Math-for-industry
Education & Research Hub

ISSN2188-1200

Workshop on "β-transformation and related topics"

Editor : Tomoyuki Shirai

九州大学マス・フォア・インダストリ研究所

MI Lecture Note Vol.62 : Kyushu University

Workshop on
“β-transformation and related topics”

Editor: Tomoyuki Shirai

About MI Lecture Note Series

The Math-for-Industry (MI) Lecture Note Series is the successor to the COE Lecture Notes, which were published for the 21st COE Program “Development of Dynamic Mathematics with High Functionality,” sponsored by Japan’s Ministry of Education, Culture, Sports, Science and Technology (MEXT) from 2003 to 2007. The MI Lecture Note Series has published the notes of lectures organized under the following two programs: “Training Program for Ph.D. and New Master’s Degree in Mathematics as Required by Industry,” adopted as a Support Program for Improving Graduate School Education by MEXT from 2007 to 2009; and “Education-and-Research Hub for Mathematics-for-Industry,” adopted as a Global COE Program by MEXT from 2008 to 2012.

In accordance with the establishment of the Institute of Mathematics for Industry (IMI) in April 2011 and the authorization of IMI’s Joint Research Center for Advanced and Fundamental Mathematics-for-Industry as a MEXT Joint Usage / Research Center in April 2013, hereafter the MI Lecture Notes Series will publish lecture notes and proceedings by worldwide researchers of MI to contribute to the development of MI.

October 2014
Yasuhide Fukumoto
Director
Institute of Mathematics for Industry

Workshop on “β-transformation and related topics”

MI Lecture Note Vol.62, Institute of Mathematics for Industry, Kyushu University
ISSN 2188-1200
Editor: Tomoyuki Shirai
Date of issue: 10 March 2015
Publisher: Institute of Mathematics for Industry, Kyushu University
Graduate School of Mathematics, Kyushu University
Motooka 744, Nishi-ku, Fukuoka, 819-0395, JAPAN
Tel +81-(0)92-802-4402, Fax +81-(0)92-802-4405
URL http://www.imi.kyushu-u.ac.jp/

Printed by
Kijima Printing, Inc.
Shirogane 2-9-6, Chuo-ku, Fukuoka, 810-0012, Japan
TEL +81-(0)92-531-7102 FAX +81-(0)92-524-4411

Preface

The present volume of Math-for-Industry Lecture Note Series collects the abstracts of all invited talks at the workshop on "β-transformation and related topics" held at Institute of Mathematics for Industry (IMI), Ito-Campus, Kyushu University, Fukuoka, Japan, March 10, 2015. The workshop is held during the visit to IMI of Professor Evgeny Verbitskiy (Leiden and Groningen) and Professor Charlene Kalle (Leiden). Professor Verbitskiy is now a visiting professor at IMI.

The purpose of this workshop is to overview recent developments around β-transformations and also to provide a forum for discussions of related topics and for exchange of ideas and information between researchers who investigate them from various points of view.

A β-transformation is a piecewise linear expanding map $T_\beta : [0,1) \to [0,1)$ $(\beta > 1)$ defined by $T_\beta(x) = \beta x - \lfloor \beta x \rfloor$. This transformation is closely related to the so-called β-expansions of real numbers which generalize the q-adic expansions of real numbers for integer q. The study on this transformation was initiated by Alfréd Rényi (1957). He showed that T_β has the unique absolutely continuous invariant probability measure ν_β, under which T_β is ergodic. William Parry (1960) gave sufficient conditions for a sequence of integers from a finite alphabet set $\{0, 1, \ldots, \lceil \beta - 1 \rceil\}$ to arise as a sequence of digits of a β-expansion, which naturally induces shift dynamical systems. He also gave an expression of the Radon-Nikodym density of ν_β. Yoichiro Takahashi (1973) and Yoichiro Takahashi and Shunji Ito (1974) studied further the symbolic dynamical structure of β-transformations. Since the transformation was introduced, over many years, there have been lots of research from various viewpoints. It still continues to be developed.

Workshop talks cover several topics related to β-transformations from both theoretical and practical points of view. C. Kalle speaks about recent results on isomorphisms between positive and negative β-transformations. S. Akiyama introduces and discusses a natural generalization of T_β which are extended to transformations on the complex plane by adding rotations. T. Kohda and Y. Jitsumatsu make emphasis on the practical aspect of β-transformations. T. Kohda gives a brief review on the background of A/D(analog-to-digital) and D/A(digital-to-analog) conversion, and discusses the advantages of β-encoders proposed by him and co-authors. Y. Jitsumatsu discusses a random binary sequence generator based on the output sequence from the β-encoder. R. Tanaka considers a Dirichlet series associated with independent 0-1 random coefficients and discuss the regularity of its distribution. H. Sumi discusses fractal structure of rational semigroups and complex version of devil's staircase and Takagi's function in the framework of multifractal formalism. E. Verbitskiy speaks about random β- and continued fraction transformations and discusses existence of invariant measures and their regularity.

We are very much grateful to all the participants, especially the invited speakers for their contribution to preparing abstracts and giving talks. We are also grateful to Ms. Tsubura Imabayashi for her help. Without her generous effort, the workshop would not have been so smoothly organized.

We hope all the participants enjoy this workshop and have a pleasant stay in Fukuoka.

This workshop is financially supported by Progress 100 (World Premier International Researcher Invitation Program), Kyushu University.

March 2015

Organizer: Tomoyuki Shirai
(IMI, Kyushu University)


Workshop on
"β-transformation and related topics"
Date: 10:00 -17:30, March 10 (Tue.), 2015
Venue: Institute of Mathematics for Industry, Kyushu University
Speakers
Charlene Kalle
Leiden University
Isomorphisms Between Positive and Negative Beta-Transformations
Evgeny Verbitskiy
IMI, Kyushu University & Leiden University
Random Expansions of Numbers
Shigeki Akiyama
University of Tsukuba
Beta Expansion with Rotation
Tohru Kohda
Kyushu University, Emeritus
β-Encoder: Symbolic Dynamics and Electronic Implementation for AD/DA Converters
Yutaka Jitsumatsu
Kyushu University
A Random Binary Sequence Generator Based on Beta Encoders
Ryokichi Tanaka
Tohoku University
Random Dirichlet Series Arising from Records
Hiroki Sumi
Osaka University
Multifractal Analysis for Complex Analogues of the Devil's Staircase and
the Takagi Function in Random Complex Dynamics
Institute of Mathematics for Industry
Kyushu University
九州大学
KYUSHU UNIVERSITY
Organized by Tomoyuki Shirai (IMI, Kyushu University)
Supported by Progress 100 -World Premier International Researcher Invitation Program-, Kyushu University
Contact=Office: Institute of Mathematics for Industry, Kyushu University, Fukuoka, 819-0395, JAPAN PHONE: 092-802-4404 FAX: 092-802-4405 E-mail: imabayashi@math.kyushu-u.ac.jp

Workshop on "β-transformation and related topics"

10 March, 2015 at IMI, Kyushu University

Program

3/10 (Tue) Chu Seminar Room 1

10:00 - 10:50 Charlene Kalle (Leiden)

Isomorphisms between positive and negative beta-transformations

11:00 - 11:40 Shigeki Akiyama (Tsukuba)

Beta expansion with rotation

13:00 - 13:40 Tohru Kohda (Kyushu, Emeritus)

β-encoders: symbolic dynamics and electronic implementation for AD/DA converters

13:50 - 14:30 Yutaka Jitsumatsu (Kyushu)

A random binary sequence generator based on beta encoders

14:50 - 15:30 Ryokichi Tanaka (Tohoku)

Random Dirichlet series arising from records

15:40 - 16:20 Hiroki Sumi (Osaka)

Multifractal analysis for complex analogues of the devil's staircase and the Takagi function in random complex dynamics

16:40 - 17:30 Evgeny Verbitskiy (Kyushu, Leiden)

Random expansions of numbers

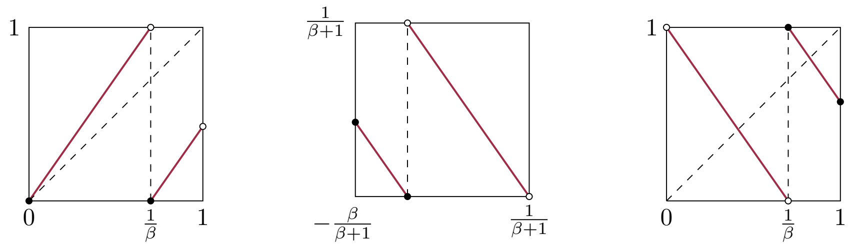
This workshop is supported by Progress 100 (World Premier International Researcher Invitation Program), Kyushu University.

Table of Contents

ISOMORPHISMS BETWEEN POSITIVE AND NEGATIVE β-TRANSFORMATIONS

CHARLENE KALLE

For real numbers $\beta > 1$, the β-transformation is the map T defined from the unit interval to itself by $Tx = \beta x \pmod 1$. See Figure 1(a) below. It was first introduced by Rényi in 1957 ([13]) and since then a lot of research has been done on the dynamical properties of the map. It is well known that the map has a unique ergodic invariant measure, μ, absolutely continuous to Lebesgue ([12]) and that the entropy for this measure is $\log \beta$ ([14], see also [5]). In this talk I will focus on the case $1 < \beta < 2$.



(a) T_β (b) The Ito-Sadahiro map (c) S_β

FIGURE 1. bla

The map T is intimately related to β-expansions, which are obtained as follows. For $x \in [0, 1)$, set $b_n(x) = 0$ if $0 \le T^{n-1}x < \frac{1}{\beta}$ and $b_n(x) = 1$ otherwise. Then

$$x = \sum_{n \ge 1} \frac{b_n(x)}{\beta^n},$$

which is called a β-expansion of x. In 2006 Daubechies et al. proposed to use β-expansions for analog-to-digital encoders ([1, 2]), due to their favourable robustness properties. In 2008, Kohda, Hironaka and Aihara further investigated the properties of such a β-encoder and proposed that the use of negative β-expansions would improve the process even more ([8]). This has led increase in research on negative β-expansions.

In its simplest form a negative β-expansion of a number x with digits 0 and 1 is an expression of the form

$$x = \sum_{n \ge 1} (-1)^n \frac{b_n}{\beta^n}, \quad \text{where } b_n \in \{0, 1\} \text{ for all } n \ge 1.$$

Dynamically such expansions can be generated by iterating the map shown in Figure 1(b). The map in this form was introduced and studied by Ito and Sadahiro in 2009 [6]. Call it S_{IS}. They found, among other things, an invariant measure for this map that is absolutely conyinuous with respect to Lebesgue. Call this measure ν_{IS}. Further investigations on the negative transformation and on the negative expansions revealed many similarities to the positive map and expansions (see [4, 3, 9, 10, 11] among others). This gave rise to a natural question: **are the positive and negative β-transformation really different, or are they essentially the same?** This is the question that we will address in this talk.

In an ergodic theoretic set up the notion of being the same means that there exists an isomorphism: There are sets $N \subseteq [0,1)$ and $M \subseteq \big(-\frac{\beta}{\beta+1}, \frac{1}{\beta+1}\big]$ such that $\mu(N) = 1 = \nu_{IS}(M)$ and $T(N) \subseteq N$ and $S_{IS}(M) \subseteq M$ and there is an invertible, measurable and measure preserving map $\phi : N \to M$ such that $\phi \circ T = S_{IS} \circ \phi$.

Instead of asking for an isomorphism between T and S_{IS}, we first replace S_{IS} by the map in Figure 1(c). These two maps are isomorphic and it is more convenient to study the existence of isomorphisms between the maps in Figure 1(a) and (c). The map in Figure 1(c), which we call S is defined from the unit interval to itself by

$$Sx = \begin{cases} 1 - \beta x, & \text{if } 0 < x < \frac{1}{\beta}, \\ \\ 2 - \beta x, & \text{if } \frac{1}{\beta} \leq x \leq 1. \end{cases}$$

In this talk we will discuss the following two results.

Theorem 0.1. *If $\beta > 1$ is a multinacci number, i.e., if $\beta > 1$ satisfies $\beta^n - \beta^{n-1} - \cdots - \beta - 1 = 0$, then the maps T and S are isomorphic.*

If β is the n-th multinacci number, then we have detailed information on the orbit of the point 1 for both maps T and S. This result can then be proven by showing that both maps are isomorphic to the same Markov shift.

Theorem 0.2. *If $1 < \beta < 2$ is not equal to a multinacci number, then the two maps T and S are not isomorphic.*

For this result we will only consider a sketch of the proof. This involves considering sets of points that have the same number of pre-images under iterates of T_β and S_β. To be more precise, we consider the sets

$$\begin{aligned} I^+_{k,n} &= \big\{x \in [0,1) \,:\, T^{-n}_\beta\{x\} = k\big\}, \\ I^-_{k,n} &= \big\{x \in [0,1) \,:\, S^{-n}_\beta\{x\} = k\big\}. \end{aligned}$$

We will show that for the invariant measure μ for T and for the invariant measure ν for S there exist numbers k and n such that $\mu(I^+_{k,n}) \neq \nu(I^-_{k,n})$. This implies that T and S cannot be isomorphic.

These results can be found in [7].

References

[1] I. Daubechies, R. DeVore, C.S. Güntürk and V. Vaishampayan. A/D Conversion with IMperfect Quantizers. *IEEE Transactions on Information Theory*, **52**(3): 874–885, 2006.

[2] I. Daubechies, C.S. Güntürk, Y. Wang and O. Yilmaz. The Golden Ratio Encoder. *IEEE Transactions on Information Theory*, **56**(10): 5097–5110, 2010.

[3] D. Dombek, Z. Masśková and E. Pelantová. Number representations using generalised $(-\beta)$-transformations. *Theor. Comp. Sci.*, **412**: 6653–6665, 2011.

[4] C. Frougny and A.C. Lai. On negative bases. *Developments in Language Theory*, **5583**: 252–263, 2009.

[5] F. Hofbauer. β-Shifts have unique maximal measure. *Monatsh. Math.*,**10**: 189–198, 1977.

[6] S. Ito and T. Sadahiro. Beta-expansions with negative bases. *Integers*, **9:A22**: 239–259, 2009.

[7] C. Kalle. Isomorphisms between positive and negative beta-transformations. *Ergodic Theory Dynam. Systems*, **34**(1): 153–170, 2014.

[8] T. Kohda, S. Hironaka and K. Aihara. Negative beta encoder. *CoRR*, 2008.

[9] L. Liao and W. Steiner. Dynamical properties of the negative beta-transformation. *Ergodic Theory Dynam. Systems*, **30**(1):123-123, 2011.

[10] Z. Masákova and E. Pelantová. Ito-Sadahiro numbers vs. Parry numbers. *Acta Polytechnica*, **51**: 59–64, 2011.

[11] F. Nakano and T. Sadahiro. A $(-\beta)$-expansion associated to Sturmian sequences. *Integers*, **12A**: 1–25, 2012.

[12] W. Parry. On the β-expansions of real numbers. *Acta Math. Acad. Sci. Hungar.*, **11**: 401–416, 1960.

[13] A. Rényi. Representations for real numbers and their ergodic properties. *Acta. Math. Acad. Sci. Hungar.*, **8**: 477–493, 1957.

[14] Y. Takahashi. Isomorphisms of β-automorphisms to Markov automorphisms. *Osaka J. Math.*, **10**: 175–184, 1973.

On Random Expansions of Numbers

Evgeny Verbitskiy
Institute of Mathematics for Industry, Kyushu University, Japan
Mathematical Institute, Leiden University, The Netherlands

It is well-known that dynamical systems can be used to generate expansions of real numbers, e.g., the so-called β-expansions

$$x = \sum_{k\geq 1} \frac{a_k}{\beta^k}, \quad \beta > 1,\ a_k \in \{0, 1, \ldots, \lfloor\beta\rfloor\}, \tag{1}$$

or the continued fraction expansions:

$$x = \cfrac{1}{a_1 \pm \cfrac{1}{a_2 \pm \cfrac{1}{a_3 \pm \cfrac{1}{\cdots}}}}, \quad a_k \in \mathbb{N}.$$

The corresponding β- and continued fractions transformations are classical objects of study in the theory of dynamical systems. Ergodic point of view (the study of properties of the invariant measures of these transformations) provides further insights into the number-theoretic properties of these expansions.

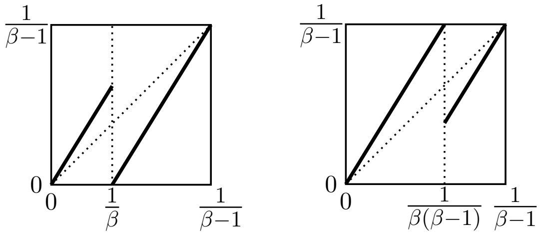
1 Random β-expansions

Any expansion (1) is called a β-expansion of x. It turns out that for non-integer base $\beta > 1$, most numbers in the interval $[0, \frac{\lfloor\beta\rfloor}{\beta-1}]$ admit **infinitely** many β-expansions. The natural question is whether one can devise a dynamical way to describe (generate) all possible β-expansions of a given number x. It turns out that such a method exists, and the underlying idea is to dynamically randomise the expansion process. In the first part of the talk I will briefly discuss some of the recent works on random β-expansions [1–5, 10].

The basic idea is to combine two maps, the well-known *greedy* and *lazy* β-transformations, denoted by T_0 and T_1, respectively. For simplicity, assume that $\beta \in (1, 2)$, then each map has two intervals of monotonicity:

$$T_0 x = \begin{cases} \beta x, & x < \frac{1}{\beta}, \\ \beta x - 1, & \text{otherwise}, \end{cases} \qquad T_1 x = \begin{cases} \beta x, & x < \frac{1}{\beta(\beta-1)}, \\ \beta x - 1, & \text{otherwise}, \end{cases}$$

and the corresponding graphs are



For every $x \in [0, \beta^{-1})$ or $(\beta^{-1}(\beta-1)^{-1}, (\beta-1)^{-1}]$, one has $T_0(x) = T_1(x)$. But for every $x \in [\beta^{-1}, \beta^{-1}(\beta-1)^{-1}]$, one does have a choice whether to apply T_0 or T_1. Let σ denote the left shift on the set of sequences $\{0,1\}^{\mathbb{N}}$. The *random β-transformation* K is defined from the set $\{0,1\}^{\mathbb{N}} \times \big[0, \frac{1}{\beta-1}\big]$ to itself as follows [1]:

$$K(\omega, x) = \begin{cases} (\omega, \beta x), & \text{if } x < \frac{1}{\beta}, \\ (\omega, \beta x - 1), & \text{if } x > \frac{1}{\beta(\beta-1)}, \\ (\sigma\omega, \beta x - \omega_1), & \text{if } x \in \big[\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}\big]. \end{cases}$$

For each $x \in \big[0, \frac{1}{\beta-1}\big]$, and every $\omega \in \{0,1\}^{\mathbb{N}}$ we are now able to construct a β-expansion of x: $x = \sum_{n=1}^{\infty} b_n \beta^{-n}$, with $b_n \in \{0,1\}$ for all $n \geq 1$ in the following fashion: let

$$b_1 = b_1(\omega, x) = \begin{cases} 0, & \text{if } x < \frac{1}{\beta}, \text{ or } x \in \big[\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}\big] \text{ and } \omega_1 = 0, \\ 1, & \text{if } x > \frac{1}{\beta(\beta-1)}, \text{ or } x \in \big[\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}\big] \text{ and } \omega_1 = 1. \end{cases}$$

For $n \geq 1$, set $b_n(\omega, x) = b_1\big(K^{n-1}(\omega, x)\big)$.

Every β-expansion of x corresponds to some $\omega \in \{0,1\}^{\mathbb{N}}$; for most x's, different ω's correspond to a different β-expansion of x. Dajani and de Vries [2,3] proved that for every $p \in [0,1]$, there exists an invariant probability measure $\nu_p = m_p \times \mu_p$, where m_p is the p-Bernoulli measure on $\{0,1\}^{\mathbb{N}}$, and μ_p is an absolutely continuous measure on $[0, \frac{1}{\beta-1}]$. Moreover, K is ergodic, and Bernoulli. Interesting is the link to Bernoulli convolutions: if $\pi_2 : \{0,1\}^{\mathbb{N}} \times \frac{1}{\beta-1} \to \frac{1}{\beta-1}$ is the projection on the second component, then

$$\rho = \nu_1 \circ \pi_2^{-1}$$

is the Bernoulli convolution – the law describing distributions of random power series

$$\sum_{k \geq 1} \omega_k \beta^{-k},$$

where ω_k are iid 0.5-Bernoulli random variables.

2 Random continued fractions

Similarly to the case of β-expansions, continued fraction expansions can be constructed using one of the following maps:

- *the Gauss continued fraction map*

$$T_0(x) = \left\{\frac{1}{x}\right\}, \quad x \in (0,1),$$

- *the backward continued fraction map*

$$T_1(x) = \left\{\frac{1}{1-x}\right\}, \quad x \in (0,1),$$

where $\{\cdot\}$ denotes the fractional part.

It is well-known that T_0 admits a unique absolutely continuous invariant probability measure μ_0 with density $\frac{1}{(1+x)\log 2}$, while T_1 only admits a σ-**finite** absolutely continuous invariant measure μ_1 with the density $\frac{1}{x}$. The source of *singularity* is the presence of an indifferent fixed point of T_1 at the origin.

