\right|}\right)^{\sigma} \quad \text { for } s \geq \sigma
$$

Since RHS is strictly positive for finite $L$, this completes the proof.
Then the following is concluded by the general theory of Markov chains (see, for example, Chapter 6.4 of [12]).

Proposition 2.6 The stationary distribution of the DASM is uniquely given by the uniform measure on $\mathcal{R}_{L}$.

We write the probability law of the DASM on $\Lambda_{L}$ in the stationary distribution as $\mathbf{P}_{L}$ and its expectation as $\mathbf{E}_{L}$.

### 2.4 Allowed configurations and spanning trees

Dhar also introduced a subset of $\mathcal{S}_{L}$ called a collection of allowed configurations $\mathcal{A}_{L}[6]$. He defined that for $h \in \mathcal{S}_{L}$, if there is a subset $F \subset \Lambda_{L}$ such that $F \neq \emptyset$ and

$$
\begin{equation*}
h(\mathbf{y})<\sum_{\mathbf{x}: \mathbf{x} \in F, \mathbf{x} \neq y}\left(-\Delta_{L}(\mathbf{x}, \mathbf{y})\right), \quad \forall \mathbf{y} \in F, \tag{2.12}
\end{equation*}
$$

then $h \in \mathcal{S}_{L}$ has a forbidden subconfiguration (FSC) on $F$. Then define

$$
\mathcal{A}_{L}=\left\{h \in \mathcal{S}_{L}: h \text { has no FSC }\right\}
$$

Lemma 2.7 For the $D A S M$ on $\Lambda_{L}$,

$$
\mathcal{R}_{L} \subset \mathcal{A}_{L}
$$

Proof. In the proof of Lemma 2.5 we have shown that $\bar{h} \in \mathcal{R}_{L}$ and all recurrent stares are reachable from this configuration $\bar{h}$. We can prove that $\bar{h} \in \mathcal{A}_{L}$ as follows. We assume that the contrary; there exists a finite nonempty set $F \subset \Lambda_{L}$ satisfying (2.12). In the DASM, however, for any $\mathbf{y} \in F, \bar{h}(\mathbf{y})=h_{\mathrm{c}}-1 / n=2 d+(m-1) / n \geq 2 d \geq \sum_{\mathbf{x}: \mathbf{x} \in F: \mathbf{x} \neq \mathbf{y}}\left(-\Delta_{L}(\mathbf{x}, \mathbf{y})\right)$, which contradicts our assumption. Since both $\mathcal{R}_{L}$ and $\mathcal{A}_{L}$ include $\bar{h}$, it is enough to show that $\mathcal{A}_{L}$ is closed under the process of avalanche to prove the lemma, since we have already proved that $\mathcal{R}_{L}$ is so in Lemma 2.2. Remark that addition of particles only increases $h$ and such procedure on an allowed configurations cannot create any FSC. Here we assume that there exists an allowed configuration $h$ such that by a single toppling at the site $\mathbf{x}$ it becomes to contain a FSC. Write $h^{\prime}=\mathrm{t}(\mathrm{x}) \mathrm{d}(\mathbf{x}) h$, that is,

$$
\begin{equation*}
h^{\prime}(\mathbf{y})=h(\mathbf{y})+\frac{1}{n} \mathbf{1}(\mathbf{y}=\mathbf{x})-\Delta_{L}(\mathbf{x}, \mathbf{y}), \quad \forall \mathbf{y} \in \Lambda_{L} . \tag{2.13}
\end{equation*}
$$

By assumption, there exists $F \neq \emptyset$ such that

$$
\begin{equation*}
h^{\prime}(\mathbf{y})<\sum_{\mathbf{z}: \mathbf{z} \in F: \mathbf{z} \neq \mathbf{y}}\left(-\Delta_{L}(\mathbf{z}, \mathbf{y})\right), \quad \forall \mathbf{y} \in F . \tag{2.14}
\end{equation*}
$$

Combining (2.13) and (2.14) gives

$$
h(\mathbf{y})<\sum_{\mathbf{z}: \mathbf{z} \in F, \mathbf{z} \neq \mathbf{y}}\left(-\Delta_{L}(\mathbf{z}, \mathbf{y})\right)+\Delta_{L}(\mathbf{x}, \mathbf{y}), \quad \forall \mathbf{y} \in F \backslash\{\mathbf{x}\}
$$

Since $\Delta_{L}(\mathbf{x}, \mathbf{y}) \leq 0$ for $\mathbf{x} \neq \mathbf{y}$, this inequality means that $h$ has a FSC on $F \backslash\{\mathbf{x}\}$ and this contradicts our assumption that $h$ is allowed. Since any avalanche consists of addition of a particle and a series of topplings, the proof is completed.

Definition 2.8 Given a pair $\left(\Lambda_{L}, \Delta_{L}\right)$, let $G_{L}^{(v)}=\Lambda_{L} \cup\{\mathbf{r}\}$ with an additional vertex $\mathbf{r}$ (the 'root'), and $G_{L}^{(e)}$ be the collection of $\left|\Delta_{L}(\mathbf{x}, \mathbf{y})\right| n=n$ edges between $\mathbf{x}, \mathbf{y} \in \Lambda_{L}, \mathbf{x} \neq \mathbf{y}$, and $\sum_{\mathbf{y}: \mathbf{y} \in \Lambda_{L}} \Delta(\mathbf{x}, \mathbf{y}) n=m$ edges between $\mathbf{x} \in \Lambda_{L}$ and $\mathbf{r}$. (See Fig.5.) Graph $G_{L}$ associated to $\left(\Lambda_{L}, \Delta_{L}\right)$ is defined as

$$
G_{L}=\left(G_{L}^{(v)}, G_{L}^{(e)}\right) .
$$

Definition 2.9 We say a graph $T$ on $G_{L}$ is a spanning tree, if the number of vertices of $T$ is $\left|G_{L}^{(v)}\right|=\left|\Lambda_{L}\right|+1$, the number of connected components is one, and the number of loops is zero.

Lemma 2.10 Let $\mathcal{T}_{L}=\left\{\right.$ spanning tree on $G_{L}$ associated to $\left.\left(\Lambda_{L}, \Delta_{L}\right)\right\}$. Then

$$
\left|\mathcal{T}_{L}\right|=n^{(2 L+1)^{d}} \operatorname{det} \Delta_{L} .
$$

Proof. See p. 133 of [20] and Theorem 6.3 in [4].
Lemma 2.11 (Majumdar and Dhar [20]) There establishes a bijection between $\mathcal{A}_{L}$ and $\mathcal{T}_{L}$.


Figure 5: A part of the graph $G_{L}=\left(G_{L}^{(v)}, G_{L}^{(e)}\right)$ associated to the DASM $\left(\Lambda_{L}, \Delta_{L}\right)$ is illustrated for the case that $d=2, n=2$ and $m=1$. In this case, each pair of the nearest-neighbor vertices are connected by $n=2$ edges and each vertex is connected to the 'root' $\mathbf{r}$ by $m=1$ edge.

Proof. First we order all edges incident on each site $\mathbf{x} \in G_{L}^{(v)}$ in some order of preference. For each configuration $h \in \mathcal{A}_{L}$, we consider a following discrete-time growth process of graph on $G_{L}$, which is called a burning process on $\left(G_{L}, h\right)$. Let $\tilde{V}_{0}=V_{0}=\{\mathbf{r}\}, E_{0}=\emptyset$ and $T_{0}=\left(V_{0}, E_{0}\right)$. Assume that we have nonempty sets $T_{t}=\left(V_{t}, E_{t}\right)$ and $\tilde{V}_{t}$ with $t \in \mathbb{N}_{0}$. Let

$$
\tilde{V}_{t+1}=\left\{\mathbf{y} \in G_{L}^{(v)} \backslash V_{t}: h(\mathbf{y}) \geq \sum_{\mathrm{x}: \mathbf{x} \in G_{L}^{(v)} \backslash V_{t}}\left(-\Delta_{L}(\mathbf{x}, \mathbf{y})\right)\right\} .
$$

For each $\mathbf{y} \in \tilde{V}_{t+1}$, consider

$$
\tilde{E}_{t+1}(\mathbf{y})=\left\{e \in G_{L}^{(e)}: e \text { connects } \mathbf{y} \text { and a site in } \tilde{V}_{t}\right\} .
$$

We must have

$$
h(\mathbf{y}) \leq \sum_{\mathrm{x}: \mathbf{x} \in G_{L}^{(v)} \backslash V_{t}}\left(-\Delta_{L}(\mathbf{x}, \mathbf{y})\right)+\left|\tilde{E}_{t+1}(\mathbf{y})\right|,
$$

since $h \in \mathcal{S}_{L}$. If $\left|\tilde{E}_{t+1}(\mathbf{y})\right|=1$, then name that edge as $e(\mathbf{y})$. If $\left|\tilde{E}_{t+1}(\mathbf{y})\right| \geq 2$, then write

$$
h(\mathbf{y})=\sum_{\mathbf{x}: \mathbf{x} \in G_{L}^{(v)} \backslash V_{t}}\left(-\Delta_{L}(\mathbf{x}, \mathbf{y})\right)+\frac{s}{n},
$$

and choose the $(s+1)$-th edge in $\tilde{E}_{t+1}(\mathbf{y})$ as $e(\mathbf{y})$. We define

$$
V_{t+1}=V_{t} \cup \tilde{V}_{t+1}, \quad E_{t+1}=E_{t} \cup\left\{e(\mathbf{y}): \mathbf{y} \in \tilde{V}_{t+1}\right\}, \quad \text { and } \quad T_{t+1}=\left(V_{t+1}, E_{t+1}\right) .
$$

By the assumption $h \in \mathcal{A}_{L}$, there is a finite time $\sigma<\infty$ such that $V_{\sigma}=G_{L}^{(v)}$ and $E_{\sigma}=G_{L}^{(s)}$. By the construction, $T_{\sigma}=\left(V_{\sigma}, E_{\sigma}\right)$ is a spanning tree on $G_{L}$. Since this growth process of $T_{t}, t \in\{0,1, \cdots, \sigma\}$ is deterministic for a given configuration $h \in \mathcal{A}_{L}$, it gives an injection from $\mathcal{A}_{L}$ to $\mathcal{T}_{L}$. This fact and Lemma 2.10 give $\left|\mathcal{A}_{L}\right| \leq\left|\mathcal{T}_{L}\right|=n^{(2 L+1)^{d}}$ det $\Delta_{L}$. On the other hand, Lemmas 2.3 and 2.7 give $n^{(2 L+1)^{d}} \operatorname{det} \Delta_{L} \leq\left|\mathcal{A}_{L}\right|$. Then we can conclude $\left|\mathcal{A}_{L}\right|=n^{(2 L+1)^{d}} \operatorname{det} \Delta_{L}$ and the burning process gives a bijection between $\mathcal{A}_{L}$ and $\mathcal{T}_{L}$.

Combining Lemmas 2.3, 2.7, 2.10, and 2.11, we have the following proposition.

Proposition 2.12 For the DASM on $\Lambda_{L}, \mathcal{R}_{L}=\mathcal{A}_{L}$.

## 3 Avalanche Propagators

### 3.1 Integral expressions for propagators

Define

$$
G_{L}(\mathbf{x}, \mathbf{y})=\mathbf{E}_{L}[T(\mathbf{x}, \mathbf{y}, h)], \quad \mathbf{x}, \mathbf{y} \in \Lambda,
$$

where $T(\mathbf{x}, \mathbf{y}, h)$ is given by (1.2) and the expectation is taken over configurations $\{h\}$ in the stationary distribution $\mathbf{P}_{L} . G_{L}(\mathbf{x}, \mathbf{y})$ is regarded as the avalanche propagator from $\mathbf{x}$ to $\mathbf{y}[6]$. Sometime in an avalanche caused by a deposit of a grain of sand at $\mathbf{x}$, this site $\mathbf{x}$ topples many times. The set of topplings between the first and the second toppling at $\mathbf{x}$ is called the first wave of toppling. There can occur many waves in one avalanche and $G_{L}(\mathbf{x}, \mathbf{x})$ gives the average number of waves of topplings in an avalanche [15].

Consider the stationary distribution $\mathbf{P}_{L}$ of the DASM. For addition of a particle at any site $\mathbf{x} \in \Lambda_{L}$, the averaged influx of grains of sand into a site $\mathbf{z} \in \Lambda_{L}$ is given by $\mathbf{1}(\mathbf{z}=\mathbf{x})+$ $\sum_{\mathbf{y}: \mathbf{y} \neq \mathbf{z}} G_{L}(\mathbf{x}, \mathbf{y})\left|\Delta_{L}(\mathbf{y}, \mathbf{z})\right| n$, and the averaged outflux of them out of $\mathbf{z}$ by $G_{L}(\mathbf{x}, \mathbf{z}) \Delta_{L}(\mathbf{z}, \mathbf{z}) n$ using the avalanche propagators. In $\mathbf{P}_{L}$, equivalence between influx and outflux must hold at any site $\mathbf{z} \in \Lambda_{L}$. This balance equation is written as

$$
\sum_{\mathrm{y}: \mathbf{y} \in \Lambda_{L}} G_{L}(\mathbf{x}, \mathbf{y}) \Delta_{L}(\mathbf{y}, \mathbf{z})=\frac{1}{n} \mathbf{1}(\mathbf{z}=\mathbf{x}) \quad \forall \mathbf{x}, \mathbf{z} \in \Lambda_{L}
$$

and thus the propagator is given using the inverse matrix of $\Delta_{L}$.
Lemma 3.1 (Dhar [6])

$$
\begin{equation*}
G_{L}(\mathbf{x}, \mathbf{y})=\frac{1}{n}\left[\Delta_{L}^{-1}\right](\mathbf{x}, \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \Lambda_{L} \tag{3.1}
\end{equation*}
$$

The matrix $\Delta_{L}$ can be diagonalized by the Fourier transformation from $\mathbf{x}=\left(x_{1}, \cdots, x_{d}\right)$ to $\mathbf{n}=\left(n_{1}, \cdots, n_{d}\right)$,

$$
U_{L}(\mathbf{n}, \mathbf{x})=U_{L}^{-1}(\mathbf{x}, \mathbf{n})=\frac{1}{(2 L+1)^{d / 2}} \exp \left(\frac{2 \pi}{2 L+1} \mathbf{x} \cdot \mathbf{n}\right)
$$

where $\mathbf{x} \cdot \mathbf{n}=\sum_{i=1}^{d} x_{i} n_{i}$, as

$$
\begin{aligned}
& \sum_{\mathbf{x}: \mathbf{x} \in \Lambda_{L}} \sum_{\mathbf{y}: \mathbf{y} \in \Lambda_{L}} U_{L}(\mathbf{n}, \mathbf{x}) \Delta_{L}(\mathbf{x}, \mathbf{y}) U_{L}^{-1}(\mathbf{y}, \mathbf{m}) \\
& \quad=2 d\left\{(1+a)-\frac{1}{d} \sum_{i=1}^{d} \cos \left(\frac{2 \pi}{2 L+1} n_{i}\right)\right\} \mathbf{1}(\mathbf{n}=\mathbf{m}) \\
& \quad \equiv \Lambda_{L}(\mathbf{n}, \mathbf{m}), \quad \mathbf{n}, \mathbf{m} \in \Lambda_{L} .
\end{aligned}
$$

Then, (3.1) is obtained as

$$
\begin{align*}
G_{L}(\mathbf{x}, \mathbf{y}) & =\frac{1}{n} \sum_{\mathbf{n}: \mathbf{n} \in \Lambda_{L}} \sum_{\mathbf{m}: \mathbf{m} \in \Lambda_{L}} U_{L}^{-1}(\mathbf{x}, \mathbf{n})\left[\Delta_{L}^{-1}\right](\mathbf{n}, \mathbf{m}) U_{L}(\mathbf{m}, \mathbf{y}) \\
& =\frac{1}{2 d n} \frac{1}{(2 L+1)^{d}} \sum_{\mathbf{n}: \mathbf{n} \in \Lambda_{L}} \frac{\mathrm{e}^{-2 \pi \sqrt{-1}(\mathbf{x}-\mathbf{y}) \cdot \mathbf{n} /(2 L+1)}}{(1+a)-(1 / d) \sum_{i=1}^{d} \cos \left(\frac{2 \pi}{2 L+1} n_{i}\right)} . \text { nonumber } \tag{3.2}
\end{align*}
$$

Lemma 3.2 There exists a limit $G(\mathbf{x}-\mathbf{y})=\lim _{L \uparrow \infty} G_{L}(\mathbf{x}, \mathbf{y}), \mathbf{x}, \mathbf{y} \in \mathbb{Z}^{d}$ and

$$
\begin{equation*}
G(\mathbf{x})=\frac{1}{2 d n} \prod_{i=1}^{d} \int_{-\pi}^{\pi} \frac{d \theta_{i}}{2 \pi} \frac{\mathrm{e}^{-\sqrt{-1} \mathbf{x}} \cdot \boldsymbol{\theta}}{(1+a)-(1 / d) \sum_{i=1}^{d} \cos \theta_{i}}, \quad \mathbf{x} \in \mathbb{Z}^{d} \tag{3.3}
\end{equation*}
$$

Proof. Consider the Euler-Maclaurin formula for $f \in \mathrm{C}^{2}(\mathbb{R})$,

$$
\begin{equation*}
\sum_{n=0}^{M} f(b+n c)=\frac{1}{c} \int_{b}^{b+M c} f(\theta) d \theta+\frac{1}{2}[f(b)+f(b+M c)]+\frac{1}{12} c^{2} \sum_{n=0}^{M-1} f^{(2)}(b+c(n+\phi)) \tag{3.4}
\end{equation*}
$$

where $M \in \mathbb{N}, b, c \in \mathbb{R}, f^{(2)}(\theta)$ is the second derivative of $f(\theta)$, and $0<\phi<1$ (see, for instance, Appendix D in [1]). Assume that

$$
f(\theta)=\frac{\mathrm{e}^{-\sqrt{-1} \alpha_{1} \theta}}{(1+a)-(1 / d)\left(\cos \theta+\alpha_{2}\right)}
$$

where $a, \alpha_{1}, \alpha_{2}$ are constants. Applying the Euler-Maclaurin formula (3.4) with $b=-2 \pi L /(2 L+$ 1), $M=2 L$ and $c=2 \pi /(2 L+1)$, we have

$$
\begin{aligned}
\sum_{n=0}^{2 L} & \frac{\mathrm{e}^{-2 \pi \sqrt{-1} \alpha_{1}(n-L) /\left(2 L_{1}+1\right)}}{(1+a)-(1 / d)\left\{\cos \left(\frac{2 \pi}{2 L+1}(n-L)\right)+\alpha_{2}\right\}} \\
= & (2 L+1) \int_{-2 \pi L /(2 L+1)}^{2 \pi L /(2 L+1)} \frac{d \theta}{2 \pi} \frac{\mathrm{e}^{-\sqrt{-1} \alpha_{1} \theta}}{(1+a)-(1 / d)\left(\cos \theta+\alpha_{2}\right)} \\
& +\frac{1}{2}\left[f\left(-\frac{2 \pi L}{2 L+1}\right)+f\left(\frac{2 \pi L}{2 L+1}\right)\right] \\
& +\frac{1}{12}\left(\frac{2 \pi}{2 L+1}\right)^{2} \sum_{n=0}^{2 L-1} f^{(2)}\left(\frac{2 \pi}{2 L+1}(n+\phi-L)\right)
\end{aligned}
$$

By dividing the both sides of the equality by $2 L+1$ and take the limit $L \uparrow \infty$, we obtain

$$
\begin{aligned}
\lim _{L \uparrow \infty} & \frac{1}{2 L+1} \sum_{n=-L}^{L} \frac{\mathrm{e}^{-2 \pi \sqrt{-1} \alpha_{1} n /\left(2 L_{1}+1\right)}}{(1+a)-(1 / d)\left\{\cos \left(\frac{2 \pi}{2 L+1} n\right)+\alpha_{2}\right\}} \\
& =\int_{-\pi}^{\pi} \frac{d \theta}{2 \pi} \frac{\mathrm{e}^{-\sqrt{-1} \alpha_{1} \theta}}{(1+a)-(1 / d)\left(\cos \theta+\alpha_{2}\right)}
\end{aligned}
$$

Repeating this procedure $d$ times, we can prove Lemma 3.2.

### 3.2 Long-distance asymptotics

Now we consider the asymptotic form in $|\mathbf{x}| \uparrow \infty$ of $G(\mathbf{x})$. Here we follow the calculation found in Section XII. 4 of [21] for the asymptotic expansion of two-point spin correlation function of the two-dimensional Ising model. By using the identity

$$
\int_{0}^{\infty} d s \mathrm{e}^{-\alpha s}=\frac{1}{\alpha}
$$

and the definition of the modified Bessel function of the first kind

$$
I_{n}(z)=\int_{-\pi}^{\pi} \frac{d \phi}{2 \pi} \mathrm{e}^{-\sqrt{-1} n \phi+z \cos \phi},
$$

we have

$$
G(\mathbf{x})=\frac{1}{2 d n} \int_{0}^{\infty} d s \mathrm{e}^{-(1+a) s} \prod_{i=1}^{d} I_{x_{i}}(s / d) .
$$

The asymptotic expansion of $I_{n}(z)$ for large $n$ is found on p. 86 in [9],

$$
I_{n}(z)=\frac{1}{\sqrt{2 \pi}} \frac{\exp \left[\left(n^{2}+z^{2}\right)^{1 / 2}-n \sinh ^{-1}(n / z)\right]}{\left(n^{2}+z^{2}\right)^{1 / 4}} \times(1+\mathcal{O}(1 / n)),
$$

and we obtain

$$
\begin{align*}
G(\mathbf{x})=\frac{1}{2 d n} & \left(\frac{1}{2 \pi}\right)^{d / 2} \int_{0}^{\infty} d s \prod_{i=1}^{d} \frac{1}{\left[x_{i}^{2}+(s / d)^{2}\right]^{1 / 4}} \exp [-g(\mathbf{x}, s)] \\
& \times\left(1+\mathcal{O}\left(\max _{i}\left\{1 / x_{i}\right\}\right)\right), \tag{3.5}
\end{align*}
$$

where

$$
g(\mathbf{x}, s)=(1+a) s-\sum_{i=1}^{d}\left[x_{i}^{2}+\left(\frac{s}{d}\right)^{2}\right]^{1 / 2}+\sum_{i=1}^{d} x_{i} \sinh ^{-1}\left(\frac{d}{s} x_{i}\right) .
$$

We can evaluate (3.5) by the saddle-point method and obtain the following result.
Theorem 3.3 Let

$$
\begin{equation*}
c_{1}(d, a)=\frac{1}{4 \pi(a+1)}\left[\frac{\sqrt{a(a+2) d}}{2 \pi(a+1)}\right]^{(d-3) / 2} \tag{3.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\xi(d, a)=\frac{1}{\sqrt{d} \sinh ^{-1} \sqrt{a(a+2)}} . \tag{3.7}
\end{equation*}
$$

Then, for the DASM with $d \geq 2, m, n \in \mathbb{N}, a=m /(2 d n)$,

$$
\begin{equation*}
\lim _{r \uparrow \infty}-\frac{1}{r} \log \left[\frac{n r^{(d-1) / 2}}{c_{1}(d, a)} G(\mathbf{x}(r))\right]=\frac{1}{\xi(d, a)}, \tag{3.8}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbf{x}(r)=\left(\frac{r}{\sqrt{d}}, \cdots, \frac{r}{\sqrt{d}}\right) \in \mathbb{Z}^{d}, \quad r>0 . \tag{3.9}
\end{equation*}
$$

Proof. Let $g^{(1)}(\mathbf{x}, s)$ and $g^{(2)}(\mathbf{x}, s)$ be the first and second derivatives of $g(\mathbf{x}, s)$ with respect to $s$,

$$
\begin{aligned}
g^{(1)}(\mathbf{x}, s) & =(1+a)-\frac{1}{d} \sum_{i=1}^{d}\left[1+\left(\frac{d}{s} x_{i}\right)^{2}\right]^{1 / 2} \\
g^{(2)}(\mathbf{x}, s) & =\frac{d}{s^{3}} \sum_{i=1}^{d} x_{i}^{2}\left[1+\left(\frac{d}{s} x_{i}\right)^{2}\right]^{-1 / 2}
\end{aligned}
$$

For each $\mathbf{x}$, let $s_{0}(\mathbf{x})$ be the saddle point at which $g^{(1)}(\mathbf{x}, s)$ vanishes,

$$
\begin{equation*}
g^{(1)}\left(\mathbf{x}, s_{0}(\mathbf{x})\right)=0 \tag{3.10}
\end{equation*}
$$

Then

$$
\begin{aligned}
G(\mathbf{x})=\frac{1}{2 d n} & \left(\frac{1}{2 \pi}\right)^{d / 2} \prod_{i=1}^{d} \frac{1}{\left(x_{i}^{2}+s_{0}(\mathbf{x})^{2} / d^{2}\right)^{1 / 4}} \exp \left[-g\left(x, s_{0}(\mathbf{x})\right)\right] \\
& \times \int_{-\infty}^{\infty} d u \exp \left[-\frac{1}{2} g^{(2)}\left(\mathbf{x}, s_{0}(\mathbf{x})\right) u^{2}\right] \times\left(1+\mathcal{O}\left(\max _{i}\left\{1 / x_{i}\right\}\right)\right) \\
=\frac{1}{2 d n} & \left(\frac{1}{2 \pi}\right)^{d / 2} \prod_{i=1}^{d} \frac{1}{\left(x_{i}^{2}+s_{0}(x)^{2} / d^{2}\right)^{1 / 4}} \exp \left[-g\left(x, s_{0}(x)\right)\right] \\
& \times\left(\frac{2 \pi}{g^{(2)}\left(\mathbf{x}, s_{0}(\mathbf{x})\right)}\right)^{1 / 2} \times\left(1+\mathcal{O}\left(\max _{i}\left\{1 / x_{i}\right\}\right)\right)
\end{aligned}
$$

Here we can prove that the higher derivatives of $g(\mathbf{x}, s)$ only give the contributions of order $\mathcal{O}\left(\max _{i}\left\{1 / x_{i}\right\}\right)$. See p. 304 in [21]. Now we consider the case

$$
x_{i}=\frac{r}{\sqrt{d}}+\varepsilon_{i}
$$

in which $\varepsilon_{i}$ 's are finite and fixed and $r \gg 1$. The equation (3.10) for the saddle point is now

$$
\sum_{i=1}^{d}\left(1+\frac{d^{2}}{s_{0}(\mathbf{x})^{2}}\left(\frac{r}{\sqrt{d}}+\varepsilon_{i}\right)^{2}\right)^{1 / 2}=(1+a) d
$$

and it is solved as

$$
s_{0}(\mathbf{x})=\sqrt{\frac{d}{a(a+2)}}\left(r+\frac{1}{\sqrt{d}} \sum_{i=1}^{d} \varepsilon_{i}+\mathcal{O}(1 / r)\right)
$$

This gives

$$
\begin{aligned}
g\left(\mathbf{x}, s_{0}(\mathbf{x})\right) & =\sum_{i=1}^{d}\left(\frac{r}{\sqrt{d}}+\varepsilon_{i}\right) \sinh ^{-1}\left[\frac{d}{s_{0}(x)}\left(\frac{r}{\sqrt{d}}+\varepsilon_{i}\right)\right] \\
& =\sqrt{d} r \sinh ^{-1} \sqrt{a(a+2)}+\sinh ^{-1} \sqrt{a(a+2)} \times \sum_{i=1}^{d} \varepsilon_{i}+\mathcal{O}(1 / r)
\end{aligned}
$$

and

$$
g^{(2)}\left(\mathbf{x}, s_{0}(x)\right)=\frac{1}{\sqrt{d}} \frac{(a(a+2))^{3 / 2}}{a+1} \frac{1}{r}+\mathcal{O}\left(1 / r^{2}\right) .
$$

Then we have the estimation

$$
G(\mathbf{x})=\frac{c_{1}(d, a)}{n} \frac{1}{r^{(d-1) / 2}} \exp \left[-\frac{r}{\xi(d, a)}-\lambda(a) \sum_{i=1}^{d} \varepsilon_{i}\right] \times(1+\mathcal{O}(1 / r)), \quad \text { as } r \uparrow \infty
$$

for $\mathbf{x}=\left(r / \sqrt{d}+\varepsilon_{1}, \cdots, r / \sqrt{d}+\varepsilon_{d}\right)$, where $c_{1}(d, a)$ and $\xi(d, a)$ are given by (3.6) and (3.7), respectively, and

$$
\begin{align*}
\lambda(a) & \equiv \frac{\sqrt{d}}{\xi(d, a)} \\
& =\sinh ^{-1} \sqrt{a(a+2)} \\
& =\log (1+a+\sqrt{a(a+2)}) . \tag{3.11}
\end{align*}
$$

If we put $\varepsilon_{i}=0,1 \leq i \leq d$, then $G(\mathbf{x})$ is reduced to be

$$
G(\mathbf{x}(r))=\bar{G}(r) \times(1+\mathcal{O}(1 / r)), \quad \text { as } r \uparrow \infty
$$

with

$$
\begin{equation*}
\bar{G}(r)=\frac{c_{1}(d, a)}{n} \frac{\mathrm{e}^{-r / \xi(d, a)}}{r^{(d-1) / 2}} . \tag{3.12}
\end{equation*}
$$

It proves the theorem.

## 4 Height-0 Density and Height-(0,0) Correlations

For

$$
\alpha, \beta \in\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, h_{\mathrm{c}}-\frac{1}{n}\right\},
$$

define

$$
\begin{align*}
P_{\alpha, L}(\mathbf{x}) & =\mathbf{E}_{L}[\mathbf{1}(h(\mathbf{x})=\alpha)], \\
P_{\alpha \beta, L}(\mathbf{x}, \mathbf{y}) & =\mathbf{E}_{L}[\mathbf{1}(h(\mathbf{x})=\alpha) \mathbf{1}(h(\mathbf{y})=\beta)], \quad \mathbf{x}, \mathbf{y} \in \Lambda_{L} . \tag{4.1}
\end{align*}
$$

$P_{\alpha, L}(\mathbf{x})$ is the probability that the site $\mathbf{x}$ has the height $\alpha n$ measured in the unit of grain of sand, $1 / n$, and $P_{\alpha \beta, L}(\mathbf{x}, \mathbf{y})$ is the ( $\alpha, \beta$ )-height correlation function [19, 5, 23].

For the two-dimensional BTW model on $B_{L}$ with open boundary condition, Majumdar and Dhar [19] proved the existence of the infinite-volume limits

$$
\begin{aligned}
P_{0} & =\lim _{L \uparrow \infty} P_{0, L}(\mathbf{x}) \\
P_{00}(\mathbf{x}(r)) & =\lim _{L \uparrow \infty} P_{00, L}(0, \mathbf{x}(r)),
\end{aligned}
$$

where $\mathbf{x}(r)=(r / \sqrt{2}, r / \sqrt{2})$. They gave an $8 \times 8$ matrix $M_{L}(r)$, whose elements depend on $L$ and $r$, such that

$$
P_{00, L}(0, \mathbf{x}(r))=\operatorname{det} M_{L}(r), \quad \forall L>\frac{r}{\sqrt{2}}
$$

and showed that every elements converge in the infinite-volume limit $L \uparrow \infty$ with a finite $r$. Then the matrix $M(r)=\lim _{L \uparrow \infty} M_{L}(r)$ is well-defined and we have the determinantal expression

$$
P_{00}(\mathbf{x}(r))=\operatorname{det} M(r)
$$

Moreover, they showed that

$$
\lim _{r \uparrow \infty} P_{00}(\mathbf{x}(r))=P_{0}^{2}
$$

and

$$
\begin{equation*}
C_{00}(\mathbf{x}(r)) \equiv \frac{P_{00}(\mathbf{x}(r))-P_{0}^{2}}{P_{0}^{2}} \simeq-\frac{1}{2} r^{-4}, \quad \text { as } r \uparrow \infty \tag{4.2}
\end{equation*}
$$

Majumdar and Dhar claimed [19] that the result (4.2) is generalized for the $d$-dimensional BTW model with $d \geq 2$ as

$$
\begin{equation*}
C_{00}(\mathbf{x}(r)) \sim r^{-2 d}, \quad \text { as } r \uparrow \infty \tag{4.3}
\end{equation*}
$$

In an earlier paper [28], all these facts also hold for the two-dimensional DASM, if we prepare $10 \times 10$ matrix $M_{L}(r)$. (See also [5] and [23] for other generalizations of [19].) Here we show the result for the height-0 density and the height- $(0,0)$ correlations of the DASM with general $d \geq 2$.

### 4.1 Nearest-neighbor correlations

First we prove the following Lemma.
Lemma 4.1 Any configuration $h \in \mathcal{S}_{L}$, in which there are two adjacent sites $\mathbf{z}_{1}, \mathbf{z}_{2} \in \Lambda_{L}$, $\left|\mathbf{z}_{1}-\mathbf{z}_{2}\right|=1$, such that $h\left(\mathbf{z}_{1}\right)<1$ and $h\left(\mathbf{z}_{2}\right)<1$, is not allowed.
Proof. Let $F=\left\{\mathbf{z}_{1}, \mathbf{z}_{2}\right\} \subset \Lambda_{L}$. Then

$$
\sum_{\mathbf{x}: \mathbf{x} \in F, \mathbf{x} \neq \mathbf{z}_{1}}\left(-\Delta_{L}\left(\mathbf{x}, \mathbf{z}_{1}\right)\right)=-\Delta_{L}\left(\mathbf{z}_{2}, \mathbf{z}_{1}\right)=1
$$

and

$$
\sum_{\mathbf{x}: \mathbf{x} \in F, \mathbf{x} \neq \mathbf{z}_{2}}\left(-\Delta_{L}\left(\mathbf{x}, \mathbf{z}_{2}\right)\right)=-\Delta_{L}\left(\mathbf{z}_{1}, \mathbf{z}_{2}\right)=1
$$

by (1.1). Then if $h\left(\mathbf{z}_{1}\right)<1$ and $h\left(\mathbf{z}_{2}\right)<1$, the condition of FSC (2.12) is satisfied.
By Propositions 2.6 and 2.12, the above lemma implies the following.
Proposition 4.2 For any $L \geq 2$,

$$
P_{\alpha \beta, L}\left(0, \pm \mathbf{e}_{i}\right)=0, \quad 1 \leq i \leq d, \quad \alpha, \beta \in\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1-\frac{1}{n}\right\}
$$

Then,

$$
P_{\alpha \beta}\left(0, \pm \mathbf{e}_{i}\right)=\lim _{L \uparrow \infty} P_{\alpha \beta, L}\left(0, \pm \mathbf{e}_{i}\right)=0, \quad 1 \leq i \leq d, \quad \alpha, \beta \in\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1-\frac{1}{n}\right\}
$$

### 4.2 Determinatal expressions of $P_{0, L}(0)$ and $P_{00, L}(0, \mathbf{x})$

Let $\mathbf{e}_{i}, 1 \leq i \leq d$ be the $i$-th unit vector in $\mathbb{Z}^{d}$. Define a real symmetric matrix with size $(2 L+1)^{d}$ as

$$
B_{L}^{(0)}(\mathbf{v}, \mathbf{w})=\left\{\begin{aligned}
-h_{\mathrm{c}}+1 / n, & \text { if } \mathbf{v}=\mathbf{w}=0, \\
-1, & \text { if } \mathbf{v}=\mathbf{w},|\mathbf{v}|=1, \mathbf{v} \neq-\mathbf{e}_{d}, \\
-1+1 / n, & \text { if } \mathbf{v}=\mathbf{w}=-\mathbf{e}_{d}, \\
1, & \text { if } \mathbf{v}=0,|\mathbf{w}|=1, \mathbf{w} \neq-\mathbf{e}_{d}, \\
1-1 / n, & \text { if } \mathbf{v}=0, \mathbf{w}=-\mathbf{e}_{d}, \\
0, & \text { otherwise },
\end{aligned}\right.
$$

where $\mathbf{v}, \mathbf{w} \in \Lambda_{L}$.
Lemma 4.3 Let $E_{L}$ be the unit matrix with size $(2 L+1)^{d}$. Then

$$
P_{0, L}(0)=\operatorname{det}\left(E_{L}+n G_{L} B_{L}^{(0)}\right) .
$$

Proof. Define a set of allowed configurations conditioned $h(0)=0$,

$$
\mathcal{A}_{L}^{(0)}=\left\{h \in \mathcal{A}_{L}: h(0)=0\right\} .
$$

By definition (4.1), Proposition 2.6 with Lemma 2.3 and Proposition 2.12 gives

$$
\begin{equation*}
P_{0, L}(0)=\frac{\left|\mathcal{A}_{L}^{(0)}\right|}{n^{(2 L+1)^{d}} \operatorname{det} \Delta_{L}} . \tag{4.4}
\end{equation*}
$$