Define $\mathcal{T} : \{0,1\}^{\mathbb{N}} \times (0,1) \to \{0,1\}^{\mathbb{N}} \times (0,1)$ as

$$\mathcal{T}(\omega, x) = (\sigma\omega, T_\omega(x)).$$

Iterating the random continued fraction map $\mathcal{T}$, gives an expansion of x: for every sequence $\omega \in \{0,1\}^{\mathbb{N}}$,

$$x = \cfrac{1}{a_1 + \cfrac{(-1)^{\omega_1}}{a_2 + \ddots + \cfrac{(-1)^{\omega_{k-1}}}{a_k + \ddots}}}$$

where the digit sequence $\{a_k = a_k(\omega, x)\}$ is determined by the integer part of the corresponding iterate of T_{ω_k}.

An interesting question is whether for a given $p \in [0,1]$ there exists a invariant $\mathcal{T}$-measure ν_p of the form $m_p \times \mu_p$, where again m_p is the $(1-p)$-Bernoulli measure on $\{0,1\}^{\mathbb{N}}$ and μ_p is a finite absolutely continuous measure on $(0,1)$. Note that the invariance of ν_p is equivalent to the following "invariance" condition for μ_p:

$$\mu_p(A) = p \cdot \mu_p(T_0^{-1}A) + (1-p) \cdot \mu_p(T_1^{-1}A) \quad \text{for all Borel } A \subseteq (0,1).$$

Clearly, for $p = 1$ the answer is positive as one the question boils down to the question about the standard Gauss map; similarly, for $p = 0$, the answer is negative. We have the following result.

Theorem 2.1 (C.Kalle, T. Kempton, E.V. [9])**.** *For any $p \in (0,1]$ there exists an absolutely continuous invariant measure μ_p whose density belongs to the class of functions with* bounded variation.

The proof is an application of a recent result of T. Inoue [6], which in some sense completes a long series of results on existence of absolutely continuous invariant measures for random interval maps with countable number of intervals of monotonicity and place dependent probabilities.

The density f_p of the measure μ_p satisfies the following equation

$$\begin{aligned} f_p(x) &= p\sum_{k=1}^{\infty} \frac{1}{(x+k)^2} f_p\left(\frac{1}{x+k}\right) + (1-p)\sum_{k=1}^{\infty} \frac{1}{(x+k)^2} f_p\left(1 - \frac{1}{x+k}\right) \\ &=: p\mathcal{L}_0 f_p(x) + (1-p)\mathcal{L}_1 f_p(x) = \mathcal{L}_p f(x), \end{aligned}$$

where $\mathcal{L}_0$, $\mathcal{L}_1$ are transfer operators for the standard and the backward continued fraction maps, respectively. Both operators preserve cones of positive smooth (analytic) functions on $[0,1]$.

Computer simulations of the invariant density f_p seem to suggest that the function f_p given in Theorem 2.1 is strictly positive and smooth for any $p \in (0,1]$. In fact, we have the following

Conjecture 2.1. *For each $0 < p < 1$ the function f_p is strictly positive and real analytic on $[0,1]$.*

Spectral properties of $\mathcal{L}_0$ are well understood, see [7] for a recent overview. Particularly useful is the relation [8, 11] between $\mathcal{L}_0$ and the the integral operator $\mathcal{K}_0$ acting on the Hilbert space $L^2(\mathbb{R}_+, \mu)$, given by

$$\mathcal{K}_0\phi(s) = \int_0^\infty \frac{J_1(2\sqrt{st})}{\sqrt{st}} \phi(t)\, d\mu(t),$$

where J_1 is the Bessel function of the first kind, and μ is the measure on $\mathbb{R}_+$ with the density

$$d\mu = \frac{t}{e^t - 1} dt.$$

The operator $\mathcal{K}_0$ has a symmetric kernel $K_0(s,t) = \frac{J_1(2\sqrt{st})}{\sqrt{st}}$, and has several nice properties, e.g., is nuclear. In a similar fashion, existence of a positive smooth fixed point of $\mathcal{L}_p$ will follow from the existence of a positive fixed point of $\mathcal{K}_p$

$$\begin{aligned} \mathcal{K}_p\phi(s) &= p\mathcal{K}_0\phi(s) + (1-p)\mathcal{K}_1\phi(s) \\ &= p\int_0^\infty \frac{J_1(2\sqrt{st})}{\sqrt{st}} \phi(t)\, d\mu(t) + (1-p)\int_0^\infty \frac{I_1(2\sqrt{st})}{\sqrt{st}} \phi(t) e^{-t}\, d\mu(t), \end{aligned} \tag{2}$$

where I_1 is the modified Bessel function of the first kind. Main technical difficulties in the analysis of $\mathcal{K}_p$ arise from the fact that the kernel $K_1 = \frac{I_1(2\sqrt{st})}{\sqrt{st}}$ albeit monotonic and positive (c.f., K_0 is oscillating), is not integrable.

References

[1] Karma Dajani and Cor Kraaikamp, *Random β-expansions*, Ergodic Theory Dynam. Systems **23** (2003), no. 2, 461–479, DOI 10.1017/S0143385702001141. MR1972232 (2004a:37010)

[2] Karma Dajani and Martijn de Vries, *Measures of maximal entropy for random β-expansions*, J. Eur. Math. Soc. (JEMS) **7** (2005), no. 1, 51–68, DOI 10.4171/JEMS/21. MR2120990 (2005k:28030)

[3] ———, *Invariant densities for random β-expansions*, J. Eur. Math. Soc. (JEMS) **9** (2007), no. 1, 157–176, DOI 10.4171/JEMS/76. MR2283107 (2007j:37008)

[4] Karma Dajani and Charlene Kalle, *Random β-expansions with deleted digits*, Discrete Contin. Dyn. Syst. **18** (2007), no. 1, 199–217, DOI 10.3934/dcds.2007.18.199. MR2276494 (2007m:37016)

[5] K. Dajani and C. Kalle, *Local dimensions for the random β-transformation*, New York J. Math. **19** (2013), 285–303. MR3084706

[6] Tomoki Inoue, *Invariant measures for position dependent random maps with continuous random parameters*, Studia Math. **208** (2012), no. 1, 11–29, DOI 10.4064/sm208-1-2. MR2891182 (2012m:37003)

[7] Marius Iosifescu, *Spectral analysis for the Gauss problem on continued fractions*, Indag. Math. (N.S.) **25** (2014), no. 4, 825–831, DOI 10.1016/j.indag.2014.02.007. MR3217038

[8] Oliver Jenkinson, Luis Felipe Gonzalez, and Mariusz Urbański, *On transfer operators for continued fractions with restricted digits*, Proc. London Math. Soc. (3) **86** (2003), no. 3, 755–778, DOI 10.1112/S0024611502013904. MR1974398 (2004d:37032)

[9] C. Kalle, T. Kempton, and E. Verbitskiy, *Random continued fractions*, Preprint (2015).

[10] T. Kempton, *On the invariant density of the random β-transformation*, Acta Math. Hungar. **142** (2014), no. 2, 403–419, DOI 10.1007/s10474-013-0377-x. MR3165489

[11] D. Mayer and G. Roepstorff, *On the relaxation time of Gauss's continued-fraction map. I. The Hilbert space approach (Koopmanism)*, J. Statist. Phys. **47** (1987), no. 1-2, 149–171, DOI 10.1007/BF01009039. MR892927 (89a:28017)

BETA EXPANSION WITH ROTATION

SHIGEKI AKIYAMA (UNIV. TSUKUBA)

A JOINT WORK WITH JONATHAN CAALIM.

The beta transform

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor$$

gives a generalization of binary and decimal expansions to a real base $\beta > 1$. Its ergodic property is well known:

- There is a unique absolutely continuous invariant probability measure (ACIM) equivalent to 1-dim Lebesgue measure [17]
- The invariant measure is made explicit ([16, 7])
- The system is exact, consequently it is mixing of any degree.
- Its natural extension is Bernoulli.

When β is not an integer, the digits $\{0, 1, \dots, \lfloor \beta \rfloor\}$ are not independent. There are many studies on the associated symbolic dynamics, in particular, when it becomes SFT, sofic, specification, etc. They are described by the forward orbit of the discontinuity $1-0$, but not so easy to give algebraic criteria of them. For example, if β is a Pisot number, then the system is sofic, but it is not easy to characterize SFT cases among them. Here Pisot number is an algebraic integer greater than one whose all other conjugates have modulus less than one.

Number theoretical generalizations had been studied by means of numeration system in complex bases, e.g., [10, 4, 2, 13]. In this talk, we wish to generalize beta expansion in a dynamical way to the complex plane introducing rotation action. Let $1 < \beta \in \mathbb{R}$ and $\zeta \in \mathbb{C} \setminus \mathbb{R}$ with $|\zeta| = 1$. Fix $\xi, \eta_1, \eta_2 \in \mathbb{C}$ with $\eta_1/\eta_2 \notin \mathbb{R}$. Then $\mathcal{X} = \{\xi + x\eta_1 + y\eta_2 \mid x \in [0,1), y \in [0,1)\}$ is a fundamental domain of the lattice $\mathcal{L}$ generated by η_1 and η_2 in $\mathbb{C}$, i.e.,

$$\mathbb{C} = \bigcup_{d \in \mathcal{L}} (\mathcal{X} + d)$$

is a disjoint partition of $\mathbb{C}$. Define a map $T : \mathcal{X} \to \mathcal{X}$ by $T(z) = \beta\zeta z - d$ where $d = d(z)$ is the unique element in $\mathcal{L}$ satisfying $\beta\zeta z \in \mathcal{X} + d$. Given a point z in $\mathcal{X}$, we obtain an expansion

$$\begin{aligned} z &= \frac{d_1}{\beta\zeta} + \frac{T(z)}{\beta\zeta} \\ &= \frac{d_1}{\beta\zeta} + \frac{d_2}{(\beta\zeta)^2} + \frac{T^2(z)}{(\beta\zeta)^2} \\ &= \sum_{i=1}^{\infty} \frac{d_i}{(\beta\zeta)^i} \end{aligned}$$

with $d_i = d(T^{i-1}(z))$. In this case, we write $d_T(z) = d_1 d_2 ....$ We call T the *rotational beta transformation* and $d_T(z)$ the *expansion* of z with respect to T. We note that the map T generalizes the notions of beta expansion [17, 16, 7] and negative beta expansion [6, 15, 8] in a natural dynamical manner to the complex plane $\mathbb{C}$.

Since T is a piecewise expanding map, by a general theory developed in [11, 12, 5, 18, 19, 3, 20], there exists an invariant probability measure μ which is absolutely continuous to the two-dimensional Lebesgue measure. The number of ergodic components is known to be finite [11, 5, 18]. An explicit upper bound in terms of the constants in Lasota-Yorke type inequality was given by Saussol [18]. However this bound may be large. We shall give two explicit constants B_1 and B_2 depending only on η_1 and η_2 that T has a unique ACIM if $\beta > B_1$. Further if $\beta > B_2$ then the ACIM is equivalent to the 2-dimensional Lebesgue measure on $\mathcal{X}$. Note that if β is small, then we can give examples of T's with at least two ergodic ACIM's. An interesting remaining problem is to improve B_1 and B_2. We feel that they are still far from best possible.

For general cases, it is difficult to make explicit the Radon-Nikodym density of the ACIM. It is of interest to study when the symbolic system associated to the rotational beta expansion is sofic, where we can compute the density explicitly.

Restricting to a rotation generated by q-th root of unity ζ with all parameters in $\mathbb{Q}(\zeta, \beta)$, it gives a sofic system when $\cos(2\pi/q) \in \mathbb{Q}(\beta)$ and β is a Pisot number. It is interesting to point out that this result gives examples of sofic rotational expansion with any finite order rotation, like 7-fold or 11-fold.

We will also also show that the condition $\cos(2\pi/q) \in \mathbb{Q}(\beta)$ is necessary by giving a family of non-sofic systems for $q = 5$. Anyway this gives a sufficient condition of soficness but it is not necessary. It is of interest to characterize sofic cases.

References

[1] S. Akiyama, *A family of non-sofic beta expansions*, To appear in Ergodic Theory and Dynamical Systems (2015).

[2] S. Akiyama, H. Brunotte, A. Pethő, and J. M. Thuswaldner, *Generalized radix representations and dynamical systems II*, Acta Arith. **121** (2006), 21–61.

[3] J. Buzzi and G. Keller, *Zeta functions and transfer operators for multidimensional piecewise affine and expanding maps*, Ergodic Theory Dynam. Systems **21** (2001), no. 3, 689–716.

[4] W. J. Gilbert, *Radix representations of quadratic fields*, J. Math. Anal. Appl. **83** (1981), 264–274.

[5] P. Góra and A. Boyarsky, *Absolutely continuous invariant measures for piecewise expanding C^2 transformation in $\mathbf{R}^N$*, Israel J. Math. **67** (1989), no. 3, 272–286.

[6] Sh. Ito and T. Sadahiro, *Beta-expansions with negative bases*, Integers **9** (2009), A22, 239–259.

[7] Sh. Ito and Y. Takahashi, *Markov subshifts and realization of β-expansions*, J. Math. Soc. Japan **26** (1974), no. 1, 33–55.

[8] C. Kalle, *Isomorphisms between positive and negative β-transformations*, Ergodic Theory Dynam. Systems **34** (2014), no. 1, 153–170.

[9] C. Kalle and W. Steiner, *Beta-expansions, natural extensions and multiple tilings associated with Pisot units*, Trans. Amer. Math. Soc. **364** (2012), no. 5, 2281–2318.

[10] I. Kátai and B. Kovács, *Canonical number systems in imaginary quadratic fields*, Acta Math. Acad. Sci. Hungar. **37** (1981), 159–164.

[11] G. Keller, *Ergodicité et mesures invariantes pour les transformations dilatantes par morceaux d'une région bornée du plan*, C. R. Acad. Sci. Paris Sér. A-B **289** (1979), no. 12, A625–A627.

[12] ———, *Generalized bounded variation and applications to piecewise monotonic transformations*, Z. Wahrsch. Verw. Gebiete **69** (1985), no. 3, 461–478.

[13] V. Komornik and P. Loreti, *Expansions in complex bases*, Canad. Math. Bull. **50** (2007), no. 3, 399–408.

[14] T.Y. Li and J. A. Yorke, *Ergodic transformations from an interval into itself*, Trans. Amer. Math. Soc. **235** (1978), 183–192.

[15] L. Liao and W. Steiner, *Dynamical properties of the negative beta-transformation*, Ergodic Theory Dynam. Systems **32** (2012), no. 5, 1673–1690.

[16] W. Parry, *On the β-expansions of real numbers*, Acta Math. Acad. Sci. Hungar. **11** (1960), 401–416.

[17] A. Rényi, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar. **8** (1957), 477–493.

[18] B. Saussol, *Absolutely continuous invariant measures for multidimensional expanding maps*, Israel J. Math. **116** (2000), 223–248.

[19] M. Tsujii, *Absolutely continuous invariant measures for piecewise real-analytic expanding maps on the plane*, Comm. Math. Phys. **208** (2000), no. 3, 605–622.

[20] ———, *Absolutely continuous invariant measures for expanding piecewise linear maps*, Invent. Math. **143** (2001), no. 2, 349–373.

E-mail address: `akiyama@math.tsukuba.ac.jp`

Institute of Mathematics, University of Tsukuba, 1-1-1 Tennodai,, Tsukuba, Ibaraki, Japan (zip:350-8571)

β-Encoders: Symbolic Dynamics and Electronic implementation for AD/DA converters
Tohru Kohda[1]

Extended Abstract: Almost all signal processing systems need analog signals to be discretized. Discretization in time and in amplitude are called *sampling* and *quantization*, respectively. These two operations constitute *analog-to-digital (A/D) conversion.* The A/D conversion includes *pulse-code modulation (PCM)* [1, 2, 3] and $\Sigma-\Delta$ modulation [4, 5, 6, 7, 8]. PCM has a precision of $O(2^{-L})$ for L iterations but has a serious problem when it is implemented in an electronic circuit, *e.g.,* if PCM has a threshold shift, then the quantization errors do not decay. In contrast, $\Sigma-\Delta$ modulation achieves a precision that decays like an inverse polynomial in L but has the practical advantage for analog circuit implementation.

In 2002, Daubechies *et al.*[15] introduced a new A/D converter using an amplifier with a factor β and a flaky quantizer with a threshold ν, known as a β-*encoder*, and showed that it has exponential accuracy even if it is iterated at each step in the successive approximation of each sample by using an imprecision quantizer with a quantization error and offset parameter, Furthermore, in a subsequent paper, Daubechies *et al.*[16] introduced a "flaky" version of an imperfect quantizer derfined as

$$Q^{\text{flaky}}_{[\nu_0,\nu_1]}(z) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if} \quad z < \nu_0, \\ 1, & \text{if} \quad z \geq \nu_1, \\ 0 \,\text{or}\, 1, & \text{if} \quad z \in [\nu_0,\nu_1], \nu_0 < \nu_1 \end{cases} \tag{1}$$

which is a model of a quantizer $Q_\nu(z)$ with a varying threshold $\nu \in [\nu_0,\nu_1], \nu_0 < \nu_1$, defined as

$$Q_\nu(x) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if} \quad x < \nu, \\ 1, & \text{if} \quad x \geq \nu \end{cases} \tag{2}$$

They made the remarkable observation that "greedy"($\nu = \nu_{\mathrm{G}} = 1$) and "lazy"($\nu = \nu_{\mathrm{L}} = (\beta-1)^{-1}$) expansions, as well as "cautious"($\nu_{\mathrm{G}} < \nu < \nu_{\mathrm{L}}$) expansions[2] in the β-encoder with such a flaky quantizer exhibit exponential accuracy in the bit rate L, and they gave the decoded values as

$$\hat{x}^{\mathrm{DDGV}}_{\mathrm{L}} = \sum_{i=1}^{L} b_i\gamma^i,\ b_i \in \{0,1\},\ \gamma = \beta^{-1}. \tag{3}$$

[1]Professor Emeritus, Kyushu University, E-mail:torukoda81@wind.ocn.ne.jp

[2]*Intermediate expansions*[30, 32] between the greedy and lazy expansions[28, 29] are called "*cautious*" by Daubechies[16].

Furthermore, Daubechies and Yilmätz[17] proposed a β-encoder that is not only robust to quantizer imperfections but also robust with to the amplification factor β, and gave the β-recovery method that relies upon embedding the value of β in the encoded bit stream for each sample value separately without measureing its value. This β-encoder is a signinificant achievement in Nyquist-rate A/D and D/A conversions in the sense that it may become a good alternative for PCM [9, 10, 11].

In our recent paper[18], we gave comprehensive reviews for A/D conversions including PCM, $\Sigma-\Delta$ modulation, and β-encoder (see Fig.1 for its single-loop feedback form) as well as symbolic dynamics. [3] Furthermore, we gave the fact that β-encoders using a flaky quantizer with the threshold ν are characterized by the symbolic dynamics of the multi-valued Rényi-Parry map, defined as[22, 23]

$$T_\beta(x) = \beta x \bmod 1 \tag{4}$$

or Parry's (β, α)-map, defined as[24]

$$T_{\beta,\alpha}(x) = \beta x + \alpha \bmod 1 \tag{5}$$

in the middle interval (see Fig.2). Dynamical systems theory[37, 38] tells us that a sample x is always confined to a subinterval of a contracted interval, as shown in Fig.3 and so its decoded sample can be defined as [18, 19, 20],

$$\widehat{x}_L^{\mathrm{KHA}} = \sum_{i=1}^{L} b_i\gamma^i + \frac{\gamma^L}{2(\beta-1)},\ b_i \in \{0,1\}, \tag{6}$$

because the decoded sample is equal to the midpoint of the subinterval. The decoded sample $\widehat{x}_L^{\mathrm{KHA}}$ also yields the characteristic equation for recovering β, which improves the quantization error by more than 3dB over the bound given by Daubechies *et al.*[16] and Daubechies and Yilmätz[17].[4]

[3] Several tutorial papers and textbooks are available (see e.g.[9, 10, 11] for digital communication, [12, 13, 14] for the basics of dynamical systems theory, and [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34] for β-transformation. See a review paper[18] and the detailed references cited therein for fundamental of quantization for digital communications and various AD/DA conversions and β-encoder fundamentals.

[4] Ward[21] has recently proposed new AD/DA algorithms for generating a binary sequence $\{b_i^{\mathrm{Ward}}\}_{i=1}^{\infty}, b_i^{\mathrm{Ward}} \in \{-1,1\}$ for a real-valued $y \in (-1,1)$ using a flaky version of an imperfect quantizer and gave its decoded value as $\widehat{y}_L^{\mathrm{Ward}} = \sum_{i=1}^{L} b_i^{\mathrm{Ward}}\gamma^i$. Ward's flaky quantizer is also realized exactly by the multi-valued Rényi-Parry map because it is topological conjugate to Parry's map:$T_{\beta,\alpha}(x)$ via the conjugacy $y = h(x) = 2x - (\beta-1)^{-1}$.