Assume that $h \in \mathcal{A}_{L}^{(0)}$. Then as shown in the proof of Lemma 2.11 we can uniquely define a burning process $T_{t}, t \in\left\{0,1, \ldots,{ }^{\exists} \sigma\right\}$ on $\left(G_{L}, h\right)$ associated that $T_{t}$ becomes a spanning tree on $G_{L}$ at time $t=\sigma$. Define a configuration $h^{\prime}$ as

$$
h^{\prime}(\mathbf{z})= \begin{cases}h(\mathbf{z})-1, & \text { if }|\mathbf{z}|=1, \mathbf{z} \neq-\mathbf{e}_{d}, \\ h(\mathbf{z})-1+1 / n, & \text { if } \mathbf{z}=-\mathbf{e}_{d}, \\ h(\mathbf{z}), & \text { otherwise }\end{cases}
$$

for $\mathbf{z} \in \Lambda_{L}$. Now we consider a new DASM which is defined by the matrix $\Delta_{L}^{\prime}$ given by

$$
\begin{equation*}
\Delta_{L}^{\prime}=\Delta_{L}+B_{L}^{(0)} \tag{4.5}
\end{equation*}
$$

and let $\mathcal{A}_{L}^{\prime}$ be a set of all allowed configurations of this DASM and $G_{L}^{\prime}$ be an associated graph to ( $\left.\Lambda_{L}, \Delta_{L}^{\prime}\right)$. Then we consider a burning process $T_{t}^{\prime}=\left(V_{t}^{\prime}, E_{t}^{\prime}\right), t \in\{0,1, \ldots, \sigma\}$ on $\left(G_{L}^{\prime}, h^{\prime}\right)$. By definition of $\Delta_{L}^{\prime}$ and $h^{\prime}$, we can make

$$
V_{t}=V_{t}^{\prime}, \quad \forall t \in\{0,1, \ldots, \sigma\},
$$

and $T_{\sigma}^{\prime}$ gives a spanning tree on $G_{L}^{\prime}$. By Lemma 2.11, this means $h^{\prime} \in \mathcal{A}_{L}^{\prime}$. Since there is a bijection between $h$ and its associated burning process $T_{t}, t \in\{0,1, \ldots, \sigma\}$, we have a bijection
between $\mathcal{A}_{L}^{(0)}$ and $\mathcal{A}_{L}^{\prime}$. By Lemmas 2.10 and $2.11,\left|\mathcal{A}_{L}^{(0)}\right|=\left|\mathcal{A}_{L}^{\prime}\right|=n^{(2 L+1)^{d}} \operatorname{det} \Delta_{L}^{\prime}$. Combining (4.4) and (4.5) gives

$$
\begin{aligned}
P_{0, L}(0) & =\frac{\operatorname{det} \Delta_{L}^{\prime}}{\operatorname{det} \Delta_{L}} \\
& =\operatorname{det}\left(\Delta_{L}^{-1} \Delta_{L}^{\prime}\right) \\
& =\operatorname{det}\left(E_{L}+\Delta_{L}^{-1} B_{L}^{(0)}\right)
\end{aligned}
$$

Then we use Lemma 3.1 and the proof is completed.
Next we consider the two-point function $P_{00, L}(0, \mathbf{x})$, where we assume that $2 \leq|\mathbf{x}|<L$. We define a real symmetric matrix with size $(2 L+1)^{d}$ as follows. For $\mathbf{v}, \mathbf{w} \in \Lambda_{L}$,

$$
B_{L}^{(0, \mathbf{x})}(\mathbf{v}, \mathbf{w})=\left\{\begin{aligned}
-h_{\mathrm{c}}+1 / n, & \text { if } \mathbf{v}=\mathbf{w}=0 \text { or if } \mathbf{v}=\mathbf{w}=\mathbf{x} \\
-1, & \text { if } \mathbf{v}=\mathbf{w},|\mathbf{v}|=1, \mathbf{v} \neq-\mathbf{e}_{d} \\
& \text { or if } \mathbf{v}=\mathbf{w},|\mathbf{v}-\mathbf{x}|=1, \mathbf{v} \neq \mathbf{x}-\mathbf{e}_{d} \\
-1+1 / n, & \text { if } \mathbf{v}=\mathbf{w}=-\mathbf{e}_{d}, \quad \text { or if } \mathbf{v}=\mathbf{w}=\mathbf{x}-\mathbf{e}_{d} \\
1, & \text { if } \mathbf{v}=0,|\mathbf{w}|=1, \mathbf{w} \neq-\mathbf{e}_{d}, \\
1-1 / n, & \begin{array}{l}
\text { or if } \mathbf{v}=\mathbf{x},|\mathbf{w}-\mathbf{x}|=1, \mathbf{w} \neq \mathbf{x}-\mathbf{e}_{d} \\
\text { if } \mathbf{v}=0, \mathbf{w}=-\mathbf{e}_{d} \\
\\
0,
\end{array} \begin{array}{l}
\text { or if } \mathbf{v}=\mathbf{x}, \mathbf{w}=\mathbf{x}-\mathbf{e}_{d} \\
\text { otherwise. }
\end{array}
\end{aligned}\right.
$$

Following the same argument as $P_{0, L}(0)$ we can prove the next lemma. (See Fig.6.)
Lemma 4.4 For $2 \leq|\mathbf{x}|<L$,

$$
P_{00, L}(0, \mathbf{x})=\operatorname{det}\left(E_{L}+n G_{L} B_{L}^{(0, \mathbf{x})}\right)
$$



Figure 6: The matrix $\Delta_{L}^{\prime \prime} \equiv \Delta_{L}+B_{L}^{(0, \mathbf{x})}$ is considered for $P_{00, L}(0, \mathbf{x})$ with $|\mathbf{x}|=r$. In the corresponding graph $G_{L}^{\prime \prime}$ the site 0 (resp. $\mathbf{x}$ ) is connected to $-\mathbf{e}_{d}$ (resp. $\mathbf{x}-\mathbf{e}_{d}$ ) by a single edge, but all other edges between 0 (resp. x) and its nearest-neighbor sites are deleted.

### 4.3 Infinite-volume limit

Since the number of nonzero elements of $B_{L}^{(0)}$ (resp. $B_{L}^{(0, \mathbf{x})}$ ) is only $6 d+1$ (resp. $2(6 d+1)$ ), we can replace the matrix $E_{L}+n G_{L} B_{L}^{(0)}$ (resp. $E_{L}+n G_{L} B_{L}^{(0, \mathbf{x})}$ ) with size $(2 L+1)^{d}$ by a matrix with size $(2 d+1)($ resp. $2(2 d+1)$ ) without changing the value of determinant. Explicit expressions are given as follows.

Let

$$
\mathbf{q}_{i}= \begin{cases}0, & \text { if } \quad i=1, \\ \mathbf{e}_{i-1}, & \text { if } \quad 2 \leq i \leq d+1, \\ -\mathbf{e}_{i-d-1}, & \text { if } \quad d+2 \leq i \leq 2 d+1\end{cases}
$$

Define a matrix $\mathcal{G}^{(L)}(\mathbf{x})=\left(\mathcal{G}_{i j}^{(L)}\right)_{1 \leq i, j \leq 2 d+1}$ with elements

$$
\begin{equation*}
\mathcal{G}_{i j}^{(L)}(\mathbf{x})=G_{L}\left(0, \mathbf{x}+\mathbf{q}_{j}-\mathbf{q}_{i}\right), \quad 1 \leq i, j \leq 2 d+1 . \tag{4.6}
\end{equation*}
$$

We also define a real symmetric matrix $\mathcal{B}=\left(\mathcal{B}_{i j}\right)_{1 \leq i, j \leq 2 d+1}$ with elements

$$
\mathcal{B}_{i j}= \begin{cases}-h_{\mathrm{c}}+1 / n, & \text { if } \quad i=j=1, \\ -1, & \text { if } 2 \leq i=j \leq 2 d, \\ -1+1 / n, & \text { if } i=j=2 d+1, \\ 1, & i=1,2 \leq j \leq 2 d, \\ 1-1 / n, & \text { if } i=1, j=2 d+1, \\ 0, & \text { otherwise. }\end{cases}
$$

Then define $2(2 d+1) \times 2(2 d+1)$ matrices

$$
\tilde{\mathcal{G}}^{(L)}(0, \mathbf{x})=\left(\begin{array}{ll}
\mathcal{G}^{(L)}(0) & \mathcal{G}^{(L)}(\mathbf{x}) \\
{ }^{t} \mathcal{G}^{(L)}(\mathbf{x}) & \mathcal{G}^{(L)}(0)
\end{array}\right), \quad \mathbf{x} \in \Lambda_{L},
$$

where ${ }^{t} \mathcal{G}^{(L)}(\mathbf{x})$ is a transpose of $\mathcal{G}^{(L)}(\mathbf{x})$, and

$$
\tilde{\mathcal{B}}=\left(\begin{array}{cc}
\mathcal{B} & 0 \\
0 & \mathcal{B}
\end{array}\right) .
$$

We have

$$
\begin{equation*}
P_{0, L}(0)=\operatorname{det}\left(E+n \mathcal{G}^{(L)}(0) \mathcal{B}\right) \tag{4.7}
\end{equation*}
$$

and

$$
\begin{equation*}
P_{00, L}(0, \mathbf{x})=\operatorname{det}\left(E+n \tilde{\mathcal{G}}^{(L)}(0, \mathbf{x}) \tilde{\mathcal{B}}\right) \tag{4.8}
\end{equation*}
$$

where $E$ denotes the unit matrix with size $2 d+1$ in (4.7) and with size $2(2 d+1)$ in (4.8), respectively.

It should be remarked that the sizes of the matrices in the RHS's are independent of the lattice size $L$ and determined only by the dimension $d$ of lattice. The dependence of $L$ is introduced only through each elements of $\mathcal{G}^{(L)}(\mathbf{x})$ given by (4.6). Lemma 3.2 guarantees the existence of infinite-volume limit $L \uparrow \infty$ of these elements and we put

$$
\begin{aligned}
\mathcal{G}_{i j}(\mathbf{x}) & =\lim _{L \uparrow \infty} \mathcal{G}_{i j}^{(L)}(\mathbf{x})=G\left(\mathbf{x}+\mathbf{q}_{j}-\mathbf{q}_{i}\right), \quad 1 \leq i, j \leq 2 d+1, \\
\mathcal{G}(\mathbf{x}) & =\left(\mathcal{G}_{i j}(\mathbf{x})\right)_{1 \leq i, j \leq 2 d+1}, \\
\tilde{\mathcal{G}}(0, \mathbf{x}) & =\lim _{L \uparrow \infty} \tilde{\mathcal{G}}^{(L)}(0, \mathbf{x})=\left(\begin{array}{ll}
\mathcal{G}(0) & \mathcal{G}(\mathbf{x}) \\
{ }^{t} \mathcal{G}(\mathbf{x}) & \mathcal{G}(0)
\end{array}\right),
\end{aligned}
$$

where $G(\mathbf{x})$ is explicitly given by (3.3). Then we have the following.
Proposition 4.5 There exist the infinite-volume limits

$$
P_{0}=\lim _{L \uparrow \infty} P_{0, L}(0), \quad P_{00}(\mathbf{x})=\lim _{L \uparrow \infty} P_{00, L}(0, \mathbf{x}), \quad \mathbf{x} \in \mathbb{Z}^{d},
$$

and they are given by

$$
P_{0}=\operatorname{det}(E+n \mathcal{G}(0) \mathcal{B})
$$

and

$$
P_{00}(\mathbf{x})=\operatorname{det}(E+n \tilde{\mathcal{G}}(0, \mathbf{x}) \tilde{\mathcal{B}}), \quad \mathbf{x} \in \mathbb{Z}^{d}
$$

### 4.4 Evaluations of determinantal expressions

From the determinantal expressions of $P_{0}$ and $P_{00}(\mathbf{x})$ given in Proposition 4.5, the following explicit evaluations of these quantities are obtained.

Theorem 4.6 (i) Define

$$
\gamma_{1}=\frac{1}{2 d} \prod_{i=1}^{d} \int_{-\pi}^{\pi} \frac{d \theta_{i}}{2 \pi} \frac{1}{(1+a)-(1 / d) \sum_{i=1}^{d} \cos \theta_{i}}
$$

and

$$
\gamma_{2}=\frac{1}{2 d} \prod_{i=1}^{d} \int_{-\pi}^{\pi} \frac{d \theta_{i}}{2 \pi} \frac{\mathrm{e}^{-2 \sqrt{-1}\left(\theta_{1}+\theta_{2}\right)}}{(1+a)-(1 / d) \sum_{i=1}^{d} \cos \theta_{i}} .
$$

Then, for the DASM with $d \geq 2, m, n \in \mathbb{N}$,

$$
\begin{align*}
P_{0} & =\frac{1-2 d a \gamma_{1}}{2 d n}\left[2\left\{1-d\left(\gamma_{1}-\gamma_{2}\right)\right\}+\left(1-4 d \gamma_{1}\right) a-2 d \gamma_{1} a^{2}\right] \\
& \times\left[2(d-1)\left(\gamma_{1}-\gamma_{2}\right)-\left(1-4 d \gamma_{1}\right) a+2 d \gamma_{1} a^{2}\right]^{2} \\
& \times\left[\left\{1-\left(\gamma_{1}-\gamma_{2}\right)\right\}^{2}-\left\{\left(2 d(1+a)^{2}-1\right) \gamma_{1}-(2 d-1) \gamma_{2}-(1+a)\right\}^{2}\right]^{d-2} \tag{4.9}
\end{align*}
$$

where $a=m /(2 d n)$.
(ii) Let

$$
\begin{equation*}
C_{00}(\mathbf{x})=\frac{P_{00}(\mathbf{x})-P_{0}^{2}}{P_{0}^{2}}, \quad \mathbf{x} \in \mathbb{Z}^{d} \tag{4.10}
\end{equation*}
$$

Then, there exists a nonzero factor $c_{2}(d, a, n)$ such that for the DASM with $d \geq 2, m, n \in \mathbb{N}$

$$
\begin{equation*}
\lim _{r \uparrow \infty}-\frac{1}{r} \log \left[\frac{r^{d-1}}{c_{2}(d, a, n)} C_{00}(\mathbf{x}(r))\right]=\frac{2}{\xi(d, a)}, \tag{4.11}
\end{equation*}
$$

where $a=m /(2 d n), \xi(d, a)$ and $\mathbf{x}(r)$ are given by (3.7) and (3.9), respectively, and that

$$
\begin{equation*}
\lim _{a \downarrow 0} \frac{c_{2}(d, a, m /(2 d a))}{a^{(d+1) / 2}}=\left(\frac{d}{2 \pi^{2}}\right)^{(d-3) / 2}\left[\frac{d\{1+(d-1) \bar{\gamma}\}}{2 \pi(d-1) \bar{\gamma}}\right]^{2}, \tag{4.12}
\end{equation*}
$$

where

$$
\bar{\gamma}=\frac{1}{2 d} \prod_{i=1}^{d} \int_{-\pi}^{\pi} \frac{d \theta_{i}}{2 \pi} \frac{1-\mathrm{e}^{-2 \sqrt{-1}\left(\theta_{1}+\theta_{2}\right)}}{1-(1 / d) \sum_{i=1}^{d} \cos \theta_{i}}
$$

In the following, we will explain how to prove this theorem. Let

$$
M^{(1)}(r)=E+n \tilde{\mathcal{G}}(0, \mathbf{x}(r)) \tilde{\mathcal{B}}, \quad r>0, \quad \mathbf{x}(r) \in \mathbb{Z}^{d}
$$

where $E$ is a unit matrix with size $2(2 d+1)$. That is,

$$
M^{(1)}(r)=\left(\begin{array}{ll}
m^{(1)} & \tilde{m}^{(1)}(r) \\
\hat{m}^{(1)}(r) & m^{(1)}
\end{array}\right)
$$

where for $1 \leq i \leq 2 d+1$

$$
\begin{aligned}
& m_{i j}^{(1)}= \begin{cases}1(i=1)+\sum_{k=1}^{2 d+1} n \mathcal{G}_{i k}(0) & \text { if } j=1, \\
-\left\{(1-1 / n)+h_{\mathrm{c}}\right\} n \mathcal{G}_{i 1}(0)-\mathcal{G}_{i 2 d+1}(0), & \text { if } 2 \leq j \leq 2 d, \\
\mathbf{1}(i=j)+n\left[\mathcal{G}_{i 1}(0)-\mathcal{G}_{i j}(0)\right], & \text { if } j=2 d+1,\end{cases} \\
& \tilde{m}_{i j}^{(1)}(r)=n \times \begin{cases}\sum_{k=1}^{2 d+1} \mathcal{G}_{i k}(\mathbf{x}(r)) & \\
-\left\{(1-1 / n)+h_{\mathrm{c}}\right\} \mathcal{G}_{i 1}(\mathbf{x}(r))-(1 / n) \mathcal{G}_{i 2 d+1}(\mathbf{x}(r)), & \text { if } j=1, \\
\mathcal{G}_{i 1}(\mathbf{x}(r))-\mathcal{G}_{i j}(\mathbf{x}(r)), & \text { if } 2 \leq j \leq 2 d, \\
(1-1 / n)\left(\mathcal{G}_{i 1}(\mathbf{x}(r))-\mathcal{G}_{i 2 d+1}(\mathbf{x}(r))\right), & \text { if } j=2 d+1,\end{cases} \\
& \hat{m}_{i j}^{(1)}(r)=n \times \begin{cases}\sum_{k=1}^{2 d+1} \mathcal{G}_{k i}(\mathbf{x}(r)) & \\
-\left\{(1-1 / n)+\eta_{\mathrm{c}}\right\} \mathcal{G}_{1 i}(\mathbf{x}(r))-(1 / n) \mathcal{G}_{2 d+1 i}(\mathbf{x}(r)), & \text { if } j=1, \\
\mathcal{G}_{1 i}(\mathbf{x}(r))-\mathcal{G}_{j i}(\mathbf{x}(r)), & \text { if } 2 \leq j \leq 2 d, \\
(1-1 / n)\left(\mathcal{G}_{1 i}(\mathbf{x}(r))-\mathcal{G}_{2 d+1 i}(\mathbf{x}(r))\right), & \text { if } j=2 d+1 .\end{cases}
\end{aligned}
$$