In order to show the self-correction property of the amplification factor β in β-encoder, Daubechies and Yilmätz[17] presented an equation governed by the sample data bit sequences as follows. Using the β-expansion sequences $\{b_i\}_{i=1}^{L}$ for $x \in [0,1)$ and $\{c_i\}_{i=1}^{L}$ for $y = 1-x$, $1 \le i \le L$ yields a root of the *algebraic equation of* β, defined by

$$P_L^{\mathrm{DY}}(\gamma) = 1 - \sum_{i=1}^{L}(b_i + c_i)\gamma^i. \tag{7}$$

On the contray, our β-recovering equation with index p_L [18, 19, 20] is

$$P_L^{\mathrm{KHA}}(\gamma, p_L) = 1 - \sum_{i=1}^{L}(b_i + c_i)\gamma^i - p_L\frac{\gamma^{L+1}}{1-\gamma},\ p_L \in \{0,1,2\}, \tag{8}$$

which is based on an L-bit truncated expansion with index p_L, defined as

$$\widehat{x}_L^{\mathrm{KHA}}(\gamma, p_L) = \sum_{i=1}^{L} b_i\gamma^i + p_L \cdot \frac{\gamma^L}{2(\beta-1)} \tag{9}$$

The associated quantization error is bounded by

$$|x - \widehat{x}_L^{\mathrm{KHA}}(\gamma, p_L)| \le \left(\frac{1+|p_L-1|}{2}\right) \cdot (\beta-1)^{-1}\gamma^L \tag{10}$$

so that the cases where $p_L = 0, 1, 2$ correspond to the leftmost, intermediate, and rightmost points of the Lth subinterval, respectively; the case where $p_L = 0$ is equal to Dabechies et al.'s decoded value.

As thoroughly discussed in our recent paper[35], the probabilistic behavior of this flaky quantizer is explained by the deterministic dynamics of a *multi-valued Rényi-Parry map* on the middle interval[18, 19, 20] (see Fig.2). This map is an eventually locally onto map of $[\nu-1, \nu)$, which is topologically conjugate to Parry's (β, α)-map $T_{\beta,\alpha}(x)$ with $\alpha = (\beta-1)(\nu-1)$. β-encoders have a closed subinterval $[\nu-1, \nu)$, which includes an *attractor*[36, 37, 38]. This *β-expansion attractor*[35] seems to be irregularly oscillatory but performs the β-expansion of each sample stably and precisely (see Fig.3). This viewpoint allows us to obtain a decoded sample(eq.6 or eq.9), which is equal

The homeomorphism $\widehat{y}_L^{\mathrm{Ward}} = h(\widehat{x}_L^{\mathrm{KHA}})$, however, does not necessarily imply equivalence in terms of the quatization errors; in fact, Ward's algorirthm doubles the maximum quantization error and quadruples its mean square error.

to the midpoint of the subinteval, and its associated characteristic equation for recovering β (eq.8), and shows that ν should be set to around the midpoint of its associated greedy and lazy values. This leads us to design β-encoders realizing *ordinary* (see Fig.4) and *negative scaled β-maps*[20] (see Fig.5) and observe β-expansion attractors embedded in these β-encoders[35].

Finally, we note that parts of this article draw on our previous work in [18, 19, 20, 35], which were supported by the Aihara Innovation Mathematical Modelling Project (Aihara Project), the Japan Society for the Promotion of Science (JSPS) through the "Fundamental Program for World-Leading Innovation R&D on Science and Technology(FIRST Program)", initiated by the Counicil for Science and technology Policy (CSTP). The FIRST Program also supported the β-encoder group to implement these β-encoders in an LSI (Large-Scale Integrated) circuit and evaluate quantization errors and their performance in practically realized LSI circuits based on a simple β-recovery method suited to operation of AD/DA conversions in LSI cicuits, .[42, 43, 44, 45].

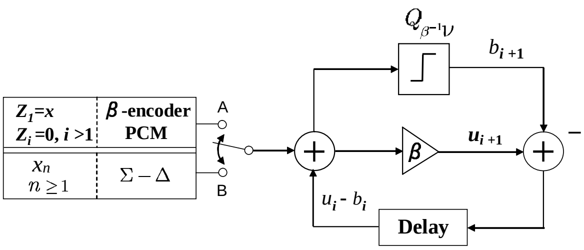


Fig.1. A discrete-time, single-loop feedback system using an amplifier with an ampflication factor β and a 1-bit quantizer $Q_{\beta^{-1}\nu}$ with a threshold ν that realizes PCM when $\beta = 2$ and $\nu = 1$; a β-encoder when $1 < \beta < 2$ and $\nu \in [1, (\beta-1)^{-1}]$, proposed by Daubechies et al.[15]; and $\Sigma - \Delta$ modulation when $\beta = 1$ and $\nu = 0$. The input is $z_1 = x \in [0, 1), z_i = 0, i > 1$ for the PCM and β-encoder, and the input is $x_n, n \geq 1$ for the $\Sigma - \Delta$ modulation. The initial conditions are given by $u_0 = b_0 = 0$. The output sequence $\{b_i\}_{i=1}^{L}, b_i \in \{0, 1\}$ gives the L-bit β-expansion for x, and the averaging sequence $\{b_i\}_{i=1}^{M}, b_i \in \{0, 1\}$ over i is the output in response to the input sequence $\{x_n\}_{n\geq 1}$ for the $\Sigma - \Delta$ modulation. The β-encoder provides the greedy, lazy, and cautious schemes for $\nu = \nu_{\mathrm{G}} = 1, \nu = \nu_{\mathrm{L}} = (\beta - 1)^{-1}$, and $\nu_{\mathrm{G}} < \nu < \nu_{\mathrm{L}}$, respectively.

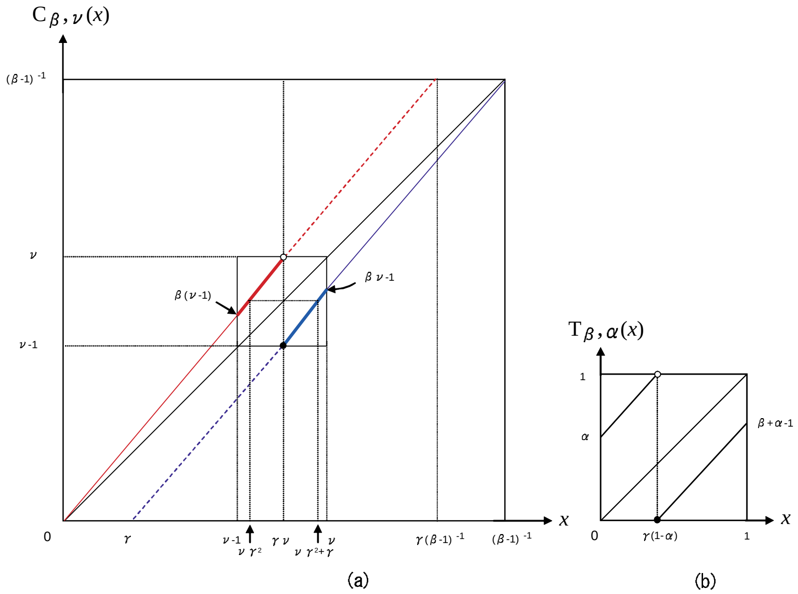


Fig.2. The expansion map $C_{\beta,\nu}(x) \stackrel{\text{def}}{=} \beta x - Q_{\beta^{-1}\nu}(x)$ realizing the Daubechies et al.'s flaky quantizer[16] $Q^{\text{flaky}}_{(\gamma\nu_{\text{G}},\gamma\nu_{\text{L}})}(z)$, $1 = \nu_{\text{G}} < \nu < \nu_{\text{L}} = (\beta-1)^{-1}$ [5] (b) renormalizing the interval $[\nu-1,\nu]$ into the unit interval $[0,1]$, which shows that such an eventually locally onto map is equivalent to the Parry (β,α)-transformation:$T_{\beta,\alpha}(x) = \beta x + \alpha \bmod 1$. The transformation $T_{\beta,\alpha}(x)$ has a finite (signed) invariant measure $\nu(E) = \int_{\text{E}} h(x)dx$, where $h(x)$ is given by $h(x) = \sum\limits_{x<T^n_{\beta,\alpha}(1)} \beta^{-n} - \sum\limits_{x<T^n_{\beta,\alpha}(0)} \beta^{-n}$.[24, 39]

[5] $C_{\beta,\nu}(x)$ has its *eventually locally onto* map with the *strongly invariant* subinterval $C^{-1}_{\beta,\nu}([0,\gamma\nu]) \cap C^{-1}_{\beta,\nu}([\gamma\nu,(\beta-1)^{-1}]) = [\nu-1,\nu]$. (Let $\tau : E \to E$ be a continuous map. Let $F \subset E$. If $\tau(F) \subset F$, then F is called *invariant*. If $\tau(F) = F$, then F is called *strongly invariant*. [14]).

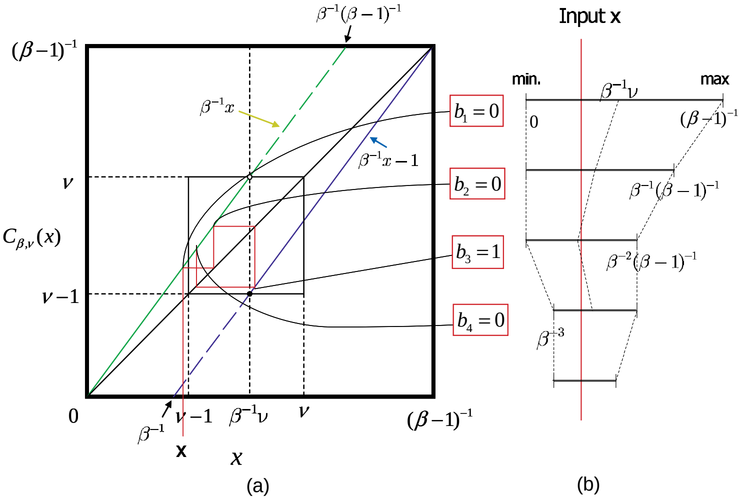


Fig.3. (a) The *multi-valued Rényi-Parry map* $C_{\beta,\nu}(x) \stackrel{\text{def}}{=} \beta x - Q_{\beta^{-1}\nu}(x)$ on the middle interval $[\beta^{-1}, \beta^{-1}(\beta-1)^{-1}]$ with its discontinuity $x = \beta^{-1}\nu$, which is *eventually locally onto* $[\nu-1, \nu)$, where $1 \leq \nu \leq (\beta-1)^{-1}$. An eventually locally onto map of $[\nu-1,\nu)$ with $\nu = 1 + \alpha/(\beta-1)$ is topologically conjugate to Parry's (β,α)-transformation $T_{\beta,\alpha}(x)$ via the conjugacy $\varphi^{-1}(x) = x + \alpha/(\beta-1)$, i.e., $\varphi(C_{\beta,\nu}(\varphi^{-1}(x))) = T_{\beta,\alpha}(x)$ when $\alpha = (\beta-1)(\nu-1)$. The map $C_{\beta,\nu}(x)$ realizes Daubechies et al.'s flaky quantizer[16] $Q^{\text{flaky}}_{[\beta^{-1},\beta^{-1}(\beta-1)^{-1}]}(z)$. (b) The contraction process by the first 4 binary β-expansions of the input x using $C_{\beta,\nu}(x)$ while the binary digits are obtained. The associated subintervals with a contraction ratio β^{-1} are given as $[0,(\beta-1)^{-1}), [0,\beta^{-1}(\beta-1)^{-1}), [0,\beta^{-2}(\beta-1)^{-1}), [\beta^{-3},\beta^{-2}(\beta-1)^{-1})$. The input x is always confined to the ith subinterval.

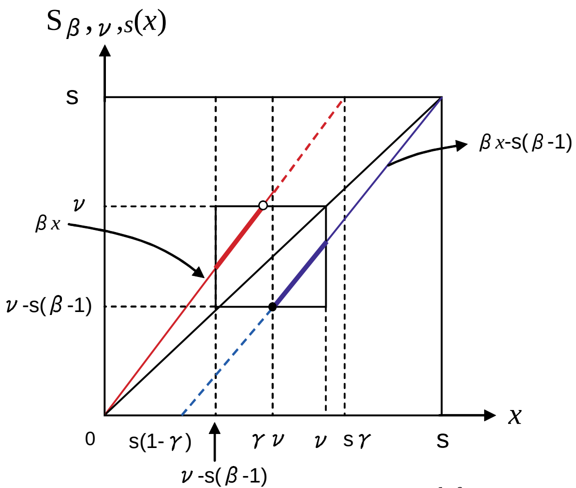


Fig.4. The *scale-adjusted ordinary β-map* $S_{\beta,\nu,s}(x) \stackrel{\text{def}}{=} \beta x - s(\beta-1)Q_{\gamma\nu}(x) = \begin{cases} \beta x & \text{when} \quad x \in [0, \gamma\nu), \\ \beta x - s(\beta-1), & \text{when} \quad x \in \gamma\nu, s), \end{cases}$ with its eventually locally onto map

$[\nu - s(\beta-1), \nu) \to [\nu - s(\beta-1), \nu)$, $\nu \in [s(\beta-1), s)$. Such an eventually locally onto map with $\nu = s(\alpha+\beta-1)$ is topologically conjugate to $T_{\beta,\alpha}(x)$ via the conjugacy $\varphi_{\mathrm{S}}^{-1}(x) = s(\beta-1)x + s\alpha$, i.e., $\varphi_S(S_{\beta,\nu,s}(x)) = T_{\beta,\alpha}(\varphi_{\mathrm{S}}(x))$.

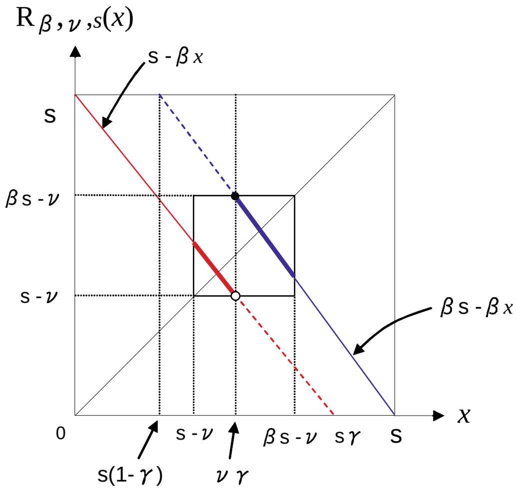


Fig.5. The *scale-adjusted negative β-map*

$$R_{\beta,\nu,s}(x) \overset{\mathrm{def}}{=} -\beta x + s[1 + (\beta-1)Q_{\gamma\nu}(x)] = \begin{cases} s - \beta x & \text{when} \quad x \in [0, \gamma\nu), \\ \beta s - \beta x, & \text{when} \quad x \in \gamma\nu, s), \end{cases}$$

with its eventually locally onto map $[s-\nu, \beta s - \nu) \to [s-\nu, \beta s - \nu)$ when $(\beta^2 - \beta + 1)/(\beta+1)s \le \nu < (2\beta-1)/(\beta+1)s$. [6] Such an eventually locally onto map with $\nu = s[(\beta-1)\alpha + \beta]/(\beta+1)$ is topologically conjugate to *Parry's transformation with negative slope*[20, 40] $T_{-\beta,\alpha}(x) \overset{\mathrm{def}}{=} -\beta x + \alpha \bmod 1$, $\beta \ge 1, 0 \le \alpha < 1$ via the conjugacy $\varphi_{\mathrm{R}}^{-1}(x) = s(\beta-1)x + s - \nu$, i.e., $\varphi_{\mathrm{R}}(R_{\beta,\nu,s}(x)) = T_{-\beta,\alpha}(\varphi_{\mathrm{R}}(x))$. The transformation $T_{-\beta,\alpha}(x)$ has a finite (signed) invariant measure $\nu(E) = \int_{\mathrm{E}} h(x)dx$, where $h(x)$ is given by $h(x) = \sum_{x < T^n_{-\beta,\alpha}(1)} (-\beta)^{-n} - \sum_{x < T^n_{-\beta,\alpha}(0)} (-\beta)^{-n}$.[41, 18]

References

[1] Clavier,A.G.,Panter,P.F., and Grieg,D.D,"Distortion in a pulse count modulation systems," *trans. AIEE*,**66**,989-1005,1947.

[2] B.M.Oliver,J.Pierce, and C.E.Shannon, "The Philosophy of PCM," *Proc. of IRE*, **36**,1324-1331,1948

[3] Bennett,W.R., "Spectra of quantized signals," *Bell. Syst.tech.J.*,**27**, 446-472,1948.

[6] There are three other eventually locally onto maps depending on ν.[18, 35]

[4] H.Inose andY.Yasuda,"A Unity bit coding method by negative feedback, " *Proc. IEEE*,**51**,1524-1535,(1963).

[5] Gray,R.M. "Oversampled sigma-delta modulation," *IEEE Trans. Commun.* **35**, 481-489,1987

[6] Lewis,S.H. and Gray,P.R., " A pipelined 5-Msample/s 9-bit analog-to-digital converter, " *IEEE. Solid-State Circuits*, **22**, 954-961.

[7] Lin,Y.-M, Kim,B and Gray P.R, " A 13-b 2.5MHz self-calibrated pipelined A/D converter in 3μm CMOS, " *IEEE J.Solid State Circuits*, **26**, 628-636,1991.

[8] Karanicolas,A.-N, Lee,H.S, and Barania,K.L, " A 15-b 1M samples/s Digitally Self-calibrated pipelined ADC " *IEEE J.Solid State Circuits*, **28**, 1207-1215,1993.

[9] Jayant, N.S. and Noll, P. "*Digital Communications of Waveforms-Principles and Applications to Speech and Video,*" Prentice-Hall, englewood Cliffs, NJ, 1984.

[10] Gray R.M., "Quantization of noise spectra," *IEEE Trans. Inf.Th.*, **36**, 1220-1243,1990.

[11] Gray R.M and Neuhoff, D.L., "Quantization," *IEEE Trans. Inf.Th.*, **44**, 2325-2383,1998.

[12] Lasota, A and Mackey, M.C., "*Chaos, Fractals and Noise*," Springer-Verlag,Ny, 1994.

[13] Boyarsky, A. and Góra,P. "*Laws of Chaos: Invariant Measures and Dynamical Systems in One Dimension*," Birkhäuser, Boston, 1997.

[14] Brucks, H.S. and Bruin, H.,"*Topics from one-dimensional dynamics,*" *London Mathematical Soiciety Student Texts*, vol.62(Cambridge University Pres), 2004.

[15] Daubechies,I.DeVore,R.Gunturuk,C and Vaishampayan,V, " Beta Expansions:A new approach to digitally corrected A/D conversion, " *Proc.IEEE Int.Symp.Circ.Sys.2002*, **2**, 784-787.

[16] Daubechies, I and Yilmätz,O, " Robust and practical analog-to-digital conversion with exponential precision, " *IEEE Trans.Inf.Th.* **52**, 3533-3545,2006.

[17] Daubechies,I. DeVore,R,Gunturk, C. and Vashampayan,V, " A/D conversion with imperfect quantizers," *IEEE Trans. Inf.Th.* **56**, 5097-5110, 2006.

[18] T. Kohda ,Y. Horio, Y.Takahashi, and K. Aihara, "Beta Encoders: Symbolic Dynamics and Electronic Implementation ", *Int.J. of Bifurcation and Chaos*, **22**, no.9,, 2012,1230031(55 pages)

[19] S.Hironaka,T.Kohda, and K,Aihara, " Markov chain of binary sequences generated by A/D conversion using β-encoder, " *Proc.of IEEE Int.Workshop on Nonlinear Dynamics of Electronic Systems*, 261-264, 2007.

[20] S.Hironaka,T.Kohda, and K,Aihara, " Negative β-encoder, " *e-print arXiv:0808.2548v2[cs.IT](2008).*

[21] Ward,R. " On robustness properties of beta encoders and golden ratio encoders, " *IEEE Trans.Inf.Th.*, **54**,4324-4344,2008.

[22] Rényi,A, "Representations for real numbers and their ergodic properties, " *Acta. Math. Hungar.*,**8**, 477-493,1957.

[23] Parry,W."On β-expansions of real numbers," *Acta Math.Acad.Sci.Hung.* **11**, 401-416.1960

[24] Parry, W, " Representations for real numbers " *Acta Math.Acad. Sci. Hug.*, **15**, 95-105,1964

[25] Takahashi, Y. " Isomorphisms of β-automorphisms to Markov automorphisms " , *Osaka J.Math.*,**10**, 175-184,1973.

[26] Ito,S. and Y.Takahashi, " Markov subshifts and realization of β-expansions " *J.Math. Soc, Japan*, **26**, 33-55,1974.

[27] Takahashi,Y. " Shift with orbit basis and realization of one-dimensional map, " *Osaka J.Math.* **20**,599-629,1983(Correction:**21**, 637,1985).

[28] Erdös,P.Joó,I,and Komoronik,V. " Characterization of the uniqueness of $1 = \sum_{i=1}^{\infty} q^{-n_i}$ and related problems, " *Bull. Soc. Math. France*, **118**, 377-390,1990

[29] Erdös,P.Horvath, and Joó,I, " On the uniqueness of the expansions $1 = \sum_{i=1}^{\infty} q^{-n_i}$ " *Acta. Math.Hung.*, **58**, 333-342,1991

[30] Sidorov,N."Arithmetic dynamics," *Topics in Dynamics and ergodic Theory,* LMS Lecture Notes, **310**, 145-189,2002.

[31] Dajani,K and Kraaikamp,C, "*Ergodic theory of numbers",The Carus Math,Monogr.*, vol.29, The Mathematical Association of America, 2002.

[32] Dajani,K and Kraaikamp,C, "From greedy to lazy expansions and their driving dynamics," *Expo.Math.*,**20**,316-327,2002.

[33] Dajani,K and Kraaikamp,C, "Random β-expansions," *Erg.Th.Dyn. Syst.*, **23**,461-479, 2003.

[34] Sidorov,N."Almost every number has a continum β-expansions," *Amer. Math.Monthly,* **110**, 838-842, 2003.

[35] T. Kohda ,Y.Horio, and Kazuyuki Aihara, " β- Expansion Attractors observed in A/D Converters" , *AIP Chaos An Interdisciplinary Journal of nonlinear Science*, **22**,no.4,2012, 047512(18 pages)

[36] Ruelle,D.,"Small random perturabations of dynamical systems and the definition of attractors," *Commun. Math.Phys.***82**, 137-151,1981.

[37] Guckenheimer J. and Holmes, P. "*Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, "(Springer-Verlag,New York,1983).

[38] Wiggins,S.,"*Introduction to Applied Nonlinear Dynamical Systems and Chaos*"(Springer-Verlag,New York,1990).

[39] Gelfond,A.O,"On a general property of number systems," *Izy.Akad. Nauk. SSSR.*, **23**, 809-814,1959

[40] Ito,S. and Sadahiro, T. "Beta-expansions with negative bases."*Integers*, **9**, 239-259, 2009.

[41] Tsujii, M. and Tanaka, H.,"Note on absolutely continuous invariant measure of β-transformations", *personal communication*, 2011.

[42] H. San, T. Kato, T. Maruyama, K. Aihara and M. Hotta, "Non-Binary Pipeline Analog-to-digital Converter Based on β-Expansion," *IEICE Trans. on Fundamentals of Electronics, Communications and Computer Sciences*, **E96-A**, no. 2, pp. 415-421, Feb. 2013.

[43] R. Suzuki, T. Maruyama, H. San, K. Aihara and M. Hotta, " Robust Cyclic-ADC Architecture Based on β-Expansion," *IEICE Trans. on Electronics*, **E96-C**, no. 4, pp. 553-559, April 2013.

[44] T. Makino, Y. Iwata, Y. Jitsumatsu, M. Hotta, H. San, and K. Aihara. " Rigorous analysis of quantization error of an A/D converter based on β-map" . *In Proc. of 2013 IEEE Int. Symp. on Circuits and Systems (ISCAS2013)*, 369-372, May 2013.

[45] T.Makino, Y.Iwata, K.Shinohara, Y.Jitsumatsu, M.Hotta, H.San, and K.Aihara, "Rigorous Estimates of Quantization error for an A/D Converter Based on a Beta-Map," *NOLTA(Nonlinear Theory and Its Applications), IEICE*, **6**-1,99-111,2015.

Workshop on β-transformation and related topics
At IMI, Kyushu University 3/10,2k15

β-encoders: Symbolic dynamics and Electronic Implementation for AD/DA converters

Tohru Kohda (Emeritus of Kyushu Univ.),
Collaborated with
Y.Horio (Tokyo Denki Univ.)
Y.Takahashi (Emeritus, the Univ. of Tokyo),
K.Aihara (the Univ. of Tokyo)

Contents

1) **β- encoder** for A/D, D/A conversion:
β-transformations and (β, α)-transformations realize flaky version of quantiser of AD-converter

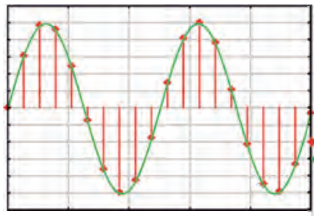
2) Renyi-Parry map with threshold has its eventually locally onto-map

i) **β-encoder** using Daubechies et.al.'s quantizer
ii) Scaled **β-encoder,** iii) Scaled negative **β-encoder**

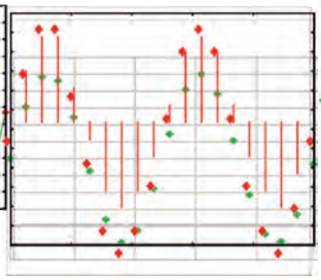
3) **β**-expansion and negative **β-expansion attractors** observed in A/D converters

Analog-to-Digital (A/D), Digital-to-Analog (D/A) conversion

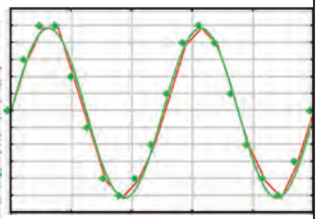
- **A/D D/A conversion are foundation for a variety of applications,**
  e.g., audio, image and communication etc.
- **Accuracy and stability**



sampling



quantization



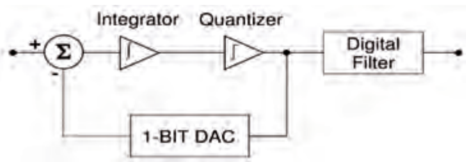
D/A conversion

quantize error
robust to fluctuation

3

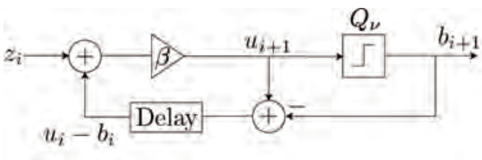
Conventional methods of A/D, D/A conversion

- **PCM has high precision, but doesn't possess stability.**
- **ΣΔ modulation**
  has stability,
  but its precision is lower than PCM .

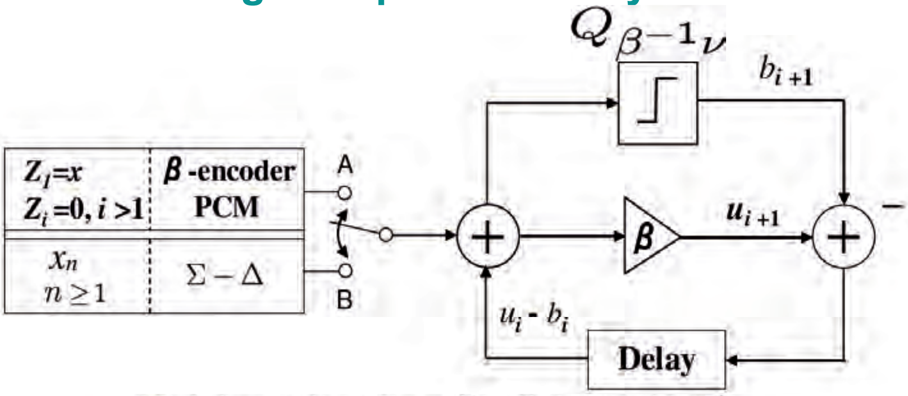


β-encoder

high precision, stability



Several kinds of AD/DA-converters **in a single-loop feedback system**



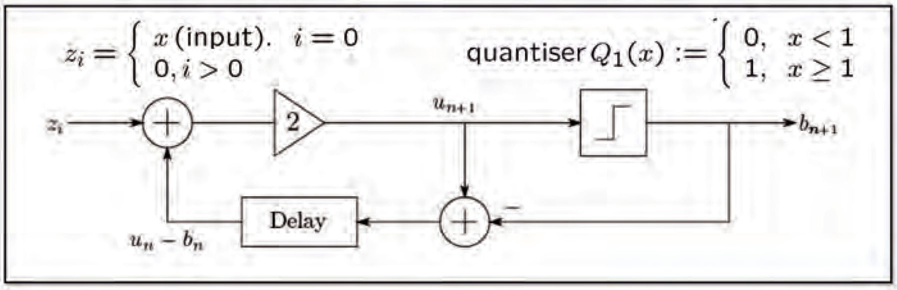
"flaky" quantizer: $Q_{\beta^{-1}\nu}(\cdot)$ with threshold $\beta^{-1}\nu$

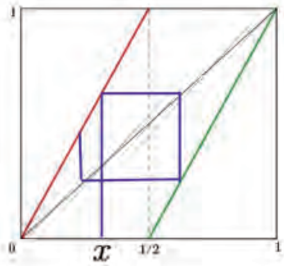
PCM when $\beta = 2, \nu = 1; L$ bits

β-**encoder** when $1 < \beta < 2, 1 < \nu < (\beta - 1)^{-1}; L$ bits

$\Sigma - \Delta$ **modulation** when $\beta = 1, \nu = 0$; 1 bit

Pulse Code Modulation (PCM)





Bernoulli shift (binary expansion):

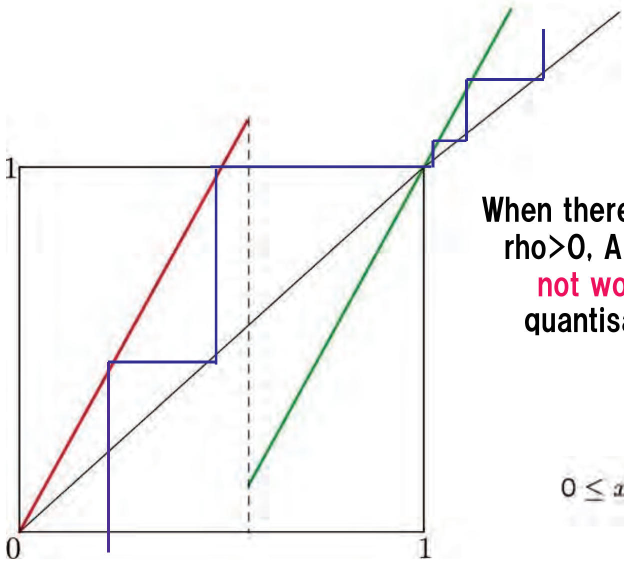
$$B(x) = \begin{cases} 2x, & x < 1/2 \\ 2x - 1, & x \geq 1/2 \end{cases}$$

$$b_i = \begin{cases} 0, & B^{i-1}(x) < 1/2 \\ 1, & B^{i-1}(x) \geq 1/2 \end{cases}$$

$$x = \sum_{i=1}^{\infty} b_i 2^{-i}$$

6

Divergence of a value x in PCM



When there is a threshold shift rho>0, A/D converter does not work well because quantisation errors don't decay

$$0 \le x - \sum_{i=1}^{\infty} b_i 2^{-i} < \frac{\rho}{2}$$

7

Background

Hardware implementation

- **Inose and Yasuda '64:** **ΣΔ modulation**
- **Gray '87 :** **Oversampled ΣΔ modulation**
- **Karanicolas '93:** **A 15-b 1-Msamples/s Digitally self-Calibrated Pipelined ADC=pipelined PCM with self-correction**

Ergodic theory

Renyi '57:f-expansion
- **Parry '60,'64: β-expansion, (β, α) expansion**

Takahashi '73, Ito & Takahashi '74, and Takahashi '83
Markov automorphism, Markov shift, orbit basis

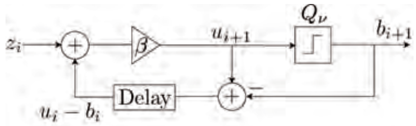
Erdoes and Joo '90:
greedy and lazy expansions

Dajani and Kraikaamp '02, and Sidorov'02:
intermediate expansion = (β, α) expansion

Hardware implementation
• Unsolved problems from mathematical standpoint

Ergodic theory

Daubechies et.al.,'06 Ⅰ: analyse stability of β-encoder:

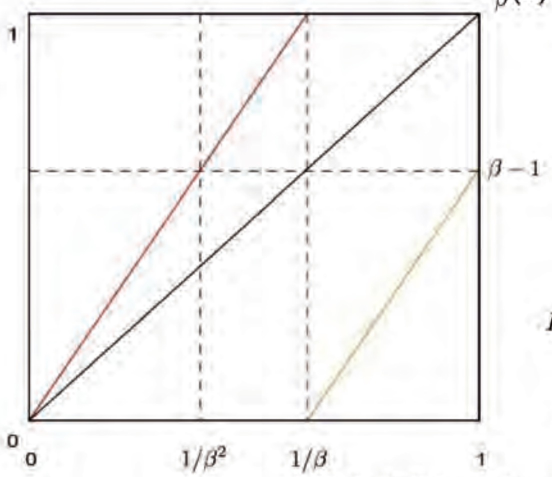


robustness to fluctuations to β of AMP and threshold v of quantiser Qv(･),
exponential accuracy in bit rate L

- **Daubechies et.al., '06 Ⅱ:** DA conversion using estimated β without knowing exact β **Milestone !**

β-transform (Renyi '57,Parry'60)

$$T_\beta(x) = \beta x \bmod 1, \beta > 1, \beta \notin \mathbf{Z}$$



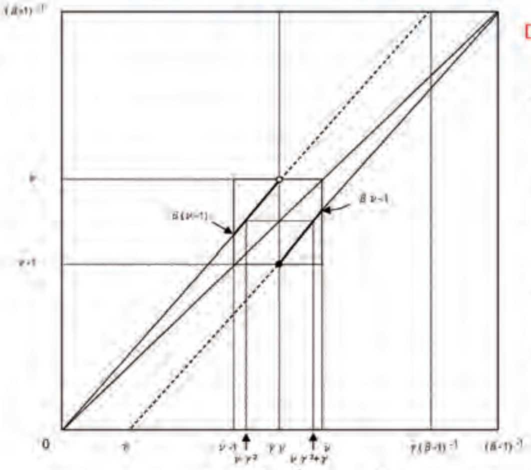
$$\beta - 1 = 1/\beta \Rightarrow$$

$$\beta = G = \frac{1+\sqrt{5}}{2}$$

$$P(\beta) = \begin{bmatrix} 1/\beta & 1-1/\beta \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow \lambda(P) = \frac{1-\sqrt{5}}{2} < 0$$

Multi-valued Renyi-**Parry's** map realizing
flaky version of an imperfect quantizer



$\gamma = \beta^{-1}$ (a)

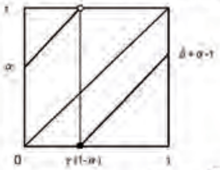
Daubechies *et.al.*'s quantiser(2k6):

$$Q^f{}_{[\nu_0,\nu_1]}(z) = \begin{cases} 0, & \text{if } z \le \nu_0 \\ 1, & \text{if } z \ge \nu_1 \\ 0 \text{ or } 1 & \text{if } z \in (\nu_0, \nu_1) \end{cases}$$

Parry's (β, α)-map

$$T_{\beta,\alpha}(x) = \beta x + \alpha \bmod 1,$$

$$\beta > 1, 0 \le \alpha < 1$$

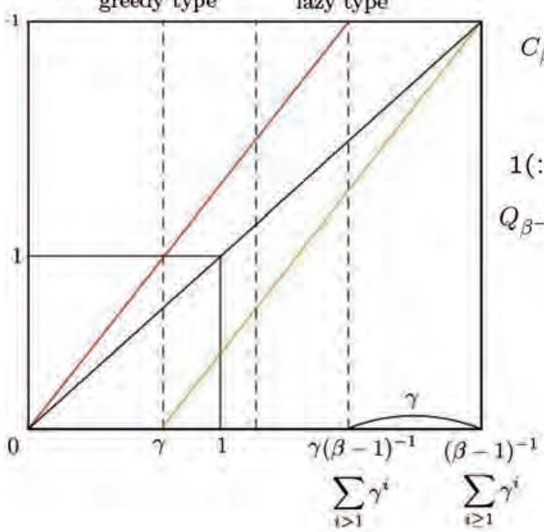


(b) $\alpha = (\beta-1)(\nu-1)$

multi-valued Renyi－Parry Map and
its eventually locally onto-map

11

Cautious map $C_{\beta,\nu}(x)$



$$C_{\beta,\nu}(x) = \beta x - Q_{\beta^{-1}\nu}(x) = \begin{cases} \beta x, & x < \beta^{-1}\nu \\ \beta x - 1, & x \ge \beta^{-1}\nu \end{cases}$$

$1(:\text{gr}) \le \nu \le (\beta-1)^{-1}(:\text{la})$

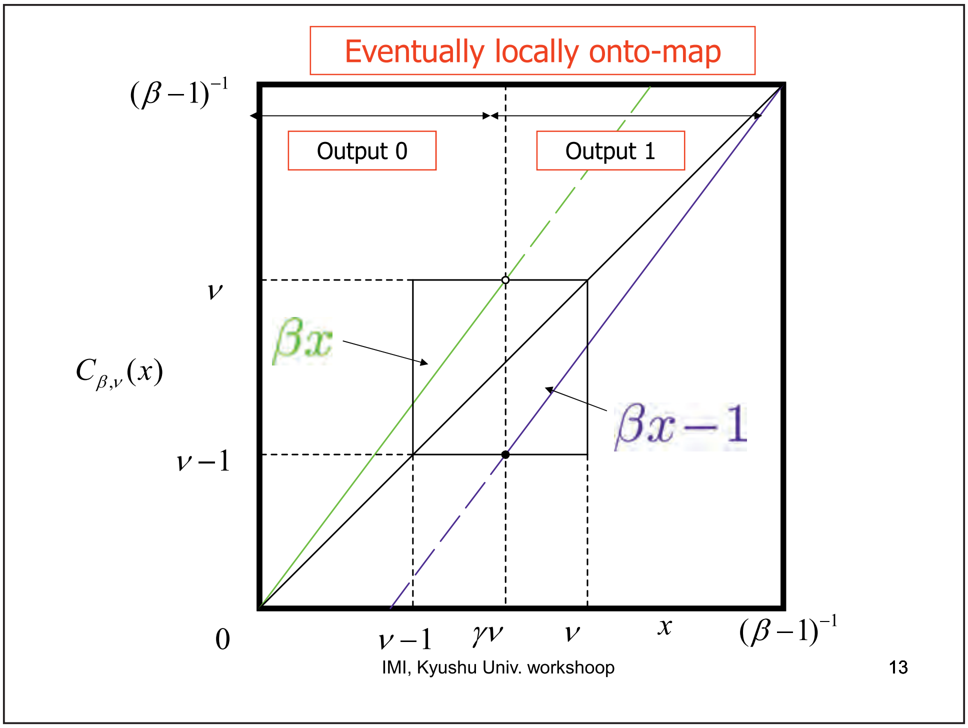
$$Q_{\beta^{-1}\nu}(x) = \begin{cases} 0, & x < \beta^{-1}\nu \\ 1, & x \ge \beta^{-1}\nu \end{cases}$$

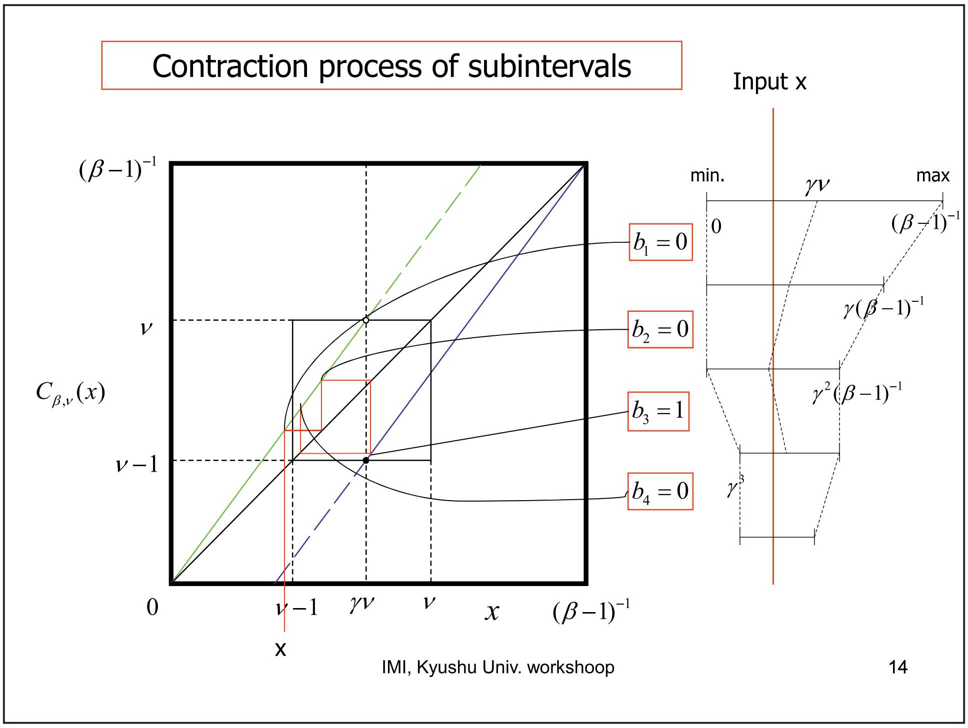
$$b_i = \begin{cases} 0, & C^{i-1}_{\beta,\nu}(x) < \beta^{-1}\nu \\ 1, & C^{i-1}_{\beta,\nu}(x) \ge \beta^{-1}\nu \end{cases}$$

$$x = \sum_{i=1}^{L} b_i\gamma^i + \gamma^L C^L_{\beta,\nu}(x)$$

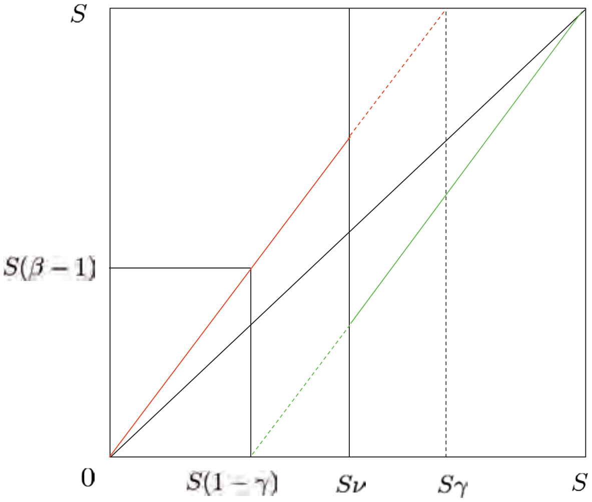
$\gamma = \beta^{-1}$

12


Eventually locally onto-map
Output 0
Output 1
$(\beta-1)^{-1}$
ν
$C_{\beta,\nu}(x)$
$\nu-1$
βx
$\beta x-1$
0
$\nu-1$
$\gamma\nu$
ν
x
$(\beta-1)^{-1}$
IMI, Kyushu Univ. workshoop
13