We find that

$$
\begin{aligned}
& m_{i 1}^{(1)}+\sum_{j=2}^{2 d+1} m_{i j}^{(1)}=1-2 \operatorname{dan} \mathcal{G}_{i 1}(0) \\
& \tilde{m}_{i 1}^{(1)}(r)+\sum_{j=2}^{2 d+1} \tilde{m}_{i j}^{(1)}=-2 \operatorname{dan} \mathcal{G}_{i 1}(\mathbf{x}(r)) \\
& \hat{m}_{i 1}^{(1)}(r)+\sum_{j=2}^{2 d+1} \hat{m}_{i j}^{(1)}=-2 \operatorname{dan} \mathcal{G}_{1 i}(\mathbf{x}(r)), \quad 1 \leq i \leq 2 d+1 .
\end{aligned}
$$

For $1 \leq i \leq 2 d+1$, let

$$
\begin{aligned}
m_{i j} & = \begin{cases}1-2 \operatorname{dan} \mathcal{G}_{i 1}(0), & \text { if } j=1, \\
m_{i j}^{(1)}, & \text { if } 2 \leq j \leq 2 d+1,\end{cases} \\
\tilde{m}_{i j}(r) & = \begin{cases}-2 \operatorname{dan} \mathcal{G}_{i 1}(\mathbf{x}(r)), & \text { if } \quad j=1, \\
\tilde{m}_{i j}^{(1)}(r), & \text { if } 2 \leq j \leq 2 d+1,\end{cases} \\
\hat{m}_{i j}(r) & = \begin{cases}-2 \operatorname{dan} \mathcal{G}_{1 i}(\mathbf{x}(r)), & \text { if } j=1, \\
\hat{m}_{i j}^{(1)}(r), & \text { if } 2 \leq j \leq 2 d+1 .\end{cases}
\end{aligned}
$$

Then

$$
\begin{align*}
P_{0} & =\operatorname{det} m^{(1)}=\operatorname{det} m \\
P_{00}(\mathbf{x}(r)) & =\operatorname{det} M^{(1)}(r)=\operatorname{det} M(r) \quad \text { with } \quad M(r)=\left(\begin{array}{ll}
m & \tilde{m}(r) \\
\hat{m}(r) & m
\end{array}\right) \tag{4.13}
\end{align*}
$$

Note that, if we introduce the the dipole potential

$$
\phi_{\left(i_{1}, j_{1}\right),\left(i_{2}, j_{2}\right)}(\mathbf{x}(r))=\mathcal{G}_{i_{1} j_{1}}(\mathbf{x}(r))-\mathcal{G}_{i_{2} j_{2}}(\mathbf{x}(r)), \quad 1 \leq i_{1}, i_{2}, j_{1}, j_{2} \leq 2 d+1
$$

the elements of the matrix $M(r)$ are expressed as follows; for $1 \leq i \leq 2 d+1$,

$$
\begin{gather*}
m_{i j}= \begin{cases}1-2 d a n \mathcal{G}_{i 1}(0), & \text { if } j=1, \\
\mathbf{1}(i=j)+n \phi_{(i, 1),(i, j)}(0), & \text { if } 2 \leq j \leq 2 d, \\
\mathbf{1}(i=2 d+1)+(1-1 / n) n \phi_{(i, 1),(i, 2 d+1)}(0), & \text { if } j=2 d+1,\end{cases}  \tag{4.14}\\
\tilde{m}_{i j}(r)=n \times \begin{cases}-2 d a \mathcal{G}_{i 1}(\mathbf{x}(r)), & \text { if } j=1, \\
\phi_{(i, 1),(i, j)}(\mathbf{x}(r)), & \text { if } 2 \leq j \leq 2 d, \\
(1-1 / n) \phi_{(i, 1),(i, 2 d+1)}(\mathbf{x}(r)), & \text { if } j=2 d+1,\end{cases} \\
\hat{m}_{i j}(r)=n \times \begin{cases}-2 d a \mathcal{G}_{1 i}(\mathbf{x}(r)), \\
\phi_{(1, i),(j, i)}(\mathbf{x}(r)), \\
(1-1 / n) \phi_{(1, i),(2 d+1, i)}(\mathbf{x}(r)), & \text { if } \quad j=2 d+1\end{cases}
\end{gather*}
$$

Now we study the asymptotics of $P_{00}(r)$ in $r \uparrow \infty$. Theorem 3.3 and its proof given in Section 3 implies that with any finite $c_{i}$ 's,

$$
G\left(\mathbf{x}(r)+\sum_{i=1}^{d} c_{i} \mathbf{e}_{i}\right)=\bar{G}(r) \exp \left(-\lambda(a) \sum_{i=1}^{d} c_{i}\right) \times(1+\mathcal{O}(1 / r)), \quad \text { as } r \uparrow \infty
$$

with $(3.6),(3.7),(3.11)$, and (3.12). Then we see

$$
\begin{aligned}
\tilde{m}(r) & =n \bar{G}(r) n(r, \lambda)(1+\mathcal{O}(1 / r)) \\
\hat{m}(r) & =n \bar{G}(r) n(r,-\lambda)(1+\mathcal{O}(1 / r)), \quad \text { as } r \uparrow \infty
\end{aligned}
$$

where $n(r, \lambda)=\left(n_{i j}(r, \lambda)\right)_{1 \leq i, j \leq 2 d+1}$ with elements,

$$
n_{i j}(r, \lambda)= \begin{cases}-2 d a, & \text { if } i=j=1, \\ \left(1-\mathrm{e}^{-\lambda}\right), & \text { if } i=1,2 \leq j \leq d+1 \\ \left(1-\mathrm{e}^{\lambda}\right), & \text { if } i=1, d+2 \leq j \leq 2 d \\ (1-1 / n)\left(1-\mathrm{e}^{\lambda}\right), & \text { if } i=1, j=2 d+1 \\ -2 d a \mathrm{e}^{\lambda}, & \text { if } 2 \leq i \leq d+1, j=1, \\ -2 d a \mathrm{e}^{-\lambda}, & \text { if } d+2 \leq i \leq 2 d+1, j=1 \\ \mathrm{e}^{\lambda}\left(1-\mathrm{e}^{-\lambda}\right), & \text { if } 2 \leq i, j \leq d+1, \\ \mathrm{e}^{\lambda}\left(1-\mathrm{e}^{\lambda}\right), & \text { if } 2 \leq i \leq d+1, d+2 \leq j \leq 2 d \\ (1-1 / n) \mathrm{e}^{\lambda}\left(1-\mathrm{e}^{\lambda}\right), & \text { if } 2 \leq i \leq d+1, j=2 d+1 \\ \mathrm{e}^{-\lambda}\left(1-\mathrm{e}^{-\lambda}\right), & \text { if } d+2 \leq i \leq 2 d+1,2 \leq j \leq d+1 \\ \mathrm{e}^{-\lambda}\left(1-\mathrm{e}^{\lambda}\right), & \text { if } d+2 \leq i, j \leq 2 d \\ (1-1 / n) \mathrm{e}^{-\lambda}\left(1-\mathrm{e}^{\lambda}\right), & \text { if } d+2 \leq i \leq 2 d+1, j=2 d+1\end{cases}
$$

We obtain a matrix $M^{\prime}(r)$ from $M(r)$ by subtracting (the first row) $\times \mathrm{e}^{\lambda}$ from the $i$-th row with $2 \leq i \leq d+1$, (the first row) $\times \mathrm{e}^{-\lambda}$ from the $i$-th row with $d+2 \leq i \leq 2 d+1$, (the $(2 d+2)$-th row) $\times \mathrm{e}^{-\lambda}$ from the $i$-th row with $2 d+3 \leq i \leq 3 d+2$, and (the $(2 d+2)$-th row) $\times \mathrm{e}^{\lambda}$ from the $i$-th row with $3 d+3 \leq i \leq 2(2 d+1)$. We have

$$
M^{\prime}(r)=\left(\begin{array}{ll}
m^{\prime}(\lambda) & \tilde{m}^{\prime}(r, \lambda) \\
\tilde{m}^{\prime}(r,-\lambda) & m^{\prime}(-\lambda)
\end{array}\right)
$$

with

$$
m_{i j}^{\prime}(\lambda)=\left\{\begin{array}{lll}
1-2 \operatorname{dan} \mathcal{G}_{11}(0), & \text { if } \quad i=j=1,  \tag{4.15}\\
n \phi_{(1,1),(1, j)}(0), & \text { if } \quad i=1,2 \leq j \leq 2 d \\
(1-1 / n) n \phi_{(1,1),(1,2 d+1)}(0), & \text { if } \quad i=1, j=2 d+1, \\
\left(1-\mathrm{e}^{\lambda}\right)-2 \operatorname{dan}\left(\mathcal{G}_{i 1}(0)-\mathrm{e}^{\lambda} \mathcal{G}_{11}(0)\right), & \text { if } 2 \leq i \leq d+1, j=1 \\
\left(1-\mathrm{e}^{-\lambda}\right)-2 \operatorname{dan}\left(\mathcal{G}_{i 1}(0)-\mathrm{e}^{-\lambda} \mathcal{G}_{11}(0)\right), & \text { if } \quad d+2 \leq i \leq 2 d+1, j=1 \\
\mathbf{1}(i=j)+n\left[\phi_{(i, 1),(i, j)}(0)-\mathrm{e}^{\lambda} \phi_{(1,1),(1, j)}(0)\right], & \text { if } \quad 2 \leq i \leq d+1,2 \leq j \leq 2 d, \\
\mathbf{1}(i=j)+n\left[\phi_{(i, 1),(i, j)}(0)-\mathrm{e}^{-\lambda} \phi_{(1,1),(1, j)}(0)\right], & \text { if } \quad d+2 \leq i \leq 2 d+1, \\
& & 2 \leq j \leq 2 d \\
(1-1 / n) & & \\
\times n\left[\phi_{(i, 1),(i, 2 d+1)}(0)-\mathrm{e}^{\lambda} \phi_{(1,1),(1,2 d+1)}(0)\right], & \text { if } \quad 2 \leq i \leq d+1, j=2 d+1 \\
\mathbf{1}(i=2 d+1)+(1-1 / n) & \\
\times n\left[\phi_{(i, 1),(i, 2 d+1)}(0)-\mathrm{e}^{-\lambda} \phi_{(1,1),(1,2 d+1)}(0)\right], & \text { if } \quad d+2 \leq i \leq 2 d+1 \\
& & j=2 d+1,
\end{array}\right.
$$

and with

$$
\tilde{m}_{i j}^{\prime}(r, \lambda)=n \bar{G}(r) \times \begin{cases}-2 d a(1+\mathcal{O}(1 / r)), & \text { if } \quad i=j=1  \tag{4.16}\\ \left(1-\mathrm{e}^{-\lambda}\right)(1+\mathcal{O}(1 / r)), & \text { if } i=1,2 \leq j \leq d+1 \\ \left(1-\mathrm{e}^{\lambda}\right)(1+\mathcal{O}(1 / r)), & \text { if } \quad i=1, d+2 \leq j \leq 2 d \\ (1-1 / n)\left(1-\mathrm{e}^{\lambda}\right)(1+\mathcal{O}(1 / r)), & \text { if } \quad i=1, j=2 d+1 \\ \mathcal{O}(1 / r), & \text { otherwise }\end{cases}
$$

so that

$$
P_{0} 0(\mathbf{x}(r))=\operatorname{det} M(r)=\operatorname{det} M^{\prime}(r), \quad r>0, \quad \mathbf{x}(r) \in \mathbb{Z}^{d}
$$

Now we expand det $M^{\prime}(r)$ along the first and the $(2 d+2)$-th rows. Let $\left|M^{\prime}(j, k)\right|$ be the determinant of $M^{\prime}(r)$ with the first and the $(2 d+2)$-th rows and the $j$-th and the $k$-th columns removed and multiplied by $-(-1)^{1+j} \times(-1)^{2 d+2+k}=(-1)^{j+k}$. Then we have

$$
\operatorname{det} M^{\prime}(r)=\sum_{j=1}^{2(2 d+1)} \sum_{k=1, k \neq j}^{2(2 d+1)} M^{\prime}(r)_{1 j} M^{\prime}(r)_{2 d+2, k}\left|M^{\prime}(j, k)\right|
$$

Remark that, by (4.15) and (4.16),

$$
\left|M^{\prime}(j, k)\right|=\mathcal{O}(1 / r), \quad \text { as } r \rightarrow \infty
$$

if $1 \leq j, k \leq 2 d+1$ or $2 d+2 \leq j, k \leq 2(2 d+1)$, and

$$
\left|M^{\prime}(j, k)\right|=\left|m^{\prime(j)}(\lambda)\right| \times\left|m^{\prime(k)}(\lambda)\right| \times(1+\mathcal{O}(1 / r)), \quad \text { as } r \rightarrow \infty
$$

if $1 \leq j \leq 2 d+1<k \leq 2(2 d+1)$ or $1 \leq k \leq 2 d+1<j \leq 2(2 d+1)$, where $\left|m^{\prime(j)}(\lambda)\right|$ is the $(1, j)$-cofactor of $m^{\prime}(\lambda)$. Then

$$
\begin{align*}
& \operatorname{det} M^{\prime}(r)=\left(\sum_{j=1}^{2 d+1} m_{1 j}^{\prime}(\lambda)\left|m^{\prime(j)}(\lambda)\right|\right)\left(\sum_{j=1}^{2 d+1} m_{1 j}^{\prime}(-\lambda)\left|m^{\prime(j)}(-\lambda)\right|\right) \\
&+\left(\sum_{j=1}^{2 d+1} \tilde{m}_{1 j}^{\prime}(r, \lambda)\left|m^{\prime(j)}(-\lambda)\right|\right)\left(\sum_{j=1}^{2 d+1} \tilde{m}_{1 j}^{\prime}(r,-\lambda)\left|m^{\prime(j)}(-\lambda)\right|\right) \\
&=\operatorname{det} m^{\prime}(\lambda) \times \operatorname{det} m^{\prime}(-\lambda)+\operatorname{det} \bar{m}(\lambda) \times \operatorname{det} \bar{m}(-\lambda) \times(n \bar{G}(r))^{2}(1+\mathcal{O}(1 / r)) \tag{4.17}
\end{align*}
$$

where $\bar{m}(\lambda)=\left(\bar{m}_{i j}(\lambda)\right)_{1 \leq i, j \leq 2 d+1}$ with elements

$$
\bar{m}_{i j}(\lambda)= \begin{cases}-2 d a, & \text { if } i=j=1 \\ 1-\mathrm{e}^{\lambda}, & \text { if } i=1,2 \leq j \leq d+1 \\ 1-\mathrm{e}^{-\lambda}, & \text { if } i=1, d+2 \leq j \leq 2 d \\ (1-1 / n)\left(1-\mathrm{e}^{-\lambda}\right), & \text { if } i=1, j=2 d+1 \\ m_{i j}^{\prime}(\lambda), & \text { otherwise }\end{cases}
$$

We find that

$$
\begin{equation*}
\operatorname{det} m^{\prime}(\lambda)=\operatorname{det} m^{\prime}(-\lambda)=\operatorname{det} m \tag{4.18}
\end{equation*}
$$

The determinantal expressions (4.13) with (3.12), (4.17), and (4.18) give

$$
\begin{aligned}
\lim _{r \uparrow \infty} P_{00}(\mathbf{x}(r)) & =\lim _{r \uparrow \infty}\left\{(\operatorname{det} m)^{2}+\operatorname{det} \bar{m}(\lambda) \operatorname{det} \tilde{m}(-\lambda)(n \bar{G}(r))^{2}\right\} \\
& =(\operatorname{det} m)^{2}=P_{0}^{2}
\end{aligned}
$$

Here we set

$$
\operatorname{det} \bar{m}(\lambda)=a \operatorname{det} m^{*}(\lambda)
$$

with a matrix $m^{*}(\lambda)=\left(m_{i j}^{*}(\lambda)\right)_{1 \leq i, j \leq 2 d+1}$ with elements

By the definition (4.10), we see

$$
C_{00}(\mathbf{x}(r))=a^{2} \frac{\operatorname{det} m^{*}(\lambda) \operatorname{det} m^{*}(-\lambda)}{(\operatorname{det} m)^{2}}(n \bar{G}(r))^{2} \times(1+\mathcal{O}(1 / r)), \quad \text { as } r \uparrow \infty .
$$

Since $\bar{G}(r)$ is given by (3.12), (4.11) of Theorem 4.6 (ii) is proved with

$$
c_{2}(d, a, n)=\left(a c_{1}(d, a)\right)^{2} \frac{\operatorname{det} m^{*}(\lambda) \times \operatorname{det} m^{*}(-\lambda)}{(\operatorname{det} m)^{2}} .
$$

Now the problem is reduced to the calculation of $\operatorname{det} m$ and $\operatorname{det} m^{*}(\lambda)$. Consider a matrix $R=\left(R_{i j}\right)_{1 \leq i, j \leq N}$ with elements

$$
R_{i j}= \begin{cases}u, & \text { if } i=j=1,  \tag{4.20}\\ b, & \text { if } i=1,2 \leq j \leq d+1, \\ c, & \text { if } i=1, d+2 \leq j \leq 2 d, \\ (1-1 / n) c, & \text { if } i=1, j=2 d+1, \\ q, & \text { if } 2 \leq i \leq d+1, j=1, \\ e, & \text { if } d+2 \leq i \leq 2 d+1, j=1, \\ f, & \text { if } 2 \leq i \leq d+1,2 \leq j \leq 2 d, j \neq i, j \neq i+d, \\ 1+v, & \text { if } 2 \leq i=j \leq d+1, \\ h, & \text { if } 2 \leq i \leq d, j=i+d, \\ (1-1 / n) f, & \text { if } 2 \leq i \leq d, j=2 d+1, \\ (1-1 / n) h, & \text { if } i=d+1, j=2 d+1, \\ s, & \text { if } d+2 \leq i \leq 2 d+1,2 \leq j \leq 2 d, j \neq i, j \neq i-d, \\ t, & \text { if } d+2 \leq i \leq 2 d+1, j=i-d, \\ 1+k, & \text { if } d+2 \leq i=j \leq 2 d, \\ (1-1 / n) s, & \text { if } d+2 \leq i \leq 2 d, j=2 d+1, \\ 1+(1-1 / n) k, & \text { if } i=j=2 d+1 .\end{cases}
$$

We perform the following procedure on $R$.
(i) Subtract (the first row) $\times q / u$ from the $i$-th row with $2 \leq i \leq d+1$.
(ii) Subtract (the first row) $\times e / u$ from the $i$-th row with $d+2 \leq i \leq 2 d+1$.
(iii) Subtract the second row from the $i$-th row with $3 \leq i \leq d+1$.
(iv) Subtract the $(d+2)$-th row from the $i$-th row with $d+3 \leq i \leq 2 d+1$.
(v) Add the $j$-th column to the second column with $3 \leq j \leq d+1$.
(vi) Add the $j$-th column to the $(d+2)$-th column with $d+3 \leq j \leq 2 d$.
(vii) Add (the $(2 d+1)$-th column) $\times 1 /(1-1 / n)$ to the $(d+2)$-th column.
(viii) Subtract (the $(d+j)$-th column) $\times(t-s) /(1+k-s)$ from the $j$-th column with $3 \leq j \leq d$.