Contraction process of subintervals
Input x
$(\beta-1)^{-1}$
ν
$C_{\beta,\nu}(x)$
$\nu-1$
$b_1=0$
$b_2=0$
$b_3=1$
$b_4=0$
min.
$\gamma\nu$
max
0
$(\beta-1)^{-1}$
$\gamma(\beta-1)^{-1}$
$\gamma^2(\beta-1)^{-1}$
γ^3
0
$\nu-1$
$\gamma\nu$
ν
x
$(\beta-1)^{-1}$
x
IMI, Kyushu Univ. workshoop
14

scaling map of β-expansion



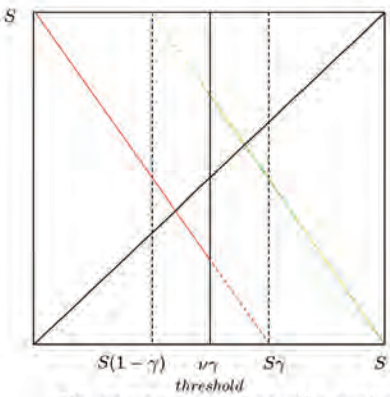
bit budget L,
threshold's tolerance:
$\sigma_{\beta,s} = s(2-\beta)$:

$$\beta_{\text{opt.}} = \frac{2L}{L+1}$$
$$s = \sigma_{\beta,s}(L+1)/2$$

Eventually locally onto-map is topologically conjugate to $T_{\beta,\alpha}(x)$

IMI, Kyushu Univ. workshoop 15

Negative beta-expansion （2k9）



$$R_{\beta,\nu,s}(x) := \begin{cases} s-\beta x, & x \in [0,\nu\gamma) \\ \beta s-\beta x, & x \in [\nu\gamma, s) \end{cases}$$

$$s > 0,\ \nu \in [s(\beta-1), s]$$

$$x = (-\gamma)^L R^L(x) - s\sum_{i=1}^{L} (b_i\beta + \overline{b_i})(-\gamma)^i$$

$$\hat{x} = s\{\frac{(-\gamma)^L}{2} - \sum_{i=1}^{L} (b_i\beta + \overline{b_i})(-\gamma)^i\}$$

Decoded value

Its eventually locally onto map is topologically conjugate to
Parry $(-\beta,\alpha)$-map: $T_{-\beta,\alpha}(x) = -\beta x + \alpha \bmod 1, \beta > 1, 0 \le \alpha < 1$

IMI, Kyushu Univ. workshoop 16

Main result I :β-decoding using interval analysis(2k7)

Theorem 1: The decoded value $\tilde{x}$ given by the interval analysis is defined by

$$\tilde{x} = \sum_{i=1}^{L} b_i \gamma^i + \boxed{\frac{\gamma^{L+1}}{2(1-\gamma)}},$$

Dust (but essential)

which gives

$$\boxed{0 \le |x - \tilde{x}| \le \frac{(\beta-1)^{-1}\gamma^L}{2} < \gamma^L \le \underline{\nu\gamma^L} \quad \text{when } \beta > 3/2.}$$

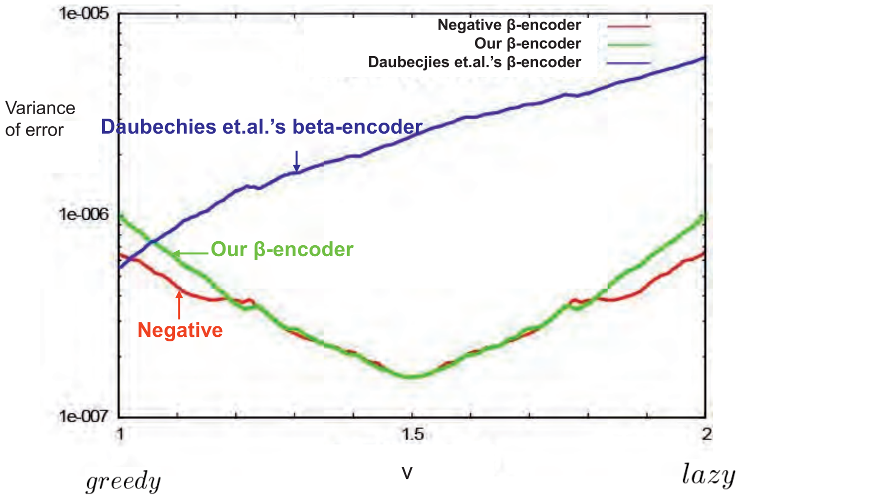
3dB improved when β> 3/2

cf.)Daubechies et.al.,: $0 \le |x - \tilde{x}_{Dau}| \le \nu\gamma^L$. $\tilde{x}_{Dau} = \sum_{i=1}^{L} b_i\gamma^i$.

IMI, Kyushu Univ. workshoop 17

Quantization Error

$(\beta = 1.5, s = (\beta-1)^{-1}, L = 16)$



IMI, Kyushu Univ. workshoop 18

Main result 2: Characteristic equation for β reconstruction（2k7）

sequence $\{b_j\}_{j=1}^{L}$ for x and sequence $\{c_j\}_{j=1}^{L}$ for $y = 1 - x$

$$\tilde{x} = \sum_{j=1}^{L} b_j \gamma^j + \underbrace{\frac{\gamma^{L+1}}{2(1-\gamma)}}_{\text{Dust term}}, \quad \tilde{y} = \sum_{j=1}^{L} c_j \gamma^j + \underbrace{\frac{\gamma^{L+1}}{2(1-\gamma)}}_{\text{Dust term}},$$

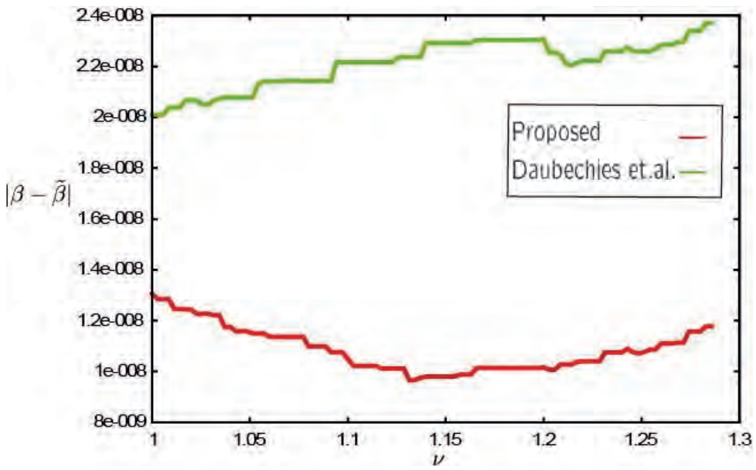
Daubechies et.al.'s idea

Since $\tilde{x} + \tilde{y} = 1$, the estimated value of γ is a root of $P(\gamma)$, referred to as characteristic equation of γ, defined by

$$\boxed{P(\gamma) = 1 - \sum_{i=1}^{L} (b_i + c_i)\gamma^i - \underbrace{\frac{\gamma^{L+1}}{1-\gamma}}_{\text{Dust term}} = 0.}$$

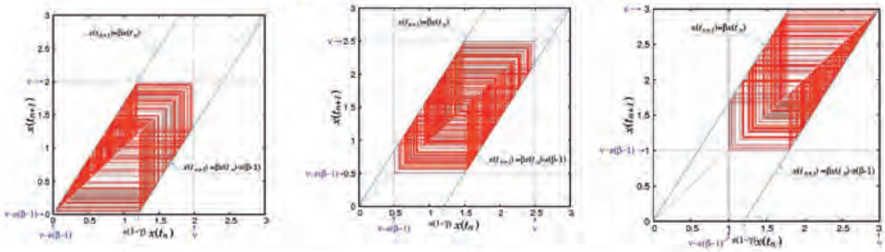
cf.) $P_{Dau}(\gamma) = 1 - \sum_{i=1}^{L} (b_i + c_i)\gamma^i = 0.$

The precision of beta estimation



For $N = 32$ and $\beta = 1.77777$, the worst precision of the estimation for β when varing x and ν.

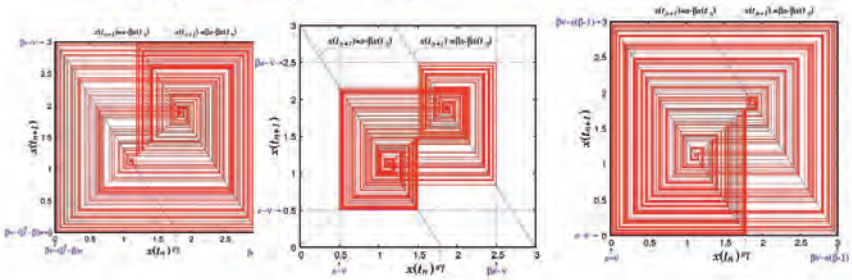
β −expansion attractors



(a) (b) (C)

Examples of the chaotic attractors obtained from the scale-adjusted β-encoder circuit.
(a) The greedy expansion attractor,
(b) the cautious expansion attractor, and
(c) the lazy expansion attractor.

Negative β −expansion attractors



(a) (b) (C)

Examples of the chaotic attractors obtained from the negative β-encoder circuit.
(a) The greedy expansion attractor,
(b) the cautious expansion attractor, and
(c) the lazy expansion attractor.

Implementation of β-encoder in an LSI (Large-Scale Integrated) circuit

- We note that parts of this article draw on our previous work:

1) T. Kohda ,Y. Horio, Y. Takahashi, and K. Aihara,
"Beta Encoders: Symbolic Dynamics and Electronic Implementation ",
Int. Journal of Bifurcation and Chaos, **22, no9**, 2012,

2) T. Kohda ,Y.Horio, and K. Aihara,"**β-** Expansion Attractors observed in A/D Converters", AIP Chaos,**22**,no.4,2012,

supported by the Aihara Project, JSPS through FIRST Program.

The FIRST Program also supported the **β-** encoder group to implement **β-**encoders in an LSI circuit and evaluate quantization errors and their performance in practically realized LSI circuits using a simple **β-**recovery method suited to operation of AD/DA converters in LSI circuits,

T.Makino, Y.Iwata, K.Shinohara, Y.Jitsumatsu, M.Hotta, H.San, and K.Aihara,
"Rigorous Estimates of Quantization error and Adaptive Decoding Scheme for an A/D Converter Based on a Beta-Map," NOLTA (Nonlinear Theory and Its Applications), IEICE, 6-1,99-111,2015.

A Random Binary Sequence Generator Based on β Encoders

Yutaka Jitsumatsu and Kazuya Matsumura
Depart. of Informatics, Kyushu University,
Motooka 744, Nishi-ku, Fukuoka, 819-0395, Japan.
Email: {jitumatu, matsumura}@me.inf.kyushu-u.ac.jp

***Abstract*—A β encoder is an Analog-to-Digital (A/D) converter whose dynamics is governed by (β, α) map. The most important feature of the β encoder is that it is robust to fluctuation of the threshold value for quantization as well as the fluctuation of the value of β. Because of this property, the β encoder can be implemented with low-precision elements and thus realized by an extremely small circuit. Hirata et.al proposed a random number generator using a β encoder followed by a kind of shift register circuit. They showed that random numbers generated by such a circuit can pass the National Institute of Standards and Technology (NIST) Statistical Test Suite. We recently proposed another method for converting the output sequences from the β encoder to binary sequences that can be regarded as independent and identically distributed (i.i.d.) random variables with equal probability. In the proposed method, we try to find a binary expansion of an input value x that is recovered from a β encoder's output sequence. We verified that binary sequences generated from a β encoder followed by the proposed method can pass the NIST Statistical Test Suite. It is shown that the proposed method is robust to the fluctuation of the value of β.**

I. Introduction

The importance of random number generation becomes significant because of the development of information and communication technologies and demand for secure communications. Pseudo-random numbers are generated by deterministic algorithms with seeds and thus completely the same numbers are produced if the same seed is used. On the other hand, there are demands, especially in a security purpose, for physical random number generator that measures some physical phenomenon. For example, a secure key distribution using bit sequences generated by the use of a semiconductor laser [4] and a random number generator based on a chaotic map [5] have been proposed. Many randomness tests have been proposed.

A random number generation method that uses a β encoder as a source of randomness has been proposed [14], [15]. The β encoder is an Analog-to-Digital (A/D) converter that is robust to fluctuation of threshold value of a quantizer [1]. Such a β encoder does not need high-precision circuit elements and is implemented by a complementary metal-oxide-semiconductor (CMOS) circuit that achieves very small area consumption as well as low power consumption [9]. It can be used at from -20 degree to 80 degree Celsius. We can observe chaos attractors in β converters [3]. However, outputs from a β encoder have strong correlations between successive bits. In [14] a random number generation by calculating the exclusive disjunction (exclusive or: EXOR) of several delayed bits of β encoder's outputs was proposed. When we use β encoders for generating random numbers, we generate one million bits, while only the first L bits of β expansion coefficients $b_i \in \{0, 1\}$ are used for expressing approximated input value as $\hat{x} = \sum_{i=1}^{L} b_i \beta^{-i}$, where L is typically less than 20.

A remarkable feature of random number generation using β encoder is that randomness is guaranteed by chaotic behavior of the attractors observed in the β encoder [3]. Hence, basically, thermal noise is not needed for β encoders to generate random numbers and it can work in an extremely low temperature environment. This is a significant difference between the proposed method and those random number generation methods whose randomness is guaranteed by thermal noise. On the other hand, β encoder is robust to fluctuation of threshold. This property makes the β encoder work also at high temperature environment.

After the computer simulation in [14], we performed experiments of random number generation using hardware β encoders implemented by CMOS technologies [15]. We found that more than 10 EXOR operations are needed to make the generated sequences pass the National Institute of Standards and Technology (NIST) statical test suite [11].

In this paper, we propose an algorithm in which the output of β encoder is converted to binary sequences that can be regarded as independent and identically distributed (i.i.d.) sequences with equal probability. The algorithm is based on an idea that using the first n outputs $b_1, b_2, \ldots, b_n$ of a β encoder, we calculate the interval $[l_n, u_n]$ in which the input value x must exist and then obtain the binary expansion $\hat{x} = \sum_{j=1}^{m} \tilde{b}_j 2^{-j}$ where m and $\tilde{b}_j \in \{0, 1\}$ are selected to satisfy $\hat{x} \le l_n$ and $u_n \le \hat{x} + 2^{-m}$. Though l_n and u_n are real-valued, we approximate them by fixed-point numbers so that the proposed method can be realized by CMOS circuit. We will show that a fixed-point arithmetic with 15 bit precision is sufficient to make the generated binary sequences pass the NIST test suite.

Output sequences from the β encoder are passed through the proposed conversion algorithm. Then, the NIST test suite is applied to these sequences. The value of β used in the β encoder is not precisely known beforehand. We performed numerical experiments of β-ary to binary conversion with estimated $\beta = 1.6, 1.7, 1.8$ and 1.9. It is confirmed that the proposed algorithm is robust to the estimation error for β.

II. Pulse Code Modulation and β encoder

In this section, we briefly review Pulse Code Modulation (PCM) and β encoder.

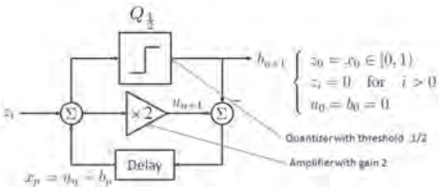


Fig. 1. PCM-based A/D converter

A. Pulse Code Modulation (PCM)

The binary expansion of a real value $z_{\mathrm{i}} \in [0, 1)$ is given by

$$z_{\mathrm{i}} = \sum_{n=1}^{\infty} \tilde{b}_n 2^{-n}, \tag{1}$$

where $\tilde{b}_n \in \{0, 1\}$ is given by

$$b_n = Q_{\frac{1}{2}}(x_{n-1}), \quad n = 1, 2, \ldots. \tag{2}$$

where $Q_{\frac{1}{2}}(x)$ is a quantizer with a threshold $\frac{1}{2}$, defined by

$$Q_{\frac{1}{2}}(x) = \begin{cases} 0, & (0 \le x < \frac{1}{2}) \\ 1, & (\frac{1}{2} \le x < 1) \end{cases}, \tag{3}$$

and $x_n = B(x_{n-1})$ $(n = 1, 2, \ldots)$, where $B(x)$ is Bernoulli shift map defined by

$$B(x) = \begin{cases} 2x, & (0 \le x < \frac{1}{2}) \\ 2x - 1, & (\frac{1}{2} \le x < 1). \end{cases} \tag{4}$$

Here, the initial value $x_0 = z_{\mathrm{i}}$ is an analog input voltage (See Fig. 1).

A drawback of PCM is that the quantizer $Q(x)$ may make a wrong decision if x_n is very close to the threshold value because a slight fluctuation occurs in the threshold voltage. For example, assume the threshold is changed from $\frac{1}{2}$ to some value ν so that $B(x)$ is replaced by

$$B'(x) = \begin{cases} 2x, & 0 \le x < \nu \\ 2x - 1, & \nu \le x < 1 \end{cases} \tag{5}$$

(See Fig. 2). Then, for $0 < \varepsilon < \nu - \frac{1}{2}$, x_n diverges as $B'(\frac{1}{2} + \varepsilon) = 1 + 2\varepsilon$, $B'(1 + 2\varepsilon) = 1 + 4\varepsilon$, $B'(1 + 4\varepsilon) = 1 + 8\varepsilon \cdots$. For avoiding such an un-stability of conversion, 1.5 bit encoders [6] and digital calibration techniques [7] have been proposed.

On the other hand, $\Sigma\Delta$ modulators have a good property that they are robust to fluctuations of threshold values in their quantizers. However, they have a drawback that oversampling rate is very high, such as one hundred or one thousand. This implies that $\Sigma\Delta$ modulation can only be used in narrow-bandwidth applications. Moreover, the quantization error of $\Sigma\Delta$ modulation decreases inverse proportionally to the number of bits in contrast to the exponential accuracy of the PCM. β encoders have the two good properties, i.e., robustness against fluctuations of threshold voltages and an exponential accuracy [1]. In the next subsection, a scale-adjusted β encoder [15] is explained.

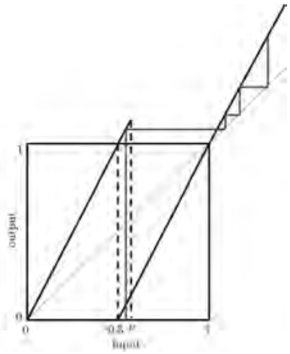


Fig. 2. A map of PCM-based A/D converter with fluctuation of threshold value : $B'(x)$

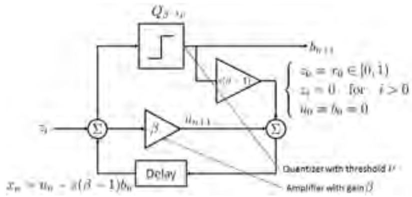


Fig. 3. Scale-adjusted β-encoder

B. β encoder

Define a scale-adjusted β map for $0 \le x \le s$ with a scale parameter s as (See Fig.3),

$$S_{\beta,\nu,s}(x) = \begin{cases} \beta x, & (0 < x < \nu\beta^{-1}) \\ \beta(x - s) + s, & (\nu\beta^{-1} < x < s) \end{cases}. \tag{6}$$

The ordinary β encoder corresponds to the case $s = \frac{1}{\beta-1}$. Note that the domain of the map is $[0, s]$ irrespective of β, while that of the ordinary β map depends on β. The output of the scale-adjusted β encoder for the input value $z_{\mathrm{i}} = x_0$ is

$$b_n = Q_{\nu\beta^{-1}}(x_{n-1}), \tag{7}$$

where

$$x_n = S_{\beta,\nu,s}(x_{n-1}), \quad n = 1, 2, \ldots, \tag{8}$$

and

$$Q_\nu(x) = \begin{cases} 0, & (x < \nu) \\ 1, & (x \ge \nu) \end{cases} \tag{9}$$

is a quantization function with threshold ν. Let

$$l_n = s(\beta - 1) \sum_{i=1}^{n} b_i \beta^i \tag{10}$$

$$u_n = l_n + s\beta^n. \tag{11}$$

Then, given the first n outputs, $b_1, b_2, \ldots, b_n$, we know that x_0 must exist in $[l_n, u_n]$.

We hereafter assume $s = 1$ for simplicity. For almost all initial value x_0, x_n generated by Eq. (8) does not fall into

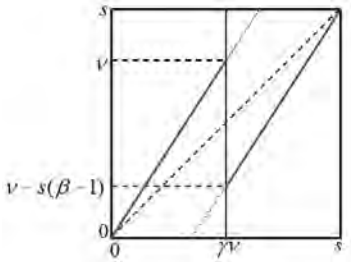


Fig. 4. A map of β-expansion

some periodic points but stays in some range. Attractors are observed in dynamics of β encoders. They are referred to as β expansion attractors [3]. We consider such attractors are used as sources of randomness for generating sequences of random numbers.

III. Random Number Generation using β encoders

Bernoulli shift map is an ideal model for fair coin tossing, i.e., a model for generating independent and identically distributed (i.i.d.) random variables. Thus, a binary expansion of an arbitrarily chosen input voltage x_0 is considered as a sequence of ideal binary i.i.d. random variables. However, the dynamics of PCM modulation is unstable because of the issue mentioned in Section II-A. In this paper, we show a method that approximately converts a sequence of β expansion coefficients for an input voltage x_0 to a sequence of binary expansion coefficients for the same initial value [16]. Such a conversion is referred to as β-ary to binary (β-ary/binary) conversion. Since β converters can be implemented in very small CMOS circuits, random number generators using a β encoder should also be implemented in CMOS circuits. Therefore, we consider a digital calculation with a finite precision to perform the proposed β-ary/binary conversion.

A. β_{D} sequences

It is easily verified that the consecutive outputs from a β encoder have a negative correlation, i.e., $\frac{1}{L}\sum_{i=1}^{L} b_i b_{i+1}$ tends to take a negative value. Fig. 5 shows an autocorrelation function of an output sequences from a β encoder with $\beta = 1.8$ and $\nu = \frac{\beta}{2(\beta-1)} = 1.125$, where a $\{0,1\}$-valued sequence is converted to a $\{-1,+1\}$-valued sequence, i.e., the graph shows $R(n) = \frac{1}{L}\sum_{i=1}^{L}(2b_i - 1)(2b_{i+n} - 1)$. Fig. 5 shows that the output sequence from the β encoder has a negative autocorrelation of delay $n = 1$, which implies that some additional process is needed to generate sequences whose distribution is close to that of i.i.d. random variables.

Hirata et.al have proposed [14] the β_D sequences that are generated by taking EXOR of several outputs. Specifically, the β_D sequence is defined as

$$b_D(k) = \bigoplus_{i=1}^{N_{xor}} b(k - d_i), \quad k \geq d_{N_{xor}} \tag{12}$$

where $d_1 < d_2 < \ldots, d_{N_{xor}}$, N_{xor} is the number of bits of which EXOR is taken.

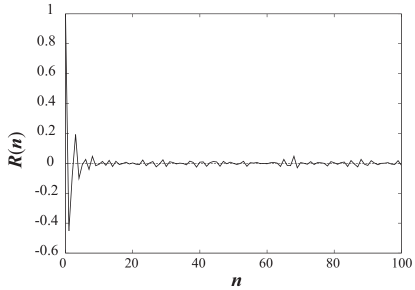


Fig. 5. Autocorrelation of β-encoder's output ($L = 10000,\ 0 \leq n < 100$)

B. β-ary/binary conversion

We recently proposed a method for converting a binary sequence generated from β encoder to another binary sequence that is approximately regarded as i.i.d. random variables [16].

Let $\{b_i\}_{i=0}^{n}$ be a β expansion, determind by Eq.(7), of some initial value $x_0 = z_{\mathrm{i}}$. In the proposed method, we calculate the interval $[l_n, u_n]$ in which the input value z_{i} exists. Then we obtain the binary expansion

$$\hat{z}_{\mathrm{i}} = \sum_{j=1}^{m} \tilde{b}_j 2^{-j} \tag{13}$$

where m and $\tilde{b}_j \in \{0,1\}$ are selected to satisfy $\hat{x} \leq l_n$ and $u_n \leq \hat{x} + 2^{-m}$.

If the perfect knowledge of β is available, then we find an interval $[l, u]$ that includes z_{i} in Method 1, explained later (See Fig.6). However, the proposed algorithm should be implemented in digital circuit. Thus, u and l should be expressed by integers.

In order to make the explanation easy, we suppose u and l are real-valued for a while. An integer implementation is explained later. The goal of the proposed method is to make a generated binary sequence that can pass the NIST statistical test suite. A binary sequence generated by the proposed method, denoted by c_j, is different from the true binary expansion $\tilde{b}_i$ because of two reasons. One is that u and l are expressed by integer numbers. The other is that there is a difference between the true β and the β used for the β-ary/binary conversion.

[Method 1]

1) Initialize: $i = j = 1, [l, u] = [0, 1], \gamma = \frac{1}{\beta}$.
2) Read b_i.
   If $b_i = 0$, then u is updated to $l + (u - l) \times \gamma$.
   If $b_i = 1$, then l is updated to $u - (u - l) \times \gamma$.
3) If $u < \frac{1}{2}$, then output $c_j = 0$ and update $j = j + 1$, $l = 2l$, and $u = 2u$.
   If $l \geq \frac{1}{2}$ then output $c_j = 1$ and update $j = j + 1$, $l = 2l - 1$, and $u = 2u - 1$.

TABLE I. VOLTAGE PARAMETERS

	V_{DDA}	V_{DDD}	$V_{\mathrm{DD\_IO}}$	$V_{\mathrm{ref}+}$	$V_{\mathrm{ref}-}$	$V_{\mathrm{ref\_}CM}$	$A_{\mathrm{in}+}$	$A_{\mathrm{in}-}$
A	1.20	1.20	1.20	0.85	0.35	0.60	0.80	0.40
B	1.20	1.20	1.20	0.85	0.35	0.60	0.60	0.60
C	1.40	1.20	1.20	0.95	0.45	0.70	0.70	0.70

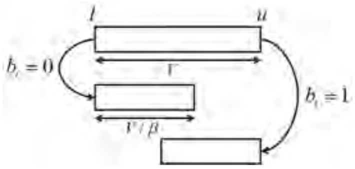


Fig. 6. How to update an interval $[l, u]$.

4) If b_i is EOF(End Of File), then quit. Otherwise, update $i = i + 1$ and go back to Step 2.

After we obtain the first n outputs from β encoder, the width of interval $[l_n, u_n]$ is exactly β^{-n}. This fact implies that the number of outputs from the converter is approximately $m = n \log_2 \beta < n$. Hence, the proposed converter does not always generate one output bit per one input bit.

In Method 1, the interval $[l, u]$ may become very small after some updates. This situation happens if the input value x subtracted by $\sum_{i'=1}^{i} b_{i'} 2^{-i'}$ exists in $[l, u]$. Hence this situation implies that the next output sequence is $0 \cdots 01$ or $10 \cdots 0$. When we express l and u by some integers, this situation makes approximation error very large. In Method 2 below, we void such a situation by doubling the size of $|u - l|$ without making decision $c_j \in \{0, 1\}$. This makes $|u-l|$ always greater than $\frac{1}{4}$. Method 2 is based on arithmetic codes [10], but the way to expand the interval is slightly different since subintervals for expressing 0 and 1 are overlapping each other. The algorithm of Method 2 is also similar to a 1.5 bit quantizer [6]. A new parameter k expresses the number of undecided output bits.

[Method 2]

1) Initialize: $i = j = 1$, $k = 0$, $[l, u] = [0, 1]$, and $\gamma = \frac{1}{\beta}$.
2) The same as Step 2. in Method 1.
3)
   - If $u < \frac{3}{4}$ and $l \geq \frac{1}{4}$, then update $l = 2l - \frac{1}{2}, u = 2u - \frac{1}{2}$, and $k = k + 1$.
   - If $u < \frac{1}{2}$, then output $c_j = c_{j+1} = \cdots = c_{j+k-1} = 1, c_{j+k} = 0$. (If $k = 0$, then output $b_j = 0$) and update $k = 0$, $l = 2l$, $u = 2u$, and $j = j + k + 1$.
   - If $l \geq \frac{1}{2}$, then output $b_j = b_{j+1} = \cdots = b_{j+k-1} = 0$, and $b_{j+k} = 1$ (If $k = 0$, then output $b_j = 1$) and update $k = 0, l = 2l - 1, u = 2u - 1$, and $j = j + k + 1$.
4) If b_i is EOF, then quit. Otherwise, update $i = i + 1$ and go back to Step 2.

Finally, we give Method 3 which is an integer calculation version of Method 2. A real number $r \in [\frac{i}{2^w}, \frac{i+1}{2^w})$ is approximated by $\frac{i}{2^w}$, where w is referred to as a window size. Real numbers l and u in $[0, 1]$ in Method 2 are replaced by integers l' and u' in $\{0, 1, \ldots, 2^w - 1\}$ in Method 3.

[Method 3]

1) Initialize: $i = j = 1$, $k = 0$, $[l', u'] = [0, 2^w - 1]$, and $\gamma = \lfloor \frac{2^w}{\beta} \rfloor$.
2) Read b_i.
   If $b_i = 0$, then update
$$u' = l' + \lfloor \frac{(u' - l') \cdot \gamma}{2^w} + \frac{1}{2} \rfloor \tag{14}$$
   If $b_i = 1$, then update
$$l' = u' - \lfloor \frac{(u' - l') \cdot \gamma}{2^w} + \frac{1}{2} \rfloor \tag{15}$$
3)
   - If $u' < \frac{3}{4} \times 2^w$ and $l' \geq \frac{1}{4} \times 2^w$, then update $l' = 2l' - 2^{w-1}, u' = 2u' - 2^{w-1}$, and $k = k + 1$.
   - If $u < \frac{2^w}{2}$, then output $b_j = b_{j+1} = \cdots = b_{j+k-1} = 1$, and $b_{j+k} = 0$ and update $k = 0, l' = 2l', u' = 2u'$, and $j = j + k + 1$.
   - If $l \geq \frac{2^w}{2}$, then output $b_j = b_{j+1} = \cdots = b_{j+k-1} = 0$, and $b_{j+k} = 1$ and update $k = 0$, $l' = 2l' - 2^w$, $u' = 2u' - 2^w$, and $j = j + k + 1$.
4) If b_i is EOF, then quit. Otherwise, update $i = i + 1$ and go back to Step 2.

Here, $\lfloor x \rfloor$ is the largest integer not greater than x. Since the calculation of Eqs.(14) and (15) are approximation of those of Method 2, the interval $[u, l]$ is not exact. Note that this algorithm has an internal state that is specified by (u', l', k), where $l' \in \{0, 1, \ldots, 2^{w-1} - 1\}$, $u' \in \{2^{w-1}, 2^{w-1} + 1, \ldots, 2^w - 1\}$, and $k \in \{0, 1, \ldots\}$.

IV. RESULTS OF EXPERIMENT

Experiments were carried out to show the validity of the proposed method. San et.al have manufactured CMOS circuits in which β encoders are embedded [8], [9]. We use the same β encoder implemented in CMOS circuit as the one used in [15]. The parameter β of the β encoder is designed to 1.83 but its effective value is slightly fluctuated and not known precisely beforehand. We generated 125 sequences; the length of each sequence is 1.05×10^6. Method 3 was applied to the output sequences from such a β encoder. The window size is $w = 20$ and $\beta = 1.8$ unless otherwise specified.

The NIST test suite [11] was applied to β_D sequences [14] and sequences obtained by β-ary/binary conversions. In a randomness test for the β_D sequence, the delay parameters in Eq.(12) were $d_1 = 6$, $d_2 = 6 + 7$, ..., $d_i = d_{i-1} + i + 5$ $(i \geq 3)$ and $N_{xor} = 4, 8$, and 16.

There are fifteen tests for the NIST test suite. A result of each test is expressed by P (Pass) or F (Fail) except for non-overlapping template matching test for which the number of templates that passes the test is shown [12].

β encoders have voltage parameters. Three patterns of voltage parameters shown in Table I are used, where V_{DDA}, V_{DDD}, and $V_{\mathrm{DDD\_IO}}$ are drive voltages for analog, digital and Input/Output (I/O) purposes, $A_{\mathrm{in}+}$ and $A_{\mathrm{in}-}$ are differential input voltage which expresses the initial value of an input voltage for A/D conversion, and $V_{\mathrm{ref}+}$, $V_{\mathrm{ref}-}$, and $V_{\mathrm{ref\_CM}}$ are reference voltages which are upper limit, lower limit, and threshold voltage. The last three values are designed to satisfy $V_{\mathrm{ref\_CM}} = \frac{1}{2}(V_{\mathrm{ref}-} + V_{\mathrm{ref}+})$.

The difference between Pattern A and Pattern B is that $A_{\mathrm{in}+} = 0.80$ and $A_{\mathrm{in}-} = 0.40$ for the former and $A_{\mathrm{in}+} = 0.60$ and $A_{\mathrm{in}-} = 0.60$ for the latter. This comparison shows the effect of input voltage on the randomness of generated sequences. The difference between Pattern B and Pattern C is that $V_{\mathrm{DDA}} = 1.20$ for the former and $V_{\mathrm{DDA}} = 1.40$ for the latter. In general, a CMOS circuit has a range of V_{DDA} that the circuit can properly work. V_{DDA} for the CMOS-implemented β encoders is designed to be 1.20, but the circuit is more stable if $V_{\mathrm{DDA}} = 1.40$.

Tables II and III show results of the NIST test suite for β_D sequences and the sequences obtained by β-ary/binary conversion, respectively. These results show that EXOR operation is needed for the former sequences to pass the NIST test suite, but is not needed for the latter sequences. Table IV shows the effect of window size w on the results of NIST test. It is shown that $w \geq 15$ is required to guarantee the generated sequences pass the NIST test and that if $w = 10$ the generated sequences do not pass most of the tests.

Since the effective value of β is not known beforehand, we verified the robustness of the proposed method to the mismatch of the values of β used in the encoder and the β-ary/binary converter. The value of β is designed to be $\beta = 1.83$. Denote the β used in the β-ary/binary converter by β'. Table.V shows that the results for $\beta' = 1.7, 1.8$, and 1.9 are almost the same. However, the result for $\beta' = 1.6$ is very poor. We conclude that the proposed method can allow a fluctuation of β at most 0.1.

V. Conclusion

In this paper, a β-ary/binary conversion method for generating random numbers using a β encoder, proposed in [16], has been explained. Experimental results have shown that sequences obtained by the β-ary/binary conversion can pass the NIST statistical test suite. The proposed method is implemented by integer calculations. It has been shown that the necessary window size is 15 and the converter is robust to the mismatch of β.

Acknowledgment

This research was supported by the Aihara Project, the FIRST program from JSPS, initiated by CSTP and JSPS KAKENHI Grant Number 25820162.

References

[1] I. Daubechies, R.A. DeVore, C.S. Güntürk, and V. A. Vaishampayan, "A/D Conversion With Imperfect Quantizers,"*IEEE Trans. Inform. Theory*, vol.52, no.3, March 2006.

[2] I. Daubechies and O. Yilmaz, "Robust and Practical Analog-to-Digital Conversion With Exponential Precision," *IEEE Trans. Inform. Theory*, vol.52, no.8, pp. 3533-3545, 2006.

[3] T. Kohda, Y. Horio, K. Aihara, "β-Expansion Attractors Observed in A/D Converters," *Chaos: An Interdisciplinary J. of Nonlinear Science*, Vol. 22, 047512, 2012.

[4] K. Yoshimura, J. Muramatsu, P. Davis, A. Uchida and T. Harayama, "Secure Key Distribution Using Correlated Randomness in Optical Devices,"*IEICE Tech. Rep.*, NLP2011-106, vol. 111, no. 276, pp. 81-84, 2011.

[5] S. Callegari, R. Rovatti, and G. Setti, "Embeddable ADC-Based True Random Number Generator for Cryptographic Applications Exploiting Nonlinear Signal Processing and Chaos," *IEEE Trans. Signal Proc.*, vol. 53, no. 2, Feb. 2005.

[6] S. Domichi, "Impacts and Countermeasures in Device Mismatches in Analog Circuits," *the Journal of IEICE*, vol. 92, no. 6, pp. 446-451, 2009.

[7] T. Matsuura, "Technical Trend of Digitally Assisted A/D Converters," *IEICE Technical Report*, CAS2011-104, vol. 111, no. 377, pp. 103-108, Jan. 2012.

[8] H. San et. al, "Non-binary Pipeline Analog-to-Digital Converters Based on β-Expansion," *IEICE Trans. Fundamentals*, vol.E96-A, no.2, pp. 415-421, Feb. 2013.

[9] R. Sugawara, H. San, K. Aihara and M. Hotta, "Experimental Implementation of Non-binary Cyclic ADCs with Radix-value Estimation Algorithm," *IEICE Trans on Electronics*, vol.E97-C, no.4, pp.308–315, April 2014.

[10] T. S. Han and K. Kobayashi, *Mathematics of Information and Coding*, Amer Mathematical Society, 2007.

[11] A. Rukhin, et.al, "A Statistical Test Suite for Random and Pseudo-random Number Generators for Cryptographic Applications," NIST, Special Publication 800-22, May 2001.

[12] H. Yoshida, T. Murakawa, and S. Kawamura, "Study on Testing for Randomness of Pseudo-Random Number Sequence with NIST SP800-22 rev.1a," *IEICE Technical Report*, vol. 112, no. 301, NLP2012-78, pp. 13-18, Nov. 2012.

[13] T. Makino, Y. Iwata, K. Shinohara, Y. Jitsumatsu, M. Hotta, H. San and K. Aihara, *Rigorous Estimates of Quantization Error for A/D Converters Based on Beta-Map*, *NOLTA Journal*, vol.6, no.1, pp.99-111, Jan. 2015.

[14] Y. Hirata, Y. Jitsumatsu, T. Kohda, and K. Aihara, "Pseudo-Random Number Generator Using β-Expansion Attractors in A/D Converters," *Proc. of the 30th Sympo. on Cryptography and Information Security (SCIS2013)*, Jan. 2013.

[15] K. Matsumura, Y. Jitsumatsu, and T. Kohda, "Implementation of Pseudo-Random Number Generator Based on β-Expansion Attractor," *IEEJ Technical Report*, ECT-14-026, pp. 135-140, Jan. 2014.

[16] K. Matsumura, T. Teraji, K. Oda, and Y. Jitsumatsu, "Random Number Generation Using β Encoder," *Proc. of the 32nd Sympo. on Cryptography and Information Security (SCIS2015)*, Jan. 2015.

TABLE II. RESULTS OF THE NIST STATISTIAL TEST SUITE FOR β_D SEQUENCES

	β-ary/binary conversion is NOT applied											
Voltage Pattern	A				B				C			
The number of EXORs	1	4	8	16	1	4	8	16	1	4	8	16
Frequency (Monobits) Test	F	P	P	P	F	P	P	P	F	P	P	P
Frequency Test within a Block	F	F	F	P	F	F	F	P	F	F	P	P
Cummulative Sums (Cusum) Test	F	P	P	P	F	P	P	P	F	P	P	P
Runs Test	F	F	F	P	F	F	F	P	F	F	F	P
Test for the Longest Run of Ones in a Block	F	F	P	P	F	F	P	P	F	F	P	P
Binary Matrix Rank Test	P	P	P	P	P	P	P	P	P	P	P	P
Discrete Frourier Transform Test	F	F	F	P	F	F	P	P	F	F	P	P
Non-Overlapping Template Matching Test	0	12	128	157	0	12	135	P	0	19	156	155
Overlapping Template Matching Test	F	F	F	P	F	F	F	P	F	F	P	P
Maurer's "Universal Statistical" Test	F	F	P	P	F	F	P	P	F	F	P	P
Approximate Entropy Test	F	F	F	P	F	F	F	P	F	F	P	P
Random Excursion Test	F	F	P	P	F	F	P	P	F	F	P	P
Random Excursions Variant Test	F	P	P	F	F	P	P	P	F	P	P	P
Serial Test	F	F	P	P	F	F	P	P	F	F	P	P
Linear Complexity Test	P	P	P	P	P	P	P	P	P	P	P	P

TABLE III. RESULTS OF THE NIST STATISTIAL TEST SUITE FOR β-ARY/BINARY CONVERSION

	β-ary/binary conversion is applied											
Voltage Pattern	A				B				C			
The number of EXORs	1	4	8	16	1	4	8	16	1	4	8	16
Frequency (Monobits) Test	P	P	P	P	P	P	P	P	P	P	P	P
Frequency Test within a Block	P	P	P	P	P	P	P	P	P	P	P	P
Cummulative Sums (Cusum) Test	P	P	P	P	P	P	P	P	P	P	P	P
Runs Test	P	P	P	P	P	P	P	P	P	P	P	P
Test for the Longest Run of Ones in a Block	P	P	P	P	P	P	P	P	P	P	P	P
Binary Matrix Rank Test	P	P	P	P	P	P	P	P	P	P	P	P
Discrete Frourier Transform Test	P	P	P	P	P	P	P	P	P	P	P	F
Non-Overlapping Template Matching Test	157	157	P	P	157	P	156	156	P	P	157	P
Overlapping Template Matching Test	P	P	P	P	P	P	P	P	P	P	P	P
Maurer's "Universal Statistical" Test	P	P	P	P	P	P	P	P	P	P	P	P
Approximate Entropy Test	P	P	P	P	P	P	P	P	P	P	P	P
Random Excursion Test	P	P	P	P	P	P	P	P	P	P	P	P
Random Excursions Variant Test	P	P	P	P	P	P	P	P	P	P	P	P
Serial Test	P	P	P	P	P	P	P	P	P	P	P	P
Linear Complexity Test	P	P	P	P	P	P	P	P	P	P	P	P