After these procedures, by changing the orders of rows and columns appropriately, we obtain the following identity.

$$
\begin{equation*}
\operatorname{det} R=u \times\left[1+v-f-\frac{t-s}{1+k-s}(h-f)\right]^{d-2} \times(1+k-s)^{d-2} \times \operatorname{det} S \tag{4.21}
\end{equation*}
$$

where $S=\left(S_{i} j\right)_{1 \leq i, j \leq 4}$ with elements

$$
\begin{array}{ll}
S_{11}=1+v+(d-1) f-d b q / u, & S_{12}=h+(d-1) f-d c q / u \\
S_{13}=(1-1 / n)(f-c q / u), & S_{14}=f-b q / u \\
S_{21}=t+(d-1) s-d b e / u, & S_{22}=1+k+(d-1) s-d c e / u \\
S_{23}=(1-1 / n)(s-c e / u), & S_{24}=s-b e / u \\
S_{31}=0, & S_{32}=1 /(n-1) \\
S_{33}=1+(1-1 / n)(k-s), & S_{34}=t-s \\
S_{41}=0, & S_{42}=0 \\
S_{43}=(1-1 / n)(h-f), & S_{44}=1+v-f
\end{array}
$$

Define

$$
g_{0}=n G(0), \quad g_{1}=n G\left(\mathbf{e}_{1}\right), \quad g_{2}=n G\left(2 \mathbf{e}_{1}\right), \quad g_{3}=n G\left(\mathbf{e}_{1}+\mathbf{e}_{2}\right)
$$

where $G(\mathbf{x})$ is given by (3.3) and $\mathbf{e}_{1}, \mathbf{e}_{2}$ are the unit vectors in the first and second directions in $\mathbb{Z}^{d}$. Since the system is isotropic, we can find that the matrix $m$ defined by (4.14) is in the form (4.20) with

$$
\begin{array}{ll}
u=1-2 d a g_{0}, & b=c=g_{0}-g_{1} \\
q=e=1-2 d a g_{1}, & f=s=g_{1}-g_{3}  \tag{4.22}\\
v=k=g_{1}-g_{0}, & h=t=g_{1}-g_{2}
\end{array}
$$

By Lemma 3.2 and the isotropy of the system gives

$$
\begin{aligned}
& 2 d(1+a) g_{0}-2 d g_{1}=1 \\
& 2 d(1+a) g_{1}-\left(g_{0}+g_{2}+2(d-1) g_{3}\right)=0
\end{aligned}
$$

which are written as

$$
\begin{align*}
& g_{1}=(1+a) g_{0}-\frac{1}{2 d}, \\
& g_{2}=\left[2 d(1+a)^{2}-1\right] g_{0}-2(d-1) g_{3}-(1+a) . \tag{4.23}
\end{align*}
$$

The formula (4.21) with (4.22) and (4.23) gives

$$
\begin{align*}
P_{0}=\operatorname{det} m & =\frac{1-2 d a g_{0}}{2 d n}\left[2\left\{1-d\left(g_{0}-g_{3}\right)\right\}+\left(1-4 d g_{0}\right) a-2 d g_{0} a\right] \\
& \times\left[2(d-1)\left(g_{0}-g_{3}\right)-\left(1-4 d g_{0}\right) a+2 d g_{0} a^{2}\right]^{2} \\
& \times\left[\left\{1-\left(g_{0}-g_{3}\right)\right\}^{2}-\left(g_{2}-g_{3}\right)^{2}\right]^{d-2} . \tag{4.24}
\end{align*}
$$

It proves (4.9) of Theorem 4.6 (i).
It should be noted that, if we put $n=1$ and take $a \downarrow 0$ limit in (4.24), we have the formula

$$
P_{0}=\frac{4(d-1)^{2}}{d}\left(1-d \bar{g}_{03}\right) \bar{g}_{03}^{2}\left[\left(1-\bar{g}_{03}\right)^{2}-\bar{g}_{23}^{2}\right]^{d-2},
$$

where

$$
\bar{g}_{03}=\lim _{a \downarrow 0}\left(g_{0}-g_{3}\right), \quad \bar{g}_{23}=\lim _{a \downarrow 0}\left(g_{2}-g_{3}\right) .
$$

In particular, $\bar{g}_{03}=1 / \pi$ and $\bar{g}_{23}=1-1 / \pi$ for $d=2[27]$, and thus we have

$$
P_{0}=\frac{2}{\pi^{2}}\left(1-\frac{2}{\pi}\right), \quad d=2 .
$$

This coincides with the value of $P_{0}$ obtained by Majumdar and Dhar [19] for the two-dimensional BTW model.

We can also find that the matrix $m^{*}(\lambda)$ defined by (4.19) is in the form (4.20) with

$$
\begin{array}{ll}
u=-2 d, & b=\left(1-\mathrm{e}^{\lambda}\right) / a^{1 / 2}, \\
c=\left(1-\mathrm{e}^{-\lambda}\right) / a^{1 / 2}, & q=\left(1-\mathrm{e}^{\lambda}\right) / a^{1 / 2}-2 d a^{1 / 2}\left(g_{1}-\mathrm{e}^{\lambda} g_{0}\right), \\
e=\left(1-\mathrm{e}^{-\lambda}\right) / a^{1 / 2}-2 d a^{1 / 2}\left(g_{1}-\mathrm{e}^{-\lambda} g_{0}\right), & f=\left(g_{1}-g_{3}\right)-\mathrm{e}^{\lambda}\left(g_{0}-g_{1}\right), \\
s=\left(g_{1}-g_{3}\right)-\mathrm{e}^{-\lambda}\left(g_{0}-g_{1}\right), & v=\left(g_{1}-g_{0}\right)-\mathrm{e}^{\lambda}\left(g_{0}-g_{1}\right), \\
k=\left(g_{1}-g_{0}\right)-\mathrm{e}^{-\lambda}\left(g_{0}-g_{1}\right), & h=\left(g_{1}-g_{2}\right)-\mathrm{e}^{\lambda}\left(g_{0}-g_{1}\right), \\
t=\left(g_{1}-g_{2}\right)-\mathrm{e}^{-\lambda}\left(g_{0}-g_{1}\right) . &
\end{array}
$$

The formula (4.21) gives

$$
\operatorname{det} m^{*}(\lambda)=-2 d\left[\left\{1-\left(g_{0}-g_{3}\right)\right\}^{2}-\left(g_{2}-g_{3}\right)^{2}\right]^{d-2} \times \operatorname{det} S,
$$

where

$$
\operatorname{det} S=b_{1}(d, a, \lambda)+b_{2}(d, a, \lambda) \frac{1}{n} .
$$

with some functions $b_{1}$ and $b_{2}$ of $d, a, \lambda$. Since (3.11) gives

$$
\mathrm{e}^{\lambda(a)}=1+a+\sqrt{a(a+2)}=1+\sqrt{2} a^{1 / 2}+\mathcal{O}(a), \quad \text { as } a \downarrow 0,
$$

we found that

$$
\begin{aligned}
b_{1}(d, a, \lambda) & =\mathcal{O}\left(a^{2}\right) \\
b_{2}(d, a, \lambda) & =\frac{4(d-1)}{d}\left(g_{0}-g_{3}\right)\left\{1-d\left(g_{0}-g_{3}\right)\right\}\left\{1+(d-1)\left(g_{0}-g_{3}\right)\right\}+\mathcal{O}\left(a^{1 / 2}\right), \text { as } a \downarrow 0
\end{aligned}
$$

Thus we obtain

$$
\lim _{a \downarrow 0} \frac{\operatorname{det} m^{*}(\lambda) \operatorname{det} m^{*}(-\lambda)}{(\operatorname{det} m)^{2}}=\left[\frac{2 d\left\{1+(d-1) \bar{g}_{03}\right\}}{(d-1) \bar{g}_{03}}\right]^{2}
$$

Since $\lim _{a \downarrow 0} c_{1}(d, a) / a^{(d-3) / 4}=\left(d /\left(2 \pi^{2}\right)\right)^{(d-3) / 4} /(4 \pi), ~(4.12)$ of Theorem 4.6 is proved.

## 5 Discussions

### 5.1 Critical exponent $\nu_{a}$

The results (3.8) of Theorem 3.3 and (4.11) of Theorem 4.6 mean that both of $G(\mathbf{x}(r))$ and $C_{00}(\mathbf{x}(r))$ decay exponentially as increasing $r$ with a correlation length $\xi(d, a)$. Since $\xi(d, a)<\infty$ for any $a>0$, the stationary state of the DASM is non-critical [28]. Moreover the theorems imply that, if we make the parameter $n$ be large with a fixed $m$, then the value of $a=m /(2 d n)$ can be small and

$$
\begin{align*}
n G(\mathbf{x}(r)) & \simeq c_{1}(d) a^{(d-3) / 4} \frac{\mathrm{e}^{-r / \xi(d, a)}}{r^{(d-1) / 2}}  \tag{5.1}\\
C_{00}(\mathbf{x}(r)) & \simeq c_{2}(d) a^{(d+1) / 2} \frac{\mathrm{e}^{-2 r / \xi(d, a)}}{r^{d-1}}, \quad \text { as } r \uparrow \infty \tag{5.2}
\end{align*}
$$

where $c_{1}(d)=\left(d /\left(2 \pi^{2}\right)\right)^{(d-3) / 4} /(4 \pi)$ and $c_{2}(d)$ is given by (4.12).
Consider a series of DASMs with increasing $n$ with a fixed $m$. Then we will have an increasing series of correlation lengths $\{\xi(d, a)\}$ and we will see the asymptotic divergence,

$$
\begin{equation*}
\xi(d, a) \simeq \frac{1}{\sqrt{2 d}} a^{-\nu_{a}} \quad \text { as } \quad a \rightarrow 0 \tag{5.3}
\end{equation*}
$$

with

$$
\begin{equation*}
\nu_{a}=\frac{1}{2} \quad \text { for all } \quad d \geq 2 \tag{5.4}
\end{equation*}
$$

We notice that, if we identify $a$ with a reduced temperature

$$
\begin{equation*}
t=\frac{\left|T-T_{\mathrm{c}}\right|}{T_{\mathrm{c}}} \tag{5.5}
\end{equation*}
$$

around a critical temperature $T_{\mathrm{c}}$ in the equilibrium spin system, (5.1) with (5.3) and (5.4) is exactly in the Ornstein-Zernike form of correlations in the mean-field theory of equilibrium phase transitions (see, for instance, Eq.(61) in Section 3.1 of [14]). This implies that we can regard (5.3) as a critical phenomenon with a parameter $a$ approaching to its critical value $a_{\mathrm{c}}=0$ and we can say that the associated critical exponent $\nu_{a}$ is exactly determined as (5.4). Vanderzande and Daerden discussed the exponent $\nu_{a}$ for the DASM on more general lattices [29].

This exponent may be identified with the critical exponent $\nu=1 / 2$ obtained by Vespignani and Zapperi by the generalized mean-field theory [30]. They claimed that they made only use of conservation laws to evaluate $\nu=1 / 2$ and thus at least on this result their mean-field theory is exact for any $d \geq 2$. The present work justifies their conjecture. We can conclude that with respect to the avalanche propagators and height- $(0,0)$ correlation functions the upper critical dimension of the ASM is two. This result does not contradict to the result by Priezzhev [24], since he studied the intersection phenomena of avalanches and for them the upper critical dimension is four.

The results (5.1) and (5.2) suggest that there exists a scaling limit such that

$$
\begin{aligned}
& \underset{\substack{r \uparrow \infty, a \downarrow 0: \\
a^{1 / 2} r=\kappa / \sqrt{2 d}}}{ } r^{d-2} n G(\mathbf{x}(r))=\mathcal{F}_{G}(\kappa), \\
& \lim _{\substack{r \uparrow \infty, a \downarrow 0: \\
a^{1 / 2} r=\kappa / \sqrt{2 d}}} r^{2 d} C_{00}(\mathbf{x}(r))=\mathcal{F}_{C}(\kappa), \quad 0<\kappa<\infty
\end{aligned}
$$

with

$$
\begin{aligned}
& \mathcal{F}_{G}(\kappa)=2^{-(d+1) / 2} \pi^{-(d-1) / 2} \kappa^{(d-3) / 2} e^{-\kappa} \\
& \mathcal{F}_{C}(\kappa)=2^{-(d+1)} \pi^{-(d-1)}\left[\frac{1+(d-1) \bar{\gamma}}{(d-1) \bar{\gamma}}\right]^{2} \kappa^{d+1} \mathrm{e}^{-\kappa}
\end{aligned}
$$

This observation is consistent with the statement

$$
\begin{equation*}
G(\mathbf{x}(r)) \sim r^{-(d-2)}, \quad \text { as } r \uparrow \infty \tag{5.6}
\end{equation*}
$$

and (4.3) claimed by Majumdar and Dhar [19] for the self-organized criticality realized in the $d$ dimensional BTW model with $d \geq 2$. (Note that for the two-dimensional BTW model, $G(\mathbf{x}(r))-$ $G(0) \simeq-(1 / 2 \pi) \log r$, as $r \uparrow \infty$.)

### 5.2 The $q \rightarrow 0$ limit of the Potts model

Majumdar and Dhar [20] discussed the relationship between the ASM and the $q \downarrow 0$ limit of the $q$-state Potts model. For $q \in\{2,3, \ldots\}$, the $q$-state Potts model on the lattice $G_{L}=\left(G_{L}^{(v)}, G_{L}^{(e)}\right)$ given by Definition 2.8 is defined as follows. At each vertex $\mathbf{v} \in G_{L}^{(v)}=\Lambda_{L} \cup\{\mathbf{r}\}$, put a spin variable $s(\mathbf{x}) \in\{1,2, \ldots, q\}$. The Hamiltonian for the configuration $s=\{s(\mathbf{v})\}_{\mathbf{v} \in G_{L}^{(v)}}$ is given by

$$
\mathcal{H}(s)=-\sum_{e=\{\mathbf{v}, \mathbf{w}\} \in G_{L}^{(e)}} \mathbf{1}(s(\mathbf{v})=s(\mathbf{w}))
$$

The partition function of the Potts model in the Gibbs ensemble with a temperature $T>0$ is defined by

$$
\begin{align*}
Z(q, T) & =\sum_{s \in\{1,2, \ldots, q\}{ }_{L}^{G_{L}^{(v)}}} e^{-\mathcal{H}(s) / T} \\
& =\sum_{s \in\{1,2 \ldots, q\}^{G_{L}^{(v)}}} \prod_{e=\{\mathbf{v}, \mathbf{w}\} \in G_{L}^{(e)}}[1+\chi \mathbf{1}(s(\mathbf{v})=s(\mathbf{w}))] \tag{5.7}
\end{align*}
$$

with $\chi=e^{1 / T}-1$. We consider a subset of $G_{L}^{(e)}$ denoted by $E \subset G_{L}^{(e)}$. Each connected component in $E$ is called a cluster. Let $c(E)$ be the number of disconnected clusters of $E ; E=\bigcup_{i=1}^{c(E)} E_{i}$, where $E_{i} \cap E_{j}=\emptyset, i \neq j$. If a vertex $\mathbf{v} \in G_{L}^{(v)}$ is not connected by any edge in $E$, we write $\mathbf{v} \notin E$. By performing binomial expansions and taking the summation over spin configurations in (5.7), we obtain the Fortuin-Kasteleyn representation of partition function,

$$
\begin{equation*}
Z(q, T)=\sum_{E \subset G_{L}^{(e)}} q^{\left|\left\{\mathbf{v} \in G_{L}^{(v)} ; \mathbf{v} \notin E\right\}\right|} q^{c(E)} \chi^{|E|}, \tag{5.8}
\end{equation*}
$$

where $|E|$ denotes the number of edges in $E$. Note that we can regard (5.8) as a function of $q \in \mathbb{R}$ and $T>0$. We consider the asymptotics of (5.8) in the limit $q \downarrow 0$. The dominant terms in this limit should be with $E$ such that $c(E)=1$ and $\left\{\mathbf{v} \in G_{L}^{(v)}: \mathbf{v} \notin E\right\}=\emptyset \Longleftrightarrow E$ contains all vertices in $G_{L}^{(v)} \Longleftrightarrow E$ is a spanning subgraph of $G_{L}$. If we further take the high-temperature limit $T \uparrow \infty \Longleftrightarrow \chi \downarrow 0$, we have only spanning subgraphs with a minimal number of edges, which are just the spanning trees. Then we have

$$
\lim _{T \uparrow \infty} \lim _{q \downarrow 0} T^{(2 L+1)^{d}} q^{-1} Z(q, T)=\left|\mathcal{T}_{L}\right|,
$$

where $\mathcal{T}_{L}$ is the collection of all spanning trees on $G_{L}$. As shown in Section 2.4, there establishes a bijection between $\mathcal{T}_{L}$ and $\mathcal{A}_{L}$ (Lemma 2.11) and $\mathcal{A}_{L}=\mathcal{R}_{L}$ (Proposition 2.12). (The relation between the $q \downarrow 0$ limit of the $q$-state Potts model with finite temperatures and the ASM is discussed in Section 7.2 in [7].) The two-dimensional $q$-state Potts model shows a continuous phase transition associated with critical phenomena at a finite temperature $0<T_{\mathrm{c}}<\infty$ without external magnetic field $B=0$, when $q=2,3$ and 4 [31].