TABLE IV. RESULTS OF THE NIST STATISTIAL TEST SUITE: COMPARISON OF WINDOW SIZE

	β-ary/binary conversion is applied								
Voltage Pattern	A			B			C		
window size	10	15	20	10	15	20	10	15	20
Frequency (Monobits) Test	F	P	P	P	P	P	F	P	P
Frequency Test within a Block	F	P	P	F	P	P	P	P	P
Cummulative Sums (Cusum) Test	F	P	P	P	P	P	F	P	P
Runs Test	F	P	P	F	P	P	F	P	P
Test for the Longest Run of Ones in a Block	P	P	P	P	P	P	P	P	P
Binary Matrix Rank Test	P	P	P	P	P	P	P	P	P
Discrete Frourier Transform Test	P	F	P	P	P	P	P	P	P
Non-Overlapping Template Matching Test	106	P	157	115	P	157	147	P	P
Overlapping Template Matching Test	F	P	P	F	P	P	P	P	P
Maurer's "Universal Statistical" Test	P	P	P	P	P	P	P	P	P
Approximate Entropy Test	F	P	P	F	P	P	F	P	P
Random Excursion Test	P	F	P	P	P	P	P	P	P
Random Excursions Variant Test	P	P	P	P	P	P	P	P	P
Serial Test	F	P	P	F	P	P	F	P	P
Linear Complexity Test	P	P	P	P	P	P	P	P	P

TABLE V. RESULTS OF THE NIST STATISTIAL TEST SUITE: COMPARISON OF β USED IN THE CONVERTER

	β-ary/binary conversion is applied											
Voltage Pattern	A				B				C			
β’	1.6	1.7	1.8	1.9	1.6	1.7	1.8	1.9	1.6	1.7	1.8	1.9
Frequency (Monobits) Test	P	P	P	P	F	P	P	P	P	P	P	P
Frequency Test within a Block	P	P	P	P	P	P	P	P	P	P	P	P
Cummulative Sums (Cusum) Test	F	P	P	P	P	P	P	P	P	P	P	P
Runs Test	P	P	P	P	P	P	P	P	P	P	P	P
Test for the Longest Run of Ones in a Block	F	P	P	P	P	P	P	P	P	P	P	P
Binary Matrix Rank Test	P	P	P	P	P	P	P	P	P	P	P	P
Discrete Frourier Transform Test	P	P	P	P	P	P	P	P	F	P	P	P
Non-Overlapping Template Matching Test	152	P	157	P	147	155	157	P	152	157	P	157
Overlapping Template Matching Test	P	P	P	P	P	P	P	P	P	P	P	P
Maurer's "Universal Statistical" Test	P	P	P	P	P	P	P	P	P	P	P	P
Approximate Entropy Test	P	P	P	P	P	P	P	P	P	P	P	P
Random Excursion Test	P	P	P	P	P	F	P	P	F	P	P	P
Random Excursions Variant Test	F	P	P	P	P	P	P	P	F	P	P	P
Serial Test	P	P	P	P	P	P	P	P	P	P	P	P
Linear Complexity Test	P	P	P	P	F	P	P	P	P	P	P	P

RANDOM DIRICHLET SERIES ARISING FROM RECORDS

RYOKICHI TANAKA

We study the distributions of the random Dirichlet series with parameters (s, β) defined by

$$S = \sum_{n=1}^{\infty} \frac{I_n}{n^s},$$

where (I_n) is an independent sequence of Bernoulli random variables taking value 1 with probability $1/n^\beta$ and 0 otherwise. Random series of this type are motivated by the record indicator sequences which have been studied in the extreme value theory in statistics. We show that the distributions have densities when $s > 0$ and $0 < \beta \leq 1$ with $s+\beta > 1$, and are purely atomic or not defined because of divergence otherwise. In particular, in the case when $s > 0$ and $\beta = 1$, we prove that the density is bounded and continuous when $0 < s < 1$, and unbounded when $s > 1$. In the case when $s > 0$ and $0 < \beta < 1$ with $s + \beta > 1$, we prove that the density is smooth. To show the absolute continuity, we obtain estimates of the Fourier transforms, employing van der Corput's method to deal with number-theoretic problems. We also give further regularity results of the densities.

References

[PPPT] Peled R., Peres Y., Pitman J., Tanaka R., *Random Dirichlet series arising from records*, in preparation.

Tohoku University, WPI-AIMR, 2-1-1 Katahira, Aoba-ku, 980-8577 Sendai, Japan

E-mail address: rtanaka@m.tohoku.ac.jp

HAUSDORFF SPECTRUM OF HARMONIC MEASURE

RYOKICHI TANAKA

Abstract. This is an introduction to the paper [T] on random walks on word hyperbolic groups and their harmonic measures.

1. Introduction

Let Γ be a finitely generated group. For a probability measure μ on it, we obtain a random walk on Γ by multiplying from right independent random elements with the law μ, and the distribution of the random walk at the time n is given by the n-th convolution power μ^{*n}. There are several important quantities which capture the asymptotic behaviours of the random walks. Define the entropy h and the drift l (also called the rate of escape, or the speed) by

$$h := \lim_{n\to\infty} -\frac{1}{n}\sum_{x\in\Gamma}\mu^{*n}(x)\log\mu^{*n}(x), \quad l := \lim_{n\to\infty}\frac{1}{n}\sum_{x\in\Gamma}|x|\mu^{*n}(x),$$

where $|\cdot|$ denotes the word norm associated with a finite symmetric set of generators of Γ. It is known that the entropy, introduced by Avez [Ave], is equal to 0 if and only if all bounded μ-harmonic functions on Γ are constants ([Der], [KV]). The entropy and the drift are connected via the logarithmic volume growth v of the group which is defined by $e^v := \lim_{n\to\infty}|B_n|^{1/n}$, where $|B_n|$ denotes the cardinality of the set B_n of words of length at most n, by the inequality

$$h \le lv, \tag{1}$$

as soon as all those quantities are well-defined ([Gui], see also e.g., [BHM1] and [Ver]). The inequality (1) is also called the fundamental inequality. In [Ver], Vershik proposed to study the equality case of (1). In this paper, we focus on hyperbolic groups in the sense of Gromov and characterise the equality of (1) in terms of the boundary behaviours of the random walks. For every hyperbolic group Γ, one can define the geometric boundary $\partial\Gamma$, which is compact and admits a metric d_ε with a parameter $\varepsilon > 0$. The harmonic measure ν is defined by the hitting distribution of the random walk starting from the identity on the boundary $\partial\Gamma$, corresponding to the step distribution μ on Γ. The boundary $\partial\Gamma$ has the Hausdorff dimension $D = v/\varepsilon$ and the D-Hausdorff measure $\mathcal{H}^D$ is finite and positive on $\partial\Gamma$ [Coo]. Here the D-Hausdorff measure $\mathcal{H}^D$ is a natural measure to compare with the harmonic measure ν. We call a probability measure μ on the group Γ admissible if the support of μ generates the whole group Γ as a semigroup. In the present paper, μ is always finitely supported and admissible unless stated otherwise.

Date: February 22, 2015. Supported by JSPS Grant-in-Aid for Young Scientists (B) 26800029.

Theorem 1.1. *For every finitely supported admissible probability measure μ on every finitely generated non-elementary hyperbolic group Γ equipped with a word metric, it holds that $h = lv$ if and only if the corresponding harmonic measure ν and the D-Hausdorff measure $\mathcal{H}^D$ are mutually absolutely continuous and their densities are uniformly bounded from above and from below.*

Blachère, Haissinsky and Mathieu established this result for every finitely supported admissible and symmetric probability measure μ [BHM2, Corollary 1.2, Theorem 1.5]. We extend it to non-symmetric measures from a completely different approach as we describe later. Recently, Gouëzel, Mathéus and Maucourant have proven that for a non-elementary hyperbolic group Γ which is not virtually free equipped with a word metric, for every finitely supported admissible probability measure μ, the equality $h = lv$ never holds [GMM2]. Together with their results, one concludes that in this setting, the harmonic measure ν and the D-Hausdorff measure $\mathcal{H}^D$ are always mutually singular. Connell and Muchnik proved that for an infinitely supported probability measure μ on Γ, the D-Hausdorff measure $\mathcal{H}^D$ (and also a Patterson-Sullivan measure) and the harmonic measure for μ can be equivalent ([CM1, Remark 0.5] and [CM2]). On the other hand, Le Prince showed that for every finitely generated non-elementary hyperbolic group Γ, there exists a finitely supported admissible and symmetric probability measure μ such that the corresponding harmonic measure ν and the D-Hausdorff measure $\mathcal{H}^D$ are mutually singular [LeP]. Ledrappier proved the corresponding result to Theorem 1.1 for non-cyclic free groups for every finitely supported admissible probability measure μ [Led, Corollary 3.15] . For free groups, it is straightforward to see that if μ depends only on the word length associated with the standard symmetric generating set, then the corresponding harmonic measure coincides with the Hausdorff measure (the uniform measure on the boundary) up to a multiplicative constant. In [GMM1], Gouëzel et al. studied a variant of the fundamental inequality (1) and also obtained some rigidity results for the equality case. Apart from Cayley graphs of groups, Lyons extensively studied the equivalence of the harmonic measure and the Patterson-Sullivan measure for universal covering trees of finite graphs [Lyo].

A novel feature of our approach is to introduce one parameter family of probability measures μ_θ, which interpolates a Patterson-Sullivan measure and the harmonic measure on the boundary $\partial\Gamma$. Let us consider for every $\theta \in \mathbb{R}$,

$$\beta(\theta) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{x \in S_n} G(1,x)^\theta,$$

where $G(x,y)$ is the Green function associated with μ, we denote by 1 the identity of the group, and by S_n the set of words of length n. The limit exists by the Ancona inequality [Anc], and we show that β is convex, in fact, analytic except for at most finitely many points and continuously differentiable at every point. Theorem 1.1 is deduced via the following dimensional properties of the harmonic measure ν.

Theorem 1.2. *Let Γ and μ be as in Theorem 1.1, and ν be the corresponding harmonic measure on the boundary $\partial\Gamma$. It holds that*

$$\lim_{r\to 0} \frac{\log \nu\,(B(\xi,r))}{\log r} = \frac{h}{\varepsilon l}, \quad \nu\text{-a.e. } \xi. \tag{2}$$

Define the set

$$E_\alpha = \left\{ \xi \in \partial\Gamma \;\middle|\; \lim_{r\to 0} \frac{\log \nu\left(B(\xi, r)\right)}{\log r} = \alpha \right\},$$

then the Hausdorff dimension of the set E_α *is given by the Legendre transform of* β*, i.e.,*

$$\dim_H E_\alpha = \frac{\alpha\theta + \beta(\theta)}{\varepsilon},$$

for every $\alpha = -\beta'(\theta)$*, where* β *is continuously differentiable on the whole* $\mathbb{R}$*, and* $B(\xi, r)$ *denotes the ball of radius* r *centred at* ξ*.*

The measure μ_θ is constructed by the Patterson-Sullivan technique. As $\theta = 0$, the measure μ_0 is a Patterson-Sullivan measure, and thus comparable with the D-Hausdorff measure $\mathcal{H}^D$, and as $\theta = 1$, the measure μ_1 is the harmonic measure ν. The probability measure μ_θ satisfies that $\mu_\theta(E_\alpha) = 1$ for $\alpha = -\beta'(\theta)$. We call $\dim_H E_\alpha$ as a function in α the *Hausdorff spectrum* of the measure ν. To determine the Hausdorff spectrum is called multifractal analysis which has been extensively studied in fractal geometry. In fact, there are many technical similarities to analyse harmonic measures and self-conformal measures ([Fen] and [PU]). For the backgrounds on this topic, see also [Fal] and references therein. We show that the measure μ_θ satisfies a Gibbs-like property with respect to $\beta(\theta)$, where $\beta(\theta)$ is an analogue of the pressure. This measure μ_θ is also characterised by the eigenmeasures of certain transfer operator built on a symbolic dynamical system associated with an automatic structure of the group. To study the measure μ_θ, we employ the results about the Martin boundary of a hyperbolic group by Izumi, Neshveyev and Okayasu [INO] and a generalised thermodynamic formalism due to Gouëzel [Gou1]. Note that the formula (2) is proved for every non-elementary hyperbolic group and for every finitely supported symmetric probability measure μ in [BHM2, Theorem 1.3], for every non-cyclic free groups and for every probability measure μ of finite first moment in [Led, Theorem 4.15], for a general class of random walks on trees in [Kai1] and for the simple random walks on the Galton-Watson trees in [LPP].

The following result is a finitistic version of Theorem 1.2, inspired by the corresponding results for the Galton-Watson trees by Lyons, Pemantle and Peres [LPP].

Theorem 1.3. *Let* Γ *and* μ *be as in Theorem 1.1, and consider the associated random walk starting at the identity on* Γ*. For every* $a \in (0, 1)$*, there exists a subset* $\Gamma_a \subset \Gamma$ *such that the random walk stays in* Γ_a *for every time with probability at least* $1 - a$*, and*

$$\lim_{n\to\infty} |\Gamma_a \cap S_n|^{1/n} = e^{h/l}.$$

In particular, if $h < lv$, then the random walk is confined in an exponentially small part of the group with positive probability. This can be compared with [Ver, p.669], where a random generation of group elements which is called the Monte Carlo method is discussed. For example, the random generation of group elements according to a random walk does not produce the whole data of the group in this case; see also [GMM2].

Let us return to Theorem 1.1. For a symmetric probability measure μ, i.e., $\mu(x) = \mu(x^{-1})$ for every $x \in \Gamma$, one can define the Green metric $d_G(x, y) = -\log F(x, y)$, where $F(x, y)$ denotes the probability that the random walk starting at x ever reaches y, and show that Γ is hyperbolic with respect to d_G according to the Ancona inequality [BHM2]. The Green metric d_G is not geodesic; nevertheless, one can use approximate trees argument and most of common techniques for the geodesic case work. The harmonic measure ν is actually a quasi-conformal (in fact, conformal) measure with respect to the metric induced in the boundary $\partial\Gamma$ by the Green metric d_G, and this fact plays an essential role to deduce Theorem 1.1 and that the local dimension of the harmonic measure ν is $h/(\varepsilon l)$ for ν-a.e. in Theorem 1.2 in the symmetric case. The symmetry of μ is required to make the Green metric d_G a genuine metric in Γ; otherwise it is not clear that one can discuss about the hyperbolicity for a non-symmetric metric. Here our alternative is to introduce a measure μ_θ, whose construction is actually inspired by a recent work of Gouëzel on the local limit theorem on a hyperbolic group [Gou1], and to obtain the Hausdorff spectrum of the harmonic measure ν. In many cases, a description of the Hausdorff spectrum is a main purpose on its own right, especially, when it is motivated by a problem in statistical physics. A fundamental observation in this paper, however, is rather the converse; namely, we use the multifractal analysis to compare the harmonic measure with a natural reference measure which is the D-Hausdorff measure on the boundary of the group Γ. More precisely, we shall see that the function β is affine on $\mathbb{R}$ if and only if those two measures are mutually absolutely continuous. Furthermore, the description of the Hausdorff spectrum implies that the harmonic measure has a rich multifractal structure as soon as it is singular with respect to the D-Hasudorff measure. In particular, the range of the Hausdorff spectrum contains the interval $[h/(\varepsilon l), v/\varepsilon]$.

Let us mention about an extension to a step distribution μ of unbounded support. The arguments in the present paper work once we have the Ancona inequality and its strengthened one. Gouëzel has proven those for every admissible probability measure μ with a super-exponential tail [Gou2]. At the same time, he has also proven a failure of description of the Martin boundary in the usual sense for an admissible probability measure with an exponential tail [ibid]. Hence the results in this paper are extended to every admissible step distribution μ with a super-exponential tail, but it is obscure whether one could extend to μ with an exponential tail in the present approach. We shall also mention about an extension to a left invariant metric which is not induced by a word length in Γ. For example, one is interested in the setting where Γ acts cocompactly on the hyperbolic space $\mathbb{H}^n$ and the metric in Γ is defined by $d(x, y) := d_{\mathbb{H}^n}(xo, yo)$ for a reference point o in $\mathbb{H}^n$. In fact, in [BHM2], they proved Theorem 1.1 for symmetric μ with every metric d which is hyperbolic and quasi-isometric to a word metric in Γ (not necessarily geodesic). Some of our results still hold for such a metric d, and in fact, most of the results are expected to remain valid; but we do not proceed to this direction in the present paper for the simplification of the proofs.

Finally, we close this introduction by pointing out some related problems in a continuous setting. On the special linear groups, the regularity problem of the harmonic measures is proposed by Kaimanovich and Le Prince [KL], and they showed that there exists a finitely supported symmetric probability measure (the support can generate a given Zariski dense subgroup) on $SL(d, \mathbb{R})$ ($d \geq 2$) such that the corresponding harmonic measure is non-atomic singular with respect to a natural smooth measure class on the Furstenberg boundary. They proved this result via the dimension inequality of the harmonic measure. Bourgain constructed a finitely supported symmetric probability measure on $SL(2,\mathbb{R})$ such that the corresponding harmonic measure is absolutely continuous with respect to Lebesgue measure on the circle [Bou]. Brieussel and the author proved analogous results in the three dimensional solvable Lie group Sol [BT], namely, they showed that random walks with finitely supported step distributions on it can produce both absolutely continuous and singular harmonic measures with respect to Lebesgue measure on the corresponding boundary. The dimension inequality is also used there to prove the existence of a finitely supported probability measure whose harmonic measure is singular. We point out, however, the dimension equality of type (2) is still missing in both cases, and in general, in the Lie group settings to the extent of our knowledge.

References

[Anc] Ancona A., *Positive harmonic functions and hyperbolicity*, Potential Theory, Prague 1987, Lecture Notes in Mathematics, vol. 1344, Springer, Berlin, 1988, 1-23.

[Ave] Avez A., *Entropie des groupes de type fini*, C. R. Acad. Sci. Paris Sér. A **275**, 1363-1366 (1972).

[BHM1] Blachère S., Haissinsky P., Mathieu P., *Asymptotic entropy and Green speed for random walks on countable groups*, Ann. Prob., Vol. 36, No.3, 1134-1152 (2008).

[BHM2] Blachère S., Haissinsky P., Mathieu P., *Harmonic measures versus quasiconformal measures for hyperbolic groups*, Ann. Sci. Éc. Norm. Supér. (4) **44**, 683-721 (2011).

[Bou] Bourgain J., *Finitely supported measures on $SL_2(\mathbb{R})$ which are absolutely continuous at infinity*, Geometric aspect of functional analysis (B. Klartag et al. eds.), 133-141, Lecture Notes in Math. 2050, Springer-Verlag, Berlin Heidelberg, 2012.

[BT] Brieussel J., Tanaka R., *Discrete random walks on the group Sol*, arXiv:1306.6180v1 [math.PR] (2013), to appear in Israel J. Math.

[CM1] Connell C., Muchnik R., *Harmonicity of quasiconformal measures and Poisson boundaries of hyperbolic spaces*, Geom. Funct. Anal. Vol. 17, 707-769 (2007).

[CM2] Connell C., Muchnik R., *Harmonicity of Gibbs measures*, Duke Math. J. Vol. 137, No. 3, 461-509 (2007).

[Coo] Coornaert M., *Mesures de Patterson-Sullivan sur le bord d'un espace hyperbolique an sens de Gromov*, Pacific J. Math., Vol 159, No 2, 241-270 (1993).

[Der] Derriennic Y., *Quelques applications du théorème ergodique sous-additif*, Conference on Random Walks (Kleebach, 1979), 183-201, Astérisque, **74**, Soc. Math. France, Paris, 1980.

[Fal] Falconer K., *Fractal Geometry, Mathematical Foundations and Applications*, Third Edition, John Wiley & Sons, Chichester 2014.

[Fen] Feng D-J., *Gibbs properties of self-conformal measures and the multi fractal formalism*, Ergodic Theory Dynam. Systems **27**, no.3, 787-812 (2007).

[Gou1] Gouëzel S., *Local limit theorem for symmetric random walks in Gromov-hyperbolic groups*, J. Amer. Math. Soc. Vol 27, No 3, 893-928 (2014).

[Gou2] Gouëzel S., *Martin boundary of measures with infinite support in hyperbolic groups*, preprint, arXiv:1302.5388v1 [math.PR] (2013).

[GMM1] Gouëzel S., Mathéus F., Maucourant F., *Sharp lower bounds for the asymptotic entropy of symmetric random walks*, preprint, arXiv:1209.3378v3 [math.PR] (2014).

[GMM2] Gouëzel S., Mathéus F., Maucourant F., *Entropy and drift in word hyperbolic groups*, arXiv:1501.05082v1 [math.PR] (2015).

[Gui] Guivarc'h Y., *Sur la loi des grands nombres et le rayon spectral d'une marche aléatoire*, Conference on Random Walks (Kleebach, 1979), 47-98, Astérisque, **74**, Soc. Math. France, Paris, 1980.

[INO] Izumi M., Neshveyev S., Okayasu R., *The ratio set of the harmonic measure of a random walk on a hyperbolic group*, Israel J. Math. **163**, 285-316 (2008).
[Kai1] Kaimanovich V A., *Hausdorff dimension of the harmonic measure on trees*, Ergodic Theory Dynam. Systems, **18**, 631-660 (1998).
[KL] Kaimanovich V A., Le Prince V.,*Matrix random products with singular harmonic measure*, Geom. Dedicata **150** (2011) 257-279.
[KV] Kaimanovich V A., Vershik A M., *Random walks on discrete groups: boundary and entropy*, Ann. Probab, Vol. 11, No. 3, 457-490 (1983).
[Led] Ledrappier F., *Some asymptotic properties of random walks on free groups*, Topics in probability and Lie groups: boundary theory, 117-152 , CRM Proc. Lecture Notes, **28**, Amer. Math. Soc., Providence, RI, 2001.
[LeP] Le Prince V., *Dimensional properties of the harmonic measure for a random walk on a hyperbolic group*, Trans. Amer. Math. Soc., Vol. 359, No. 6, 2881-2898 (2007).
[Lyo] Lyons R., *Equivalence of boundary measures on covering trees of finite graphs*, Ergodic Theory Dynam. Systems, **14**, 575-597 (1994).
[LPP] Lyons R., Pemantle R., Peres Y., *Ergodic theory on Galton-Watson trees: speed of random walk and dimension of harmonic measure*, Ergodic Theory Dynam. Systems, **15**, 593-619 (1995).
[PU] Przytycki F., Urbański M., *Conformal Fractals: Ergodic Theory Methods*, London Mathematical Society Lecture Note Series 371, Cambridge University Press, Cambridge 2010.
[T] Tanaka R., *Hausdorff spectrum of harmonic measure*, arXiv:1411.2312v1 [math.PR] (2014).
[Ver] Vershik A M., *Dynamic theory of growth in groups: Entropy, boundaries, examples*, Russian Math. Surveys, **55**:4, 667-733 (2000).

Tohoku University, WPI-AIMR, 2-1-1 Katahira, Aoba-ku, 980-8577 Sendai, Japan

E-mail address: `rtanaka@m.tohoku.ac.jp`

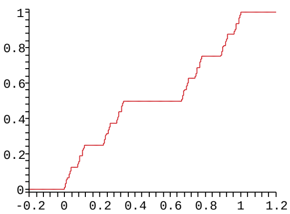
Multifractal Analysis for the Complex Analogues of the Devil's Staircase and the Takagi Function in Random Complex Dynamics

Hiroki Sumi
Osaka University
sumi@math.sci.osaka-u.ac.jp
http://www.math.sci.osaka-u.ac.jp/~sumi/
with
Johannes Jaerisch
Waseda University
jjaerisch@gmail.com
http://cr.math.sci.osaka-u.ac.jp/~jaerisch/
March 10, 2015

1

1 Introduction

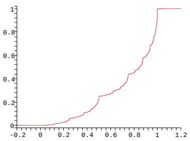
(1) The devil's staircase is equal to the restriction of function of probability of tending to $+\infty$ w.r.t. the random dynamics on $\mathbb{R}$ s.t. we take $h_1(x) = 3x$ with prob. $1/2$ and we take $h_2(x) = 3x - 2$ with prob. $1/2$.

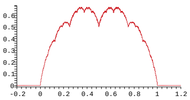


2

(2) Lebesgue's singular function L_p with parameter $p \in (0,1)$ is equal to the restriction of function of probability of tending to $+\infty$ w.r.t. the random dynamics on $\mathbb{R}$ s.t.
we take $h_1(x) = 2x$ with prob. p and
we take $h_2(x) = 2x - 1$ with prob. $1 - p$.

(3) The Takagi function (on $[0,1]$) is equal to the function $x \mapsto \frac{1}{2} \cdot \frac{\partial L_p(x)}{\partial p}|_{p=1/2}$, $x \in [0,1]$.





3

We want to consider complex analogues of the above story. We consider the following setting.

- $\hat{\mathbb{C}} := \mathbb{C} \cup \infty \cong S^2$ (Riemann sphere).
- Let $s \in \mathbb{N}$.
- Let $f_i : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, i = 1, \ldots, s+1$, be rational maps with $\deg(f_i) \geq 2$.
- Probability parameter space of dimension s:

$$\mathcal{W} := \left\{ \vec{p} = (p_1, p_2, \ldots, p_s) \in (0,1)^s \mid \sum_{i=1}^{s} p_i < 1 \right\}.$$

4

- For each $\vec{p} \in \mathcal{W}$ we consider the **random dynamical system** on $\hat{\mathbb{C}}$ such that at every step we choose f_i with probability p_i, i.e., a Markov process whose state space is $\hat{\mathbb{C}}$ and whose transition probability is given by

$$p(x, A) := \sum_{i=1}^{s+1} p_i 1_A(f_i(z)), z \in \hat{\mathbb{C}}, A \subset \hat{\mathbb{C}}.$$

- Let $C(\hat{\mathbb{C}}) := \{\varphi : \hat{\mathbb{C}} \to \mathbb{C} \mid \varphi$ is conti.$\}$ endowed with sup. norm.
- The transition operator $M_{\vec{p}} : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$ is given by

$$M_{\vec{p}}(\varphi)(z) := \sum_{i=1}^{s+1} p_i \cdot \varphi(f_i(z)), \quad \varphi \in C(\hat{\mathbb{C}}), z \in \hat{\mathbb{C}}.$$

5

- Let

$$G := \{f_{i_1} \circ \cdots \circ f_{i_n} \mid n \in \mathbb{N}, i_1, \ldots, i_n \in \{1, \ldots, s+1\}\}.$$

This is a semigroup whose semigroup operation is the functional composition.
This G is called the **rational semigroup** generated by $\{f_1, \ldots, f_{s+1}\}$.

- Let
$F(G) := \{z \in \hat{\mathbb{C}} \mid \exists$ nbd U of z s.t.
$\{h : U \to \hat{\mathbb{C}}\}_{h \in G}$ is equiconti. on $U\}$.
This is called the **Fatou set** of G.

- Let $J(G) := \hat{\mathbb{C}} \setminus F(G)$. This is called the **Julia set** of G.

6

Assumptions for G:

- G is hyperbolic, i.e., $\overline{\cup_{h\in G}\{\text{all critical values of } h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\}} \subset F(G)$.
- $(f_i^{-1}(J(G))) \cap (f_j^{-1}(J(G))) = \emptyset$ for all $i \neq j$.
- $\exists$ at least two minimal sets of G.

  Here, we say that a non-empty compact set $K \subset \hat{\mathbb{C}}$ is a **minimal set** of G if

$$K = \overline{\cup_{h\in G}\{h(z)\}} \text{ for each } z \in K.$$

7

Theorem 1.1 (S, [S11-1, S13-1]). *Fix $\vec{p} \in \mathcal{W}$ and let L be a minimal set of G. For each $z \in \hat{\mathbb{C}}$, let $T_{L,\vec{p}}(z) \in [0,1]$ be the probability of tending to L starting with the initial value $z \in \hat{\mathbb{C}}$. Then we have the following.*

(1) *$\exists \alpha > 0$ s.t. $T_{L,\vec{p}} \in C^{\alpha}(\hat{\mathbb{C}}) :=$ the space of α-Hölder conti. fcns on $\hat{\mathbb{C}}$ endowed with α-Hölder norm. Moreover, $M_{\vec{p}}(T_{L,\vec{p}}) = T_{L,\vec{p}}$.*

(2) *$\exists V$:nbd of $\vec{p}$ in $\mathcal{W}$, $\exists \alpha > 0$ s.t. $\vec{q} \mapsto T_{L,\vec{q}} \in C^{\alpha}(\hat{\mathbb{C}})$ is real-analytic in V.*

(3) *The set of varying points of $T_{L,\vec{p}}$ is equal to $J(G)$, which is a thin fractal set (e.g. $\dim_H(J(G)) < 2$).*

$T_{L,\vec{p}}$ is a complex analogue of the devil's staircase or Lebesgue's singular functions.

8

Complex analogues of the Takagi function

Fix $\vec{p} \in \mathcal{W}$ and let L be a minimal set of G.

Definition 1.2.
For $\vec{n} = (n_1, \ldots, n_s) \in (\mathbb{N} \cup \{0\})^s$ and $z \in \hat{\mathbb{C}}$ we set

$$C_{\vec{n}}(z) := \frac{\partial^{|\vec{n}|} T_{L,(a_1,\ldots,a_s,1-\sum_{i=1}^{s} a_i)}(z)}{\partial^{n_1} a_1 \partial^{n_2} a_2 \cdots \partial^{n_s} a_s}|_{\vec{a}=\vec{p}}.$$

(note: $C_{(1,0,\ldots,0)}$ is a complex analogue of the Takagi function.)
Also, define the $\mathbb{C}$-vector space

$$\mathcal{T} := \mathrm{span}\{C_{\vec{n}} \mid \vec{n} \in (\mathbb{N} \cup \{0\})^s\} \subset C^{\alpha}(\hat{\mathbb{C}}).$$

9

- For $C \in \mathcal{T}$ and $z \in \hat{\mathbb{C}}$ consider **pointwise Hölder exponent of C at z:**
$$\mathrm{Höl}(C, z) := \sup\{\beta \in \mathbb{R} \mid \limsup_{y \to z, y \neq z} \frac{|C(y) - C(z)|}{d(y, z)^{\beta}} < \infty\}.$$
- By the separation condition in the setting, we have $\forall z \in J(G)$, $\exists! i(z) \in \{1, \ldots, s+1\}$ s.t. $f_{i(z)}(z) \in J(G)$. We define $f : J(G) \to J(G)$ by $f(z) = f_{i(z)}(z)$.
- Define potentials
$\zeta : J(G) \to \mathbb{R}, \zeta(z) := -\log \|f'_{i(z)}(z)\|$ and
$\psi : J(G) \to \mathbb{R}, \psi(z) := \log p_{i(z)}$.

Theorem 1.3 ([JS14, JS15]). *Let $C \in \mathcal{T} \setminus \{0\}$, $z \in J(G)$. Then*

$$\mathrm{Höl}(C, z) = \liminf_{n \to \infty} \frac{\sum_{k=0}^{n-1} \psi \circ f^k(z)}{\sum_{k=0}^{n-1} \zeta \circ f^k(z)}.$$

10

Corollary 1.4. *Let $C \in \mathcal{T} \setminus \{0\}$. Then C is continuous on $\hat{\mathbb{C}}$ and varies precisely on $J(G)$ (which is a thin fractal set). In particular, $\mathcal{T} = \oplus_{\vec{n} \in (\mathbb{N} \cup \{0\})^s} \mathbb{C} C_{\vec{n}}$ is a direct sum.*

Theorem 1.5. (*Multifractal formalism*) *Let $C \in \mathcal{T} \setminus \{0\}$. Then the level sets*

$$\{z \in J(G) \mid \text{Höl}(C, z) = \alpha\}, \ \alpha \in \mathbb{R},$$

satisfy the multifractal formalism.
That is, the Hausdorff dimension function
$\alpha \mapsto \dim_H(\{z \in J(G) \mid \text{Höl}(C, z) = \alpha\})$ is a real analytic strictly concave and positive function on a bounded open interval (α_-, α_+), except very rare cases.

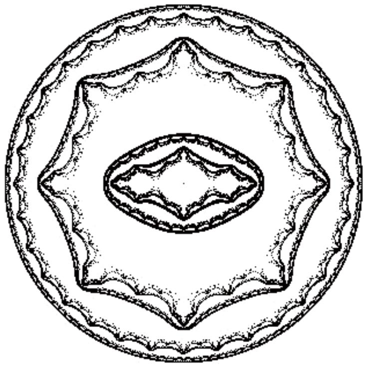
11

Example 1.6. Let $g_1(z) = z^2 - 1, g_2(z) = \frac{z^2}{4}$ and $f_1 = g_1 \circ g_1, f_2 := g_2 \circ g_2$. Let $\vec{p} = (1/2, 1/2)$. Then $\{\infty\}$ is a minimal set of $G = \{f_{i_1} \circ \cdots \circ f_{i_n} \mid n \in \mathbb{N}, \forall i_j \in \{1, 2\}\}$.

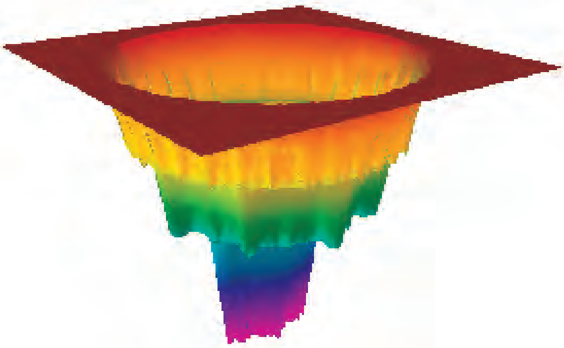
- The function $T_{\infty, \vec{p}} : \hat{\mathbb{C}} \to [0, 1]$ of prob. of tending to ∞ is a complex analogue of the devil' s staircase (or Lebesgue's singular functions) and it is called a **devil's coliseum**.
- Also, let $C_{(1)}(z) = \frac{\partial T_{\infty,(a,1-a)}(z)}{\partial a} \mid a = 1/2$. Then the function $C_{(1)} : \hat{\mathbb{C}} \to \mathbb{R}$ is a **complex analogue of the Takagi function**.
- Both $T_{\infty, \vec{p}}$ and $C_{(1)}$ are Hölder continuous on $\hat{\mathbb{C}}$ and vary precisely on $J(G)$, which is a thin fractal set (e.g. $\dim_H J(G) < 2$). **Multifractal formalism works**.

12

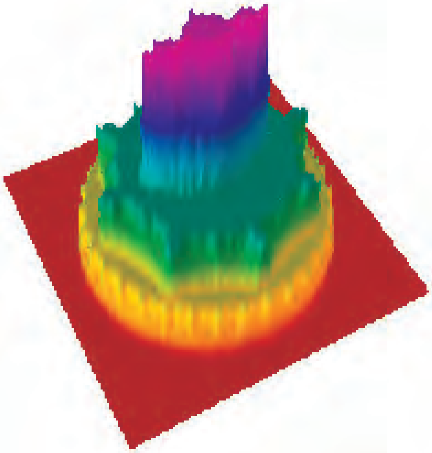
$g_1(z) := z^2 - 1,\ g_2(z) := \frac{z^2}{4},\ h_1 := g_1^2,\ h_2 := g_2^2.\ G := \langle h_1, h_2 \rangle.\ G \in \mathcal{G}_{dis}.$
The figure of $J(G)$. $\sharp \mathrm{Con}(J(G)) > \aleph_0$.



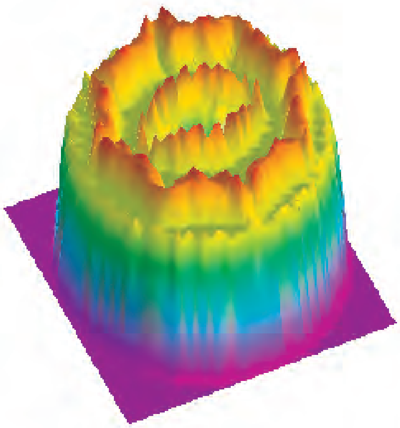
$g_1(z) := z^2 - 1,\ g_2(z) := \frac{z^2}{4},\ h_1 := g_1^2,\ h_2 := g_2^2,\ \tau := \frac{1}{2}\delta_{h_1} + \frac{1}{2}\delta_{h_2}.$
The graph of $z \mapsto T_{\tau,\infty}(z)$.
(Devil's Coliseum (Complex analogue of devil's staircase).)



The graph of $z \mapsto 1 - T_{7,\infty}(z)$.



$g_1(z) := z^2 - 1,\ g_2(z) := \frac{z^2}{4},\ h_1 := g_1^2,\ h_2 := g_2^2.$
The graph of $z \mapsto \frac{\partial T}{\partial p}(h_1, h_2, \frac{1}{2}, z)$.
(A complex analogue of the Takagi function)



References

[AK06] P. Allaart and K. Kawamura, *Extreme values of some continuous nowhere differentiable functions*, Math. Proc. Cambridge Philos. Soc. 140 (2006), no. 2, 269–295.

[AK11] P. Allaart and K. Kawamura, *The Takagi function: a survey.* Real Anal. Exchange 37 (2011/12), no. 1, 154.

[Br00] R. Brück, *Connectedness and stability of Julia sets of the composition of polynomials of the form $z^2 + c_n$*, J. London Math. Soc. **61** (2000), 462-470.

[BBR99] R. Brück, M. Büger and S. Reitz, *Random iterations of polynomials of the form $z^2 + c_n$: Connectedness of Julia sets*, Ergodic Theory Dynam. Systems, **19**, (1999), No.5, 1221–1231.

[FS91] J. E. Fornaess and N. Sibony, *Random iterations of rational functions*, Ergodic Theory Dynam. Systems, **11**(1991), 687–708.

[GQL03] Z. Gong, W. Qiu and Y. Li, *Connectedness of Julia sets for

13

a quadratic random dynamical system, Ergodic Theory Dynam. Systems, (2003), **23**, 1807-1815.

[GR96] Z. Gong and F. Ren, *A random dynamical system formed by infinitely many functions,* Journal of Fudan University, **35**, 1996, 387–392.

[HM96] A. Hinkkanen and G. J. Martin, *The Dynamics of Semigroups of Rational Functions I*, Proc. London Math. Soc. (3)**73**(1996), 358-384.

[JS13] J. Jaerisch and H. Sumi, *Dynamics of infinitely generated nicely expanding rational semigroups and the inducing method*, preprint, http://arxiv.org/abs/1501.06772.

[JS14] J. Jaerisch and H. Sumi, *Multifractal formalism for expanding rational semigroups and random complex dynamical systems*, preprint, http://arxiv.org/abs/1311.6241.

[JS15] J. Jaerisch and H. Sumi, *Holder regularity of the complex analogues of the Takagi function*, in preparation.

14

[MT83] K. Matsumoto and I. Tsuda, *Noise-induced order*, J. Statist. Phys. 31 (1983) 87-106.

[SeSh91] T. Sekiguchi and Y. Shiota, *A generalization of Hata-Yamaguti's results on the Takagi function*, Japan J. Appl. Math. **8**, pp203-219, 1991.

[St12] R. Stankewitz, Density of repelling fixed points in the Julia set of a rational or entire semigroup, II, Discrete and Continuous Dynamical Systems Ser. A, 32 (2012), 2583 - 2589.

[SS11] R. Stankewitz and H. Sumi, *Dynamical properties and structure of Julia sets of postcritically bounded polynomial semigroups*, Trans. Amer. Math. Soc., 363 (2011), no. 10, 5293–5319.

[S97] H. Sumi, *On dynamics of hyperbolic rational semigroups*, J. Math. Kyoto Univ., Vol. 37, No. 4, 1997, 717-733.

[S98] H. Sumi, *On Hausdorff dimension of Julia sets of hyperbolic rational semigroups*, Kodai Math. J., Vol. 21, No. 1, pp. 10-28,

15

1998.

[S00] H. Sumi, *Skew product maps related to finitely generated rational semigroups,* Nonlinearity, **13**, (2000), 995–1019.

[S01] H. Sumi, *Dynamics of sub-hyperbolic and semi-hyperbolic rational semigroups and skew products*, Ergodic Theory Dynam. Systems, (2001), **21**, 563–603.

[S05] H. Sumi, *Dimensions of Julia sets of expanding rational semigroups*, Kodai Mathematical Journal, Vol. 28, No. 2, 2005, pp390–422. (See also http://arxiv.org/abs/math.DS/0405522.)

[S06] H. Sumi, *Semi-hyperbolic fibered rational maps and rational semigroups*, Ergodic Theory Dynam. Systems, (2006), **26**, 893–922.

[S09] H. Sumi, *Interaction cohomology of forward or backward self-similar systems*, Adv. Math., 222 (2009), no. 3, 729–781.

[S10-1] H. Sumi, *Dynamics of postcritically bounded polynomial semigroups III: classification of semi-hyperbolic semigroups and*

16

random Julia sets which are Jordan curves but not quasicircles, Ergodic Theory Dynam. Systems, (2010), **30**, No. 6, 1869–1902.

[S10-2] H. Sumi, *Rational semigroups, random complex dynamics and singular functions on the complex plane*, survey article, Selected Papers on Analysis and Differential Equations, Amer. Math. Soc. Transl. (2) Vol. 230, 2010, 161–200.

[S11-1] H. Sumi, *Random complex dynamics and semigroups of holomorphic maps*, Proc. London Math. Soc., (2011), 102 (1), 50–112.

[S11-2] H. Sumi, *Dynamics of postcritically bounded polynomial semigroups I: connected components of the Julia sets*, Discrete Contin. Dyn. Sys. Ser. A, Vol. 29, No. 3, 2011, 1205–1244.

[S13-1] H. Sumi, *Cooperation principle, stability and bifurcation in random complex dynamics*, Adv. Math., 245 (2013), 137–181.

[S13-2] H. Sumi, *Dynamics of postcritically bounded polynomial semigroups II: fiberwise dynamics and the Julia sets*, J. London

17

Math. Soc. (2) 88 (2013) 294–318.

[S14] H. Sumi, *Random complex dynamics and devil's coliseums*, preprint 2014, http://arxiv.org/abs/1104.3640.

[SU10] H. Sumi and M. Urbański, *Real analyticity of Hausdorff dimension for expanding rational semigroups*, Ergodic Theory Dynam. Systems (2010), Vol. 30, No. 2, 601-633.

[SU11] H. Sumi and M. Urbański, *Measures and dimensions of Julia sets of semi-hyperbolic rational semigroups*, Discrete and Continuous Dynamical Systems Ser. A., Vol 30, No. 1, 2011, 313–363.

[SU12] H. Sumi and M. Urbański