Usual critical phenomena of spin models are specified by the behavior of two-point correlation functions for the energy density $G_{\epsilon}(r, t, b, L)$ and for the order-parameter density $G_{\sigma}(r, t, b, L)$. Here $r$ denotes the distance of two points, $t$ the reduced temperature (5.5), $b$ the reduced external field

$$
b=\frac{|B|}{T_{\mathrm{c}}},
$$

and $L$ the size of the lattice on which the model is defined. It is conjectured in the scaling theory that, if $L$ is sufficiently large and we observe the system in the very vicinity of the critical point; $t \ll 1, b \ll 1$, the correlation functions behave as

$$
\begin{align*}
G_{\epsilon}(r, t, b, L) & =L^{2 x_{\epsilon}} \mathcal{F}_{\epsilon}\left(\frac{r}{L}, t L^{y_{t}}, b L^{y_{b}}\right), \\
G_{\sigma}(r, t, b, L) & =L^{2 x_{\sigma}} \mathcal{F}_{\sigma}\left(\frac{r}{L}, t L^{y_{t}}, b L^{y_{b}}\right), \tag{5.9}
\end{align*}
$$

with the scaling exponents $x_{\epsilon}, x_{\sigma}, y_{\epsilon}, y_{\sigma}$, and the scaling functions $\mathcal{F}_{\epsilon}, \mathcal{F}_{\sigma}$. If the system is of $d$-dimensional, the hyperscaling relations $x_{\epsilon}+y_{t}=d, x_{\sigma}+y_{b}=d$ hold (see, for instance, [13, 14]). From the scaling forms (5.9), we expect the power-law behavior of correlation functions at the critical point ( $t=b=0, L \uparrow \infty$ ) such that

$$
G_{\epsilon}(r) \sim r^{-2 x_{\epsilon}}, \quad G_{\sigma}(t) \sim r^{-2 x_{\sigma}}, \quad \text { as } r \uparrow \infty,
$$

and in the off-critical regions with $L \uparrow \infty$, the correlation length $\xi=\xi(t, b)$ behaves as

$$
\begin{aligned}
& \xi(t, 0) \sim t^{-\nu_{t}} \quad \text { with } \quad \nu_{t}=\frac{1}{y_{t}}, \\
& \xi(0, b) \sim b^{-\nu_{b}} \quad \text { with } \quad \nu_{b}=\frac{1}{y_{b}}, \quad \text { as } t \downarrow 0, b \downarrow 0 .
\end{aligned}
$$

For the two-dimensional $q$-state Potts model, the critical exponents are determined as functions of $q$ through the parameter

$$
u=u(q)=\frac{2}{\pi} \cos ^{-1}\left(\frac{\sqrt{q}}{2}\right)
$$

as [31]

$$
\begin{array}{ll}
x_{\epsilon}=\frac{1+u}{2-u}, & y_{t}=2-x_{\epsilon}=\frac{3(1-u)}{2-u}, \\
x_{\sigma}=\frac{1-u^{2}}{4(2-u)}, & y_{b}=2-x_{\sigma}=\frac{(3-u)(5-u)}{4(2-u)} .
\end{array}
$$

They give the limits

$$
x_{\epsilon} \rightarrow 2, \quad y_{t} \rightarrow 0, \quad x_{\sigma} \rightarrow 0, \quad y_{b} \rightarrow 2, \quad \text { as } q \downarrow 0 \Longleftrightarrow u \uparrow 1 .
$$

Majumdar and Dhar [20] noted by their results (4.3) and (5.6) for the BTW models that the avalanche propagator $G(\mathbf{x}(r))$ and the height- $(0,0)$ correlation function $C_{00}(\mathbf{x}(r))$ in ASM play the roles of the order-parameter density correlation function $G_{\sigma}(r)$ and the energy density correlation function $G_{\epsilon}(r)$ in the critical phenomena, respectively. In particular, in the twodimensional case, the power-law exponents are respectively given as

$$
\left.2 x_{\sigma}\right|_{q \downarrow 0}=0=\left.(d-2)\right|_{d=2},\left.\quad 2 x_{\epsilon}\right|_{q \downarrow 0}=4=\left.2 d\right|_{d=2} .
$$

Our interpretation of the present result (5.4) is that introduction of dissipation to the ASM may correspond to imposing an external magnetic field $B$ to the Potts models and hence $\nu_{a}=1 / 2$ is identified with

$$
\left.\nu_{b}\right|_{q \downarrow 0}=\left.\frac{1}{y_{b}}\right|_{q \downarrow 0}=\frac{1}{2} .
$$

We remark that the critical exponents for the specific heat $\alpha$, for the order parameter $\beta$, and for the magnetic-field susceptibility $\gamma$ of the

$$
\alpha=\frac{2(1-2 u)}{3(1-u)} \rightarrow-\infty, \quad \beta=\frac{1+u}{12} \rightarrow \frac{1}{6}, \quad \gamma=\frac{7-4 u+u^{2}}{6(1-u)} \rightarrow \infty, \quad \text { as } q \downarrow 0 \Longleftrightarrow u \uparrow 1 .
$$

We suspect some interpretation of the value $\left.\beta\right|_{q \downarrow 0}=1 / 6$ in the DASM.

### 5.3 Recent topics on height correlations

In Section 4 the one-point and the two-point correlations of height-0 sites were calculated for the DASM with general $d \geq 2$. In the two-dimensional case, the three-point and the four-point correlations were also calculated for height-0 sites and general property of 'the height-0 field of ASMs' have been extensively studied from the view point of a $c=-2$ conformal field theory $[18,8]$.

For the two-dimensional BTW model, in which the values of stable height of sandpile are $h=0,1,2$, and 3 , the height correlations have been calculated also for $h \geq 1$. Priezzhev determined $P_{\alpha}$ for $\alpha \in\{0,1,2,3\}$, where the results with $\alpha \geq 1$ are expressed using multivariate integrals of determinantal integrands [23]. Poghosyan et al. [22] claimed that the height-0 state is the only one showing pure power-law-correlations and that general form of height correlations for $h \geq 1$ contains logarithmic functions. They showed that for $\alpha \geq 1$

$$
C_{0 \alpha}(\mathbf{x}(r))=\frac{P_{0 \alpha}(\mathbf{x}(r))-P_{0} P_{\alpha}}{P_{0} P_{\alpha}} \simeq \frac{1}{r^{4}}\left(c_{1} \log r+c_{2}\right), \quad \text { as } r \uparrow \infty
$$

with some constants $c_{1}, c_{2}$. Moreover, they predicted that $C_{\alpha \beta}(\mathbf{x}(r)) \sim \log ^{2} r / r^{4}$ if $\alpha \geq 1$ and $\beta \geq 1$. These results are discussed with the logarithmic conformal field theory. See also [11]. We will see a lot of interesting open problems concerning height correlations for the BTW models and the DASMs in higher dimensions.

Acknowledgements This manuscript was prepared for the workshop "Probabilistic models with determinantal structure", (April 30th and May 1st, 2015) held at the Faculty of Mathematics - Institute of Mathematics for Industry, Ito Campus, Kyushu University. The present author would like to thank T. Shirai for the invitation and for his hospitality. He thanks T. Shirai, E. Verbitskiy, and T. Hara for useful discussions in the workshop. This work is supported in part by the Grant-in-Aid for Scientific Research (C) (No.26400405) of Japan Society for the Promotion of Science.

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## Diffusions associated with GAF <br> Hirofumi Osada (Kyushu University)

This manuscript is an announcement and based on the talk
Diffusions associated with Gaussian analytic functions on Workshop on (2015/4/30/Thu-2015/5/1/Fri Kyushu)
"Probabilistic models with determinantal structure". Proofs of Main theorems will be given elsewhere.

We construct unlabeled diffusion reversible to random point fields given by zero points of GAF. The standard planar GAF is the random entire function with Gaussian coefficients:

$$
f(z)=\sum_{k=0}^{\infty} \frac{\xi_{k}}{\sqrt{k!}} z^{k}
$$

- $\left\{\xi_{k}\right\}$ is i.i.d. standard complex Gaussian.
- The zero points of $f$ are regarded as configuration on $\mathbb{C}\left(\mathbb{R}^{2}\right)$.
- Let $\mu_{\mathrm{GAF}}$ be its distribution. Rotation \& translation invariant.


## The standard planar GAF

Problem 1. We discuss three problems:

- What is the natural $\mu_{G A F}$-reversible diffusion $X=\left\{X_{t}\right\}$. Here

$$
X_{t}=\sum_{i=1}^{\infty} \delta_{X_{t}^{i}} \quad \text { (unlabeled diffusion) }
$$

- How to construct $X=\left\{X_{t}\right\}$ ?
- What is the SDE representation of $\mathbf{X}_{t}=\left(X_{t}^{i}\right)$ ?
- Let $S$ be the configuration space. Let $s=\sum_{i} \delta_{s_{i}} \in S$.
- Let $\mathbb{D}$ is the standard square field on $S$ :

$$
\mathbb{D}[f, g](\mathrm{s})=\frac{1}{2} \sum_{i=1}^{\infty} \frac{\partial \tilde{f}}{\partial s_{i}} \cdot \frac{\partial \tilde{g}}{\partial s_{i}}
$$

Here $f$ is a local and smooth function on S , and $\tilde{f}\left(s_{1}, \ldots,\right)$ is a symmetric function such that $f(\mathrm{~s})=\tilde{f}\left(s_{1}, \ldots,\right)$.

Main theorem: Set Up

- Let $\mathcal{D}_{0}$ be the set of local smooth functions. Let

$$
\mathcal{E}^{\mu_{\mathrm{GAF}}}(f, g)=\int_{\mathrm{S}} \mathbb{D}[f, g] d \mu_{\mathrm{GAF}} \quad \text { on } L^{2}\left(\mathrm{~S}, \mu_{\mathrm{GAF}}\right)
$$

with domain $\mathcal{D}_{0}^{\mu \mathrm{GAF}}=\left\{f \in L^{2}\left(\mu_{\mathrm{GAF}}\right) ; f \in \mathcal{D}_{0}, \mathcal{E}^{\mu_{\mathrm{GAF}}}(f, f)<\infty\right\}$.


- Proof of Thm 1 consists of "Ghosh's quantitative bound of GAF" and "a generalization of [O. '13]".

Let $\left(\mathcal{E}^{\mu}{ }_{\mathrm{GAF}}, \mathcal{D}^{\mu_{\mathrm{GAF}}}\right)$ be the closure on $L^{2}\left(\mu_{\mathrm{GAF}}\right)$.
Thm 2 (O.15). (Construction of dynamics)
(1) $\mu_{\mathrm{GAF}}$-reversible unlabeled diffusions X

$$
x_{t}=\sum_{i=1}^{\infty} \delta_{X_{t}^{i}}
$$

associated with ( $\left.\mathcal{E}^{\mu \mathrm{GAF}}, \mathcal{D}^{\mu \mathrm{GAF}}\right)$ on $L^{2}\left(\mu_{\mathrm{GAF}}\right)$ exists.
(2) $\mathrm{X}=\left(X_{t}^{i}\right)_{i \in \mathbb{N}}$ is a $\mathbb{C}^{\mathbb{N}}$-valued diffusion.
(3) Each tagged particle $X_{t}^{i}$ does not collide each other.

## Main theorem: GAF diffusion

- Thm 2 follows from a general theory in [O.'96,'04,'10,'13]" and the closability in Thm 1.
- We have not yet obtained the infinite-dimensional stochastic differential equation describing the labeled dynamics $\mathbf{X}=\left(X_{t}^{i}\right)$.
This is a problem to calculate the logarithmic derivative of $\mu_{\mathrm{GAF}}$.
I have been developing a general theory for interacting Brownian motions in infinite dimentions, and like to apply to GAF. I would explain about this.
- We solve ISDEs of the form

$$
\begin{equation*}
d X_{t}^{i}=d B_{t}^{i}+b\left(X_{t}^{i}, \mathbf{X}_{t}^{\diamond i}\right) d t \quad(i \in \mathbb{N}) \tag{1}
\end{equation*}
$$

Here $\mathbf{X}_{t}=\left(X_{t}^{1}, \ldots,\right) \in\left(\mathbb{R}^{2}\right)^{\mathbb{N}}$-valued, and $\mathbf{X}_{t}^{\diamond i}=\left(X_{t}^{j}\right)_{j \in \mathbb{N} \backslash i i\}}$.

- The coefficient $b(x, \mathbf{y})$ is symmetric in $\mathbf{y}=\left(y_{i}\right)_{i \in \mathbb{N}}$ for each $x \in \mathbb{R}^{2}$.
$\mathbf{B}_{t}=\left(B_{t}^{1}, \ldots,\right)$ is $\left(\mathbb{R}^{2}\right)^{\mathbb{N}}$-valued standard Brownian motion.
We will construct weak solution (X,B).

Our method can be applied to the case with $\sigma\left(X_{t}^{i}, \mathbf{X}_{t}^{\diamond i}\right) d B_{t}^{i}$.
For simplicity we talk about (1) only.

- Because of the symmetry of $b(x, y)$ in $\mathbf{y}$, we can rewrite

$$
\begin{equation*}
d X_{t}^{i}=d B_{t}^{i}+b\left(X_{t}^{i}, X_{t}^{\diamond i}\right) d t \quad(i \in \mathbb{N}) \tag{2}
\end{equation*}
$$

Here we regard $b(x, \cdot)$ as a function on the configuration space.

- Gibbsian examples for suitable $\alpha$ and $d:(i \in \mathbb{N})$
- (LJ 6-12): $d=3$ Lennard-Jones 6-12 potential
- (Riesz): $\alpha>d+2$ Riesz potential (Gibbsian case)

$$
\begin{align*}
& d X_{t}^{i}=d B_{t}^{i}+\frac{\beta}{2} \sum_{j=1, j \neq i}^{\infty}\left\{\frac{12\left(X_{t}^{i}-X_{t}^{j}\right)}{\left|X_{t}^{i}-X_{t}^{j}\right|^{14}}-\frac{6\left(X_{t}^{i}-X_{t}^{j}\right)}{\left|X_{t}^{i}-X_{t}^{j}\right|^{8}}\right\} d t  \tag{LJ6-12}\\
& d X_{t}^{i}=d B_{t}^{i}+\frac{\beta}{2} \sum_{j \neq i}^{\infty} \frac{X_{t}^{i}-X_{t}^{j}}{\left|X_{t}^{i}-X_{t}^{j}\right|^{\alpha}} d t
\end{align*}
$$

(Riesz)

- We recall the examples: $(i \in \mathbb{N})$ and $\varrho(x)=\frac{\sqrt{-x}}{\pi} 1_{(-\infty, 0]}(x)$.

$$
\begin{align*}
& d X_{t}^{i}=d B_{t}^{i}+\frac{\beta}{2} \lim _{r \rightarrow \infty} \sum_{\left|X_{t}^{i}-X_{t}^{j}\right|<r, j \neq i} \frac{1}{X_{t}^{i}-X_{t}^{j}} d t  \tag{Sine}\\
& d X_{t}^{i}=d B_{t}^{i}+\frac{\beta}{2} \lim _{r \rightarrow \infty}\left\{\left(\sum_{\substack{j \neq i,\left|X_{t}^{j}\right|<r}} \frac{1}{X_{t}^{i}-X_{t}^{j}}\right)-\int_{|x|<r} \frac{\varrho(x)}{-x} d x\right\} d t \quad \text { (Airy) } \\
& d X_{t}^{i}=d B_{t}^{i}+\frac{a}{2 X_{t}^{i}} d t+\frac{\beta}{2} \sum_{j \neq i}^{\infty} \frac{1}{X_{t}^{i}-X_{t}^{j}} d t  \tag{Bessel}\\
& d X_{t}^{i}=d B_{t}^{i}+\lim _{r \rightarrow \infty} \sum_{\substack{\left|X_{t}^{i}-X_{t}^{j}\right|<r \\
j \neq i}} \frac{X_{t}^{i}-X_{t}^{j}}{\left|X_{t}^{i}-X_{t}^{j}\right|^{2}} d t
\end{align*} \quad \text { (Bessel) }
$$

(Ginibre)

Algebraic construction in 1D. Let $d=1$ and $\beta=2$.

- Sine, Airy, and Bessel can be constructed by space-time correlation functions. So there are two very different constructions for 1D system woth $\beta=2$ arising from Random matrix theory.

Thm 3 (O.-Tanemura '14). Let $\mu$ be Sine, Airy or Bessel RPFs. Stochastic dynamics constructed by stochastic analysis and the spacetime correlation functions are equal.

- The importance is the following. From algebraic construction we can obtain quantative infomation such as moment bounds of linear statistics. From analytic construction, we can obtain qualitative information such as semi-martingale property of tagged particles, non-collision property, non-explosion property, Itô formula, and so on.
- At present, such a algebraic construction is restricted $d=1, \beta=$ 2 and dynamics coming from Random matrix theory (logarithmic interactions).

Algebraic construction in 1D.
As an example, we explain Airy.

- Space-time correlation functions are given by the extended Airy kernel:

$$
K_{\mathrm{Ai}}(s, x ; t, y)= \begin{cases}\int_{0}^{\infty} d u e^{-u(t-s) / 2} \mathrm{Ai}(u+x) \mathrm{Ai}(u+y), & t \geq s \\ -\int_{-\infty}^{0} d u e^{-u(t-s) / 2} \operatorname{Ai}(u+x) \operatorname{Ai}(u+y), & t<s\end{cases}
$$

The unlabeled process $Z_{t}=\sum_{i=1}^{\infty} \delta_{Z_{t}^{i}}$ is given by its moment generating function $\left(\mathbf{f}=\left(f_{1}, \ldots, f_{M}\right), \mathbf{t}=\left(t_{1}, \ldots, t_{M}\right), t_{i}<t_{i+1}\right)$

$$
\psi^{\mathrm{t}}[\mathbf{f}]=E\left[\exp \left\{\sum_{m=1}^{M} \int_{\mathbb{R}} f_{m}(x) Z_{t_{m}}(d x)\right\}\right]
$$

defined as a Fredholm determinant

$$
\Psi^{\mathrm{t}}[\mathbf{f}]=\operatorname{Det}_{(s, t) \in I^{2},(x, y) \in \mathbb{R}^{2}}\left[\delta_{s t} \delta(x-y)+K_{\mathrm{Ai}}(s, x ; t, y) \chi_{t}(y)\right]
$$

Here $I=\left\{t_{1}, \ldots, t_{M}\right\}$ and $\chi_{t_{m}}(y)=e^{f_{m}(y)}-1$,

Ginibre interacting Brownian motions in infinite-dimensions.

- We write Ginibre in non-consice form SDEs:

$$
\begin{aligned}
& d X_{t}^{1}=d B_{t}^{1}+\lim _{r \rightarrow \infty} \sum_{j \neq 1,\left|X_{t}^{1}-X_{t}^{j}\right|<r}^{\infty} \frac{X_{t}^{1}-X_{t}^{j}}{\left|X_{t}^{1}-X_{t}^{j}\right|^{2}} d t \\
& d X_{t}^{2}=d B_{t}^{2}+\lim _{r \rightarrow \infty} \sum_{j \neq 2,\left|X_{t}^{2}-X_{t}^{j}\right|<r}^{\infty} \frac{X_{t}^{2}-X_{t}^{j}}{\left|X_{t}^{2}-X_{t}^{j}\right|^{2}} d t \\
& d X_{t}^{3}=d B_{t}^{3}+\lim _{r \rightarrow \infty} \sum_{j \neq 3,\left|X_{t}^{3}-X_{t}^{j}\right|<r}^{\infty} \frac{X_{t}^{3}-X_{t}^{j}}{\left|X_{t}^{3}-X_{t}^{j}\right|^{2}} d t \\
& d X_{t}^{4}=d B_{t}^{4}+\lim _{r \rightarrow \infty} \sum_{j \neq 4,\left|X_{t}^{4}-X_{t}^{j}\right|<r}^{\infty} \frac{X_{t}^{4}-X_{t}^{j}}{\left|X_{t}^{4}-X_{t}^{j}\right|^{2}} d t
\end{aligned}
$$

- • •

Ginibre interacting Brownian motions in infinite-dimensions.

- Ginibre in non-consice form SDEs in the 2'nd representation:

$$
\begin{aligned}
& d X_{t}^{1}=d B_{t}^{1}-X_{t}^{1} d t+\lim _{r \rightarrow \infty} \sum_{j \neq 1,\left|X_{t}^{j}\right|<r}^{\infty} \frac{X_{t}^{1}-X_{t}^{j}}{\left|X_{t}^{1}-X_{t}^{j}\right|^{2}} d t \\
& d X_{t}^{2}=d B_{t}^{2}-X_{t}^{2} d t+\lim _{r \rightarrow \infty} \sum_{j \neq 2,\left|X_{t}^{j}\right|<r}^{\infty} \frac{X_{t}^{2}-X_{t}^{j}}{\left|X_{t}^{2}-X_{t}^{j}\right|^{2}} d t \\
& d X_{t}^{3}=d B_{t}^{3}-X_{t}^{3} d t+\lim _{r \rightarrow \infty} \sum_{j \neq 3,\left|X_{t}^{j}\right|<r}^{\infty} \frac{X_{t}^{3}-X_{t}^{j}}{\left|X_{t}^{3}-X_{t}^{j}\right|^{2}} d t \\
& d X_{t}^{4}=d B_{t}^{4}-X_{t}^{4} d t+\lim _{r \rightarrow \infty} \sum_{j \neq 4,\left|X_{t}^{j}\right|<r}^{\infty} \frac{X_{t}^{4}-X_{t}^{j}}{\left|X_{t}^{4}-X_{t}^{j}\right|^{2}} d t
\end{aligned}
$$

## Cofiguration spaces

## Set up:

- $S=\mathbb{R}^{d}$ : Space, where particles move,
- $S_{r}=\{|x| \leq r\}$,
- $\mathrm{S}=\left\{\mathrm{s}=\sum_{i} \delta_{s_{i}}, \mathrm{~s}\left(S_{r}\right)<\infty(\forall r)\right\}:$

Configuration space over $S$.
Polish space with vague topology.
The space of unlabeled particles.

- $S^{\mathbb{N}}$ is the space of labeled particles.
$\bullet \mathrm{s}=\sum_{i} \delta_{s_{i}}$ denotes unlabeled particles.
$\mathrm{s}=\left(s_{i}\right) \in S^{\mathbb{N}}$ denotes labeled particles.
- Since $S^{\mathbb{N}}$ is too large, we use S instead.
- $\mathrm{B}_{t}=\sum_{i=1}^{\infty} \delta_{B_{t}^{i}}$ is S-valued Brownian motion.
- $\mathrm{B}_{t}=\left(B_{t}^{i}\right)_{i \in \mathbb{N}}$ is $S^{\mathbb{N}}$-valued Brownian motion.


## Canonical square field

For a fun $f$ on S let $f(\mathrm{~s})=: \tilde{f}\left(s_{1}, \ldots\right)$, where $\tilde{f}$ is symmetric, $\mathrm{s}=\sum \delta_{s_{i}}$. Let $\mathcal{D}_{0}$ be the set of bounded, local, smooth functions $f$ on S .
i.e. $f$ is $\sigma\left[\pi_{r}\right]$-measurable for some $r<\infty, \tilde{f}$ is smooth.

Let $\mathbb{D}$ be the canonical square field on S :

$$
\mathbb{D}[f, g](\mathrm{s})=\frac{1}{2} \sum_{i} \nabla_{i} \tilde{f} \cdot \nabla_{i} \tilde{g} .
$$

Here $\nabla_{i}=\left(\frac{\partial}{\partial s_{i 1}}, \ldots, \frac{\partial}{\partial s_{i d}}\right)$.
The rhs is independent of particular choice of label.

- For a RPF $\mu$ we set

$$
\begin{aligned}
& \mathcal{E}^{\mu}(f, g)=\int_{\mathrm{S}} \mathbb{D}[f, g] \mu(d \mathrm{~s}) \\
& \mathcal{D}_{0}^{\mu}=\left\{f \in \mathcal{D}_{0} ; \mathcal{E}^{\mu}(f, f)<\infty, f \in L^{2}(\mu)\right\}
\end{aligned}
$$

- If we take $\mu=\Lambda$, Poisson RPF with Lebesgue intensiy, then the bilinear form associates Brownian motion $\mathrm{B}_{t}=\sum_{i} \delta_{B_{t}^{i}}$.
In this sense $\mathbb{D}$ is the canonical square field.


## From RPF to unlabeled diffusion

Outline of the proof:

$$
\mu \Rightarrow\left(\mathcal{E}^{\mu}, \mathcal{D}_{0}^{\mu}, L^{2}(\mu)\right) \Rightarrow X_{t}=\sum_{i=1}^{\infty} \delta_{X_{t}^{i}} \Rightarrow \mathbf{X}=\left(X_{t}^{i}\right)_{i \in \mathbb{N}} \Rightarrow \text { ISDE }
$$

- The first arrow is automatic. For a given RPF $\mu$, we can associated a positive bilinear form through the square field $\mathbb{D}$.
- If $\left(\mathcal{E}^{\mu}, \mathcal{D}_{0}^{\mu}, L^{2}(\mu)\right)$ is closable and its closue is quasi-regular, then by Dirichlet form theory an associated $\mu$-reversible diffusion $X_{t}$ exists.
- For this we introduce a notion of quasi-Gibbs measure.

If $\mu$ is quasi-Gibbs with upper semi-continuous potential $\Psi$, then the bilinear form id closable. In addition, $\mu$ satisies a marginal condition (local boundedness of correlation functions, say), then the form becomes quasi-regular. Hence by the general theory of Dirichlet form there exists the associated unlabeled diffusion $X_{t}$.

U-Quasi-Gibbs meas.

- Quasi-Gibbs measures:
- $\pi_{r}, \pi_{r}^{c}: \mathrm{S} \rightarrow \mathrm{S}:$ projections $\pi_{r}(\mathrm{~s})=\mathrm{s}\left(\cdot \cap S_{r}\right), \pi_{r}^{c}(\mathrm{~s})=\mathrm{s}\left(\cdot \cap S_{r}^{c}\right)$
- For a RPF $\mu$ we set $\mu_{r, \xi}^{m}(\cdot)=\mu\left(\pi_{r} \in \cdot \mid \mathrm{s}\left(S_{r}\right)=m, \pi_{r}^{c}(\mathrm{~s})=\pi_{r}^{c}(\xi)\right)$
- Let $\Psi: S \rightarrow \mathbb{R} \cup\{\infty\}$ (interaction).

$$
\mathcal{H}_{r}=\sum_{s_{i}, s_{j} \in S_{r}, i<j} \Psi\left(s_{i}-s_{j}\right)
$$

Def: $\quad \mu$ is $\Psi$-quasi-Gibbs measure if $\exists c_{r, \xi}^{m}$ s.t.

$$
c_{r, \xi}^{m-1} e^{-\mathcal{H}_{r}} d \wedge_{r}^{m} \leq \mu_{r, \xi}^{m} \leq c_{r, \xi}^{m} e^{-\mathcal{H}_{r}} d \wedge_{r}^{m}
$$

Here $\Lambda_{r}^{m}=\Lambda\left(\cdot \mid \mathrm{s}\left(S_{r}\right)=m\right)$ and $\Lambda_{r}$ is the Poisson RPF with $1_{S_{r}} d x$.

- Gibbs measures $\Rightarrow$ Quasi-Gibbs measures: If $\mu$ satisfies DLA eq.

$$
\begin{equation*}
\mu_{r, \xi}^{m}=c_{r, \xi}^{m} e^{-\mathcal{H}_{r}-\sum_{x_{i} \in S, \xi_{j} \in S_{r}^{c}} \Psi\left(x_{i}, \xi_{j}\right)} d \wedge_{r}^{m} \tag{DLA}
\end{equation*}
$$

then $\mu$ is a canonical Gibbs m. (DLA) does not make sense for

$$
\Psi(x, y)=-\log |x-y|
$$

Application of quasi-Gibbs property to dynamics

$$
\mu \Rightarrow\left(\mathcal{E}^{\mu}, \mathcal{D}_{0}^{\mu}, L^{2}(\mu)\right) \Rightarrow \mathrm{X}_{t}=\sum_{i=1}^{\infty} \delta_{X_{t}^{i}} \Rightarrow \mathbf{X}=\left(X_{t}^{i}\right)_{i \in \mathbb{N}} \Rightarrow \text { ISDE }
$$

## Unlabeled diffusions

(A1) $\mu$ is a $\psi$-quasi-Gibbs $m$ with upper-semicont

Thm 4 (O.'96 (CMP) (closability)).

$$
(\mathrm{A} 1) \Rightarrow\left(\mathcal{E}^{\mu}, \mathcal{D}_{0}^{\mu}\right) \text { is closable on } L^{2}(\mu)
$$

- Thm implies the existence of the associated $L^{2}$ Markovian semigroup.

Thm $1(\mathrm{~A} 1) \Rightarrow\left(\mathcal{E}^{\mu}, \mathcal{D}_{0}^{\mu}\right)$ is closable on $L^{2}(\mu)$.
Proof. Outline of (1): - Let

$$
\mathcal{E}^{\mu_{r, \xi}^{m}}(f, g)=\int_{\mathrm{S}} \mathbb{D}[f, g] d \mu_{r, \xi}^{m} \quad \text { (reflecting } \mathrm{BC} \text { ) }
$$

Then $\left(\mathcal{E}^{\mu_{r, \xi}^{m}}, \mathcal{D}_{0}^{\mu_{r, \xi}^{m}}\right)$ is closable on $L^{2}\left(\mu_{r, \xi}^{m}\right)$ by (A1).

- Recall the disintegration: $\mu(\cdot)=\sum_{m=1}^{\infty} \mu_{r, \xi}^{m}(\cdot) \mu(d \xi)$.

Then $\left(\hat{\mathcal{E}}_{r}^{\mu}, \mathcal{D}_{0}^{\mu}\right)$ are closable on $L^{2}(\mu)$. Here

$$
\left.\hat{\mathcal{E}}_{r}^{\mu}(f, g)=\int_{\mathrm{S}} \sum_{m=1}^{\infty} \mathcal{E}^{\mu_{r, \xi}^{m}}(f, g) d \mu \quad \text { (reflecting } \mathrm{BC}\right) .
$$

- By the monotone convergence theorem of closable forms we see

$$
\widehat{\mathcal{E}}^{\mu}(f, f)=\lim _{r \rightarrow \infty} \widehat{\mathcal{E}}_{r}^{\mu}(f, f), \quad \widehat{\mathcal{D}}_{0}=\left\{f ; \lim _{r \rightarrow \infty} \widehat{\mathcal{E}}_{r}^{\mu}(f, f)<\infty\right\}
$$

is closable. Hence $\left(\mathcal{E}^{\mu}, \mathcal{D}_{0}^{\mu}\right)$ is closable.

Application of quasi-Gibbs property to dynamics: existence of diffusions
(A2) $\sum_{k=1}^{\infty} k \mu\left(S_{r}^{k}\right)<\infty, \sigma_{r}^{k} \in L^{2}\left(S_{r}^{k}, d x\right)$
Here $\widehat{S}_{r}^{k}=\left\{\mathrm{s}\left(S_{r}\right)=k\right\}, \sigma_{r}^{k}$ is $k$-density fun on $S_{r}^{k}$.
Thm 5 (O.'96 (CMP) (existence of diffusions)).
Assume (A2). Assume that $\left(\mathcal{E}^{\mu}, \mathcal{D}_{0}^{\mu}\right)$ is closable on $L^{2}(\mu)$.
Then $\exists$ diffusion $X_{t}=\sum_{i} \delta_{X_{t}^{i}}$ associated with the closure

$$
\left(\mathcal{E}^{\mu}, \mathcal{D}^{\mu}\right) \text { of }\left(\mathcal{E}^{\mu}, \mathcal{D}_{0}^{\mu}\right) \text { on } L^{2}(\mu) .
$$

Proof. This follows from a concrete construction of cut off function, which yields the quasi-regularity of Dirichlet forms. The general theory gives the diffusion.
Remark 1.• In general, the closures of the limit Dirichlet forms

$$
\left(\widehat{\mathcal{E}}^{\mu}, \widehat{\mathcal{D}}\right) \quad \text { and } \quad\left(\mathcal{E}^{\mu}, \mathcal{D}^{\mu}\right)
$$

are not equal. We will prove the coincidence of these by using the strong uniqueness of the solutions of the associated ISDEs.

- Lang's dynamics ('79) are given by the Dirichlet form ( $\hat{\mathcal{E}}^{\mu}, \widehat{\mathcal{D}}$ ).

O's ('96) dynamics are given by ( $\mathcal{E}^{\mu}, \mathcal{D}^{\mu}$ ). O.-Tanemura prove these are the same if tagged particles have no explosions.

Let $\Psi_{2}(x, y)=-\log |x-y|$ be the 2-dim Coulomb potential.
Thm 6 (O. AOP '13, O.-Honda '14, O.-Tanemura '14).
(1) Ginibre RPF is a $2 \Psi_{2}$-quasi Gibbs measure.
(2) Sine $_{\beta}$ RPF are $\beta \Psi_{2}$-quasi Gibbs $m$ for $\beta=1,2,4$.
(3) Bessel ${ }_{2}^{a} R P F$ is a $2 \Psi_{2}$-quasi Gibbs $m$.
(4) Airy ${ }_{\beta}$ RPF are $\beta \Psi_{2}$-quasi Gibbs $m$ for $\beta=1,2,4$.

- GAF is not quasi-Gibbsian. Indeed, Ghosh proved:

Thm 7 (Ghosh '12). Let $\mu=\mu_{\text {GAF }}$. Then there exists constant $c_{r, \xi}^{m}$ such that

$$
\frac{1}{c_{r, \xi}^{m}} e^{-\mathcal{H}_{r}} d \wedge_{r}^{m}[\mathrm{Ce}(\xi)] \leq \mu_{r, \xi}^{m} \leq c_{r, \xi}^{m} e^{-\mathcal{H}_{r}} d \wedge_{r}^{m}[\mathrm{Ce}(\xi)]
$$

for $\mu$-a.s. $\xi$. Here Ce is $\pi_{r}^{c}$-measurable, and

$$
\Lambda_{r}^{m}[M]=\Lambda_{r}^{m}\left(\cdot \mid \sum_{i=1}^{m} s_{i}=M\right)
$$

General theorems on infinite-dim SDEs

$$
\mu \Rightarrow\left(\mathcal{E}^{\mu}, \mathcal{D}_{0}^{\mu}, L^{2}(\mu)\right) \Rightarrow X_{t}=\sum_{i=1}^{\infty} \delta_{X_{t}^{i}} \Rightarrow \mathbf{X}=\left(X_{t}^{i}\right)_{i \in \mathbb{N}} \Rightarrow \operatorname{ISDE}
$$

## Labeled dynamics

(A1) $\mu$ is a $\psi$-quasi-Gibbs $m$ with upper-semicont $\Psi$.
(A2) $\sum_{k=1}^{\infty} k \mu\left(\mathrm{~S}_{r}^{k}\right)<\infty, \sigma_{r}^{k} \in L^{2}\left(S_{r}^{k}, d x\right)$
(A3) $\left\{X_{t}^{i}\right\}$ do not collide each other (non-collision)
(A4) each tagged particle $X_{t}^{i}$ never explode (non-explosion)
By (A3) and (A4) the labeled dynamics

$$
\mathbf{X}_{t}=\left(X_{t}^{1}, X_{t}^{2}, \ldots\right)
$$

can be constructed from the unlabeled dynamics

$$
X_{t}=\sum_{i \in \mathbb{N}} \delta_{X_{t}^{i}}
$$

Indeed, the particles keep the initial label forever.

Sufficient condition of (A3) \& (A4)
Let $S_{s, i}=S_{s} \cap S_{i}$ :
$\mathrm{S}_{s}=\{\mathrm{s} \in \mathrm{S} ; \mathrm{s}(\{x\})=0$ for all $x \in S\}, \quad \mathrm{S}_{i}=\{\mathrm{s} \in \mathrm{S} ; \mathrm{s}(S)=\infty\}$.

- (A3) is equaivalent to

$$
\begin{equation*}
\operatorname{Cap}^{\mu}\left(\mathrm{S}_{s, i}^{c}\right)=0 \tag{3}
\end{equation*}
$$

Let $\rho^{n}$ be a $n$-correlation function of $\mu$.
Lem 1. Suppose $\mu$ is quasi-Gibbs with $\Psi$. Let $\rho^{2}$ be 2-correlation function of $\mu$. Suppose one of the following holds. Then (A3) holds.
(1) $d \geq 2$ and $\rho^{2}$ are locally bounded.
(2) $d=1$ and

$$
\rho^{2}(x, y) \leq C h(|x-y|) \text { locally near }\{x=y\}
$$

Here $h(t)$ such that

$$
\int_{0+}^{1} \frac{1}{h(t)} d t=\infty
$$

Corollary 1. Sine $_{\beta}$, Airy $_{\beta}$, Bessel $_{\beta}(\beta \geq 1)$, Ginibre RPFs satsfy (A2).

## General theorems on infinite-dim SDEs

- By (A3) we represent one-labeled process $\left(X_{t}^{1}, \sum_{j=2}^{\infty} \delta_{X_{t}^{j}}\right)$ by the Dirichlet space

$$
\left(\mathcal{E}^{\mu^{[1]}}, \mathcal{D}^{\mu^{[1]}}, L^{2}\left(\mu^{[1]}\right)\right)
$$

Applying Takeda criteria based on Lyons-Zheng decomposition we deduce (A4) from $\exists T>0$

$$
\begin{equation*}
\liminf _{r \rightarrow \infty}\left\{\int_{|x| \leq r+R} \rho^{1}(x) d x\right\}\left\{\int_{\frac{r}{\sqrt{(r+R) T}}} \mathrm{~g}(u) d u\right\}=0 \quad \text { for all } T \tag{4}
\end{equation*}
$$

Lem 2. (A4) follows from (4).

SDE representation

$$
\mu \Rightarrow\left(\mathcal{E}^{\mu}, \mathcal{D}_{0}^{\mu}, L^{2}(\mu)\right) \Rightarrow \mathrm{X}_{t}=\sum_{i=1}^{\infty} \delta_{X_{t}^{i}} \Rightarrow \mathbf{X}=\left(X_{t}^{i}\right)_{i \in \mathbb{N}} \Rightarrow \mathrm{ISDE}
$$

ISDE representation

Log derivative of $\mu$ : precise correspondence between RPFs \& potentials

- Let $\mu_{x}$ be the (reduced) Palm m. of $\mu$ conditioned at $x$

$$
\mu_{x}(\cdot)=\mu\left(\cdot-\delta_{x} \mid \mathbf{s}(x) \geq 1\right)
$$

- Let $\mu^{1}$ be the 1 -Campbell measure on $\mathbb{R}^{d} \times \mathrm{S}$ :

$$
\mu^{1}(A \times B)=\int_{A} \rho^{1}(x) \mu_{x}(B) d x
$$

- $\mathrm{d}^{\mu} \in L_{\text {loc }}^{1}\left(\mathbb{R}^{d} \times \mathrm{S}, \mu^{1}\right)$ is called the log derivative of $\mu$ if

$$
\int_{\mathbb{R}^{d} \times S} \nabla_{x} f d \mu^{1}=-\int_{\mathbb{R}^{d} \times S} f \mathrm{~d}^{\mu} d \mu^{1} \quad \forall f \in C_{0}^{\infty}\left(\mathbb{R}^{d}\right) \otimes \mathcal{D}_{0}
$$

Here $\nabla_{x}$ is the nabla on $\mathbb{R}^{d}$.

- Very informally

$$
\mathrm{d}^{\mu}=\nabla_{x} \log \mu^{1}
$$

## Log derivative

A very informal calculation shows:

- If $\mu^{1}(d x d s)=m\left(x, s_{1}, \ldots\right) d x \prod_{i} d s_{i}$, then

$$
\begin{aligned}
& -\int \nabla_{x} f\left(x, s_{1}, \ldots\right) \mu^{1}\left(d x d s_{1} \cdots\right) \\
= & -\int \nabla_{x} f\left(x, s_{1}, \ldots\right) m\left(x, s_{1}, \ldots\right) d x \prod_{i} d s_{i} \\
= & \int f\left(x, s_{1}, \ldots\right) \nabla_{x} m\left(x, s_{1}, \ldots\right) d x \prod_{i} d s_{i} \\
= & \int f\left(x, s_{1}, \ldots\right) \frac{\nabla_{x} m\left(x, s_{1}, \ldots\right)}{m\left(x, s_{1}, \ldots\right)} m\left(x, s_{1}, \ldots\right) d x \prod_{i} d s_{i} .
\end{aligned}
$$

Hence

$$
\mathrm{d}^{\mu}=\frac{\nabla_{x} m\left(x, s_{1}, \ldots\right)}{m\left(x, s_{1}, \ldots\right)}=\nabla_{x} \log m\left(x, s_{1}, \ldots\right)
$$

General theorems on infinite-dim SDEs
(A1) $\mu$ is a $\Psi$-quasi-Gibbs $m$ with upper-semicont $\Psi$.
(A2) $\sum_{k=1}^{\infty} k \mu\left(\mathrm{~S}_{r}^{k}\right)<\infty, \sigma_{r}^{k} \in L^{2}\left(S_{r}^{k}, d x\right)$
(A3) $\left\{X_{t}^{i}\right\}$ do not collide each other
(A4) each tagged particle $X_{t}^{i}$ never explode
(A5) The log derivative $\mathrm{d}^{\mu} \in L_{\text {loc }}^{1}\left(\mu^{1}\right)$ exists $\Rightarrow$ (SDE representation)
Thm 8. (O.12(PTRF)) (A1)-(A5) $\Rightarrow \exists \mathrm{S}_{0} \subset \mathrm{~S}$ such that $\mu\left(\mathrm{S}_{0}\right)=1$, and that, for $\forall \mathrm{s} \in \mathfrak{u}^{-1}\left(\mathrm{~S}_{0}\right)$, there exists a solution (X,B) satisfying

$$
\begin{aligned}
& d X_{t}^{i}=d B_{t}^{i}+\frac{1}{2} \mathrm{~d}^{\mu}\left(X_{t}^{i}, \sum_{j \neq i} \delta_{X_{t}^{j}}\right) d t, \quad\left(X_{0}^{i}\right)_{i \in \mathbb{N}}=\mathrm{s} \\
& \mathbf{X}_{t} \in \mathfrak{u}^{-1}\left(\mathrm{~S}_{0}\right) \quad \text { for all } t
\end{aligned}
$$

Here $\mathfrak{u}: S^{\mathbb{N}} \rightarrow$ S such that $\mathfrak{u}\left(\left(s_{i}\right)\right)=\sum_{i} \delta_{s_{i}}$.
Corollary 2. Suppose that there exists a RPF $\mu$ satisfying (A1)-(A4) and

$$
\nabla_{x} \log \mu^{[1]}(x, \mathrm{~s})=2 b(x, \mathrm{~s})
$$

Then ISDE (1) has a weak solution.

## General theorems on infinite-dim SDEs

Proof:

- $S^{\mathbb{N}}$ does not have good measures $\Rightarrow$ no Dirichlet forms on $S^{\mathbb{N}} \Rightarrow$ Introduce a sequence of spaces with Campbel measures $\mu^{[M]}$ :

$$
S^{M} \times \mathrm{S}, \quad d \mu^{[M]}=\rho^{M}\left(\mathbf{x}_{M}\right) \mu_{\mathbf{x}_{m}}(d \mathrm{~s}) d \mathbf{x}_{M}
$$

Here $\rho^{M}$ is a $M$-correlation function of $\mu$ and $\mu_{\mathbf{x}_{m}}$ is the reduced Palm measure conditioned at $\mathbf{x}_{M}$.

Let $\mathbb{D}^{[M]}$ be the natural square field of $S^{M} \times \mathrm{S}$. Let

$$
\begin{aligned}
\mathcal{E}^{[M]}(f, g) & =\int_{S^{M} \times S} \mathbb{D}^{[M]}[f, g] d \mu^{[M]} \\
L^{2}\left(\mu^{[M]}\right), & C_{0}^{\infty}\left(S^{M}\right) \otimes \mathcal{D}_{\circ}
\end{aligned}
$$

Lem 3. These bilinear forms are closable, and their closures are quasi-regular Dirichlet forms. Hence associated diffusion ( $\mathbf{X}_{t}^{M}, \mathrm{X}_{t}^{M *}$ ) exists:

$$
\left(\mathrm{X}_{t}^{M}, \mathrm{X}_{t}^{M *}\right)=\left(X_{t}^{M, 1}, \ldots, X_{t}^{M, M}, \sum_{i=M+1}^{\infty} \delta_{X_{t}^{M, i}}\right)
$$

Coupling of Dirichlet forms:

- Let fix a label $\ell$. Let

$$
\mathrm{X}_{t}=\sum_{i=1}^{\infty} \delta_{X_{t}^{i}}
$$

be the unlabeld diffusion associated with the original unlabeled Dirichlet form

$$
\left(\mathcal{E}^{\mu}, \mathcal{D}^{\mu}, L^{2}(\mu)\right) .
$$

Thm 9. Associated diffusions have consistency
$\left(X_{t}^{M, 1}, \ldots, X_{t}^{M, M}, X_{t}^{M, M+1}, \ldots\right)=\left(X_{t}^{1}, \ldots, X_{t}^{M}, X_{t}^{M+1}, \ldots\right)$ in law or equivalently

$$
\left(\mathrm{X}_{t}^{M}, \mathrm{X}_{t}^{M *}\right)=\left(X_{t}^{1}, \ldots, X_{t}^{M}, \sum_{i=M+1}^{\infty} \delta_{X_{t}^{i}}\right) \quad \text { in law }
$$

From this coupling and Fukushima decomposition (Itô formula) we prove that ( $X_{t}^{i}$ ) satisfies the ISDE. We use the $M$-labeled process ( $\mathrm{X}_{t}^{M}, \mathrm{X}_{t}^{M *}$ ), to apply Itô formula to coordinate functions $x_{1}, \ldots, x_{M}$.

## Coupling of Dirichlet forms:

- The key point here is that, instead of large space

$$
S^{\mathbb{N}}
$$

we use a system of countably infinite good infinite dimensional sapce

$$
S^{1} \times \mathrm{S}, S^{2} \times \mathrm{S}, S^{3} \times \mathrm{S}, S^{4} \times \mathrm{S}, S^{5} \times \mathrm{S}, S^{6} \times \mathrm{S}, S^{7} \times \mathrm{S}, \cdots
$$

- By the diffusion $X$ on the original unlabeled space

$$
\mathrm{S},
$$

we construct a coupling of diffusions ( $\mathrm{X}^{M}, \mathrm{X}^{M *}$ ) on these inifinite many spaces $S^{M} \times \mathrm{S}$.

- From this coupling, we have the ISDE representation. Indeed, we can apply Itoô formula to each coordinate functions $f(\mathrm{x})=x_{k}$. We use $\mathcal{E}^{[M]}(f, g)$ for $1 \leq k \leq M$.
- The log derivative gives the precise correspondence between RPFs $\mu$ and potentials ( $\Phi, \Psi$ ).
- We next give examples of logarithmic derivatives

$$
\begin{aligned}
& \mathrm{d}^{\mu}=\nabla_{x} \log \mu^{1} \\
& \text { Thm } 10 \text { (O. PTRF } 12 / \text { Honda-O. SPA 15/O.-Tanemura). } \\
& \mathrm{d}^{\mu_{\operatorname{gin}}}(x, \mathrm{~s})=\lim _{r \rightarrow \infty} 2 \sum_{\left|x-s_{i}\right|<r} \frac{x-s_{i}}{\left|x-s_{i}\right|^{2}} \\
& \mathrm{~d}^{\mu_{\operatorname{gin}}}(x, \mathrm{~s})=-2 x+\lim _{r \rightarrow \infty} 2 \sum_{\left|s_{i}\right|<r} \frac{x-s_{i}}{\left|x-s_{i}\right|^{2}} \\
& \mathrm{~d}^{\mu_{\sin , \beta}(x, \mathrm{~s})}=\lim _{r \rightarrow \infty} \beta \sum_{\left|x-s_{i}\right|<r} \frac{1}{x-s_{i}} \\
& \mathrm{~d}^{\mu_{\text {bes }, 2}^{a}(x, \mathrm{~s})}=\frac{a}{x}+2 \sum_{\left|x-s_{i}\right|<r} \frac{1}{x-s_{i}} \\
& \mathrm{~d}^{\mu_{\mathrm{Ai}, \beta}(x, \mathrm{~s})}=\beta \lim _{r \rightarrow \infty}\left\{\left(\sum_{\left|x-s_{i}\right|<r} \frac{1}{x-s_{i}}\right)-\int_{|x|<r} \frac{\varrho(x)}{-x} d x\right\} \\
& \varrho(x)=\frac{\sqrt{-x}}{\pi} 1_{(-\infty, 0)}(x)
\end{aligned}
$$

## Calculation of logarithmic derivative

- Assume that $n$-point cor funs $\left\{\rho^{N, n}\right\}$ satisfy for each $r, n \in \mathbb{N}$

$$
\begin{align*}
& \lim _{N \rightarrow \infty} \rho^{N, n}(\mathrm{x})=\rho^{n}(\mathrm{x}) \quad \text { uniformly on } S_{r}^{n}  \tag{5}\\
& \sup _{N \in \mathbb{N}} \sup _{\mathrm{x} \in S_{r}^{n}} \rho^{N, n}(\mathrm{x}) \leq C_{1}^{-n} n^{C_{2} n}, \quad 0<C<\infty, 0<C_{2}<1 \tag{6}
\end{align*}
$$

- We assume that $\mu^{N}$ have log derivative $\mathrm{d}^{N}$ such that

$$
\begin{equation*}
\mathrm{d}^{N}(x, \mathrm{y})=u^{N}(x)+\mathrm{g}_{s}^{N}(x, \mathrm{y})+w_{s}^{N}(x, \mathrm{y}) \tag{7}
\end{equation*}
$$

Here $g, g^{N}, v, v^{N}: S^{2} \rightarrow \mathbb{R}^{d}$ and $w: S \rightarrow \mathbb{R}^{d}$ and set $\left(\mathrm{y}=\sum_{i} \delta_{y_{i}}\right)$

$$
\begin{aligned}
& \mathrm{g}_{s}(x, \mathrm{y})=\int_{|x-y|<s} v(x, y) d y+\sum_{\left|x-y_{i}\right|<s} g\left(x, y_{i}\right) \\
& \mathrm{g}_{s}^{N}(x, \mathrm{y})=\int_{|x-y|<s} v^{N}(x, y) d y+\sum_{\left|x-y_{i}\right|<s} g^{N}\left(x, y_{i}\right) \\
& w_{s}^{N}(x, \mathrm{y})=\int_{s \leq|x-y|} v^{N}(x, y) d y+\sum_{s \leq\left|x-y_{i}\right|} g^{N}\left(x, y_{i}\right) \in L_{\text {|oc }}^{\hat{p}}\left(\mu^{1}\right) .
\end{aligned}
$$

- Let $1<p<\hat{p}<\infty$. Assume that

$$
\begin{align*}
& \limsup _{N \rightarrow \infty} \int_{S_{r} \times 5}\left|\mathrm{~d}^{N}-u^{N}\right|^{\hat{p}} d \mu^{N, 1}<\infty \quad \text { for all } r \in \mathbb{N}  \tag{8}\\
& \lim _{N \rightarrow \infty} u^{N}=u \quad \text { in } L_{\mathrm{loc}}^{\widehat{p}}(S, d x)  \tag{9}\\
& \lim _{N \rightarrow \infty} \mathrm{~g}_{s}^{N}=\mathrm{g}_{s} \quad \text { in } L_{\mathrm{Poc}}^{\hat{p}}\left(\mu^{1}\right) \quad \text { for all } s,  \tag{10}\\
& \lim _{s \rightarrow \infty} \lim _{N \rightarrow \infty} \sup _{S_{r} \times S}\left|w_{s}^{N}(x, \mathrm{y})-w(x)\right|^{\hat{p}} d \mu^{N, 1}=0 . \tag{11}
\end{align*}
$$

Recall that

$$
\mathrm{g}_{s}(x, y)=\int_{|x-y|<s} v(x, y) d y+\sum_{\left|x-y_{i}\right|<s} g\left(x, y_{i}\right)
$$

Thm 11. Assume (5)-(11). Then $\mathrm{d}^{\mu}$ exists in $L_{\text {loc }}^{p}\left(\mu^{1}\right)$ given by

$$
\begin{equation*}
\mathrm{d}^{\mu}(x, \mathrm{y})=u(x)+\lim _{s \rightarrow \infty} \mathrm{~g}_{s}(x, \mathrm{y})+w(x) \tag{12}
\end{equation*}
$$

Calculation of logarithmic derivative

- Ginibre RPF, we take

$$
\begin{aligned}
u^{N}(x) & =u(x)=-2 x, \quad v^{N}(x, y)=v(x, y)=0, \quad w(x)=2 x, \\
g^{N}(x, y) & =g(x, y)=\frac{2(x-y)}{|x-y|^{2}} .
\end{aligned}
$$

-Airy RPF:

$$
\begin{aligned}
u^{N}(x) & =\beta\left\{\int_{\mathbb{R}} \frac{\rho_{\beta, x}^{N, 1}(y)}{x-y} d y\right\}-N^{1 / 3}-\frac{N^{-1 / 3}}{2} x \\
u(x) & =\beta \lim _{s \rightarrow \infty}\left\{\int_{|s|<s} \frac{\rho_{\beta, x}^{1}(y)}{x-y} d y-\int_{|y|<s} \frac{\varrho(y)}{-y} d y\right\} \\
v^{N}(x, y) & =-\beta \frac{\rho_{\beta, x}^{N, 1}(y)}{x-y}, \quad v(x, y)=-\beta \frac{\rho_{\beta, x}^{1}(y)}{x-y} \\
w(x) & =0, \quad g^{N}(x, y)=g(x, y)=\frac{\beta}{x-y} .
\end{aligned}
$$

MI レクチャーノートシリーズ刊行にあたり

本レクチャーノートシリーズは，文部科学省 21 世紀COE プログラム「機能数理学の構築と展開」（H．15－19 年度）において作成した COE Lecture Notes の続刊であり，文部科学省大学院教育改革支援プログラム「産業界が求める数学博士と新修士養成」（H19－21 年度）および，同グローバルCOE プログラ ム「マス・フォア・インダストリ教育研究拠点」（H．20－24 年度）において行 われた講義の講義録として出版されてきた。平成 23 年 4 月のマス・フォア・ インダストリ研究所（IMI）設立と平成 25 年 4 月の IMIの文部科学省共同利用•共同研究拠点として「産業数学の先進的•基礎的共同研究拠点」の認定を受け，今後，レクチャーノートは，マス・フォア・インダストリに関わる国内外の研究者による講義の講義録，会議録等として出版し，マス・フォア・インダ ストリの本格的な展開に資するものとする。

平成 26 年 10 月
マス・フォア・インダストリ研究所
所長 福本康秀

# Workshop on ＂Probabilistic models with determinantal structure＂ 

発 行 2015年8月20日<br>編 集 白井朋之<br>発 行 九州大学マス・フォア・インダストリ研究所<br>九州大学大学院数理学府<br>〒819－0395福岡市西区元岡744<br>九州大学数理－IMI 事務室<br>TEL 092－802－4402 FAX 092－802－4405<br>URL http：／／www．imi．kyushu－u．ac．jp／<br>印 刷 城島印刷株式会社<br>〒810－0012福岡市中央区白金2丁目9番6号<br>TEL 092－531－7102 FAX 092－524－4411

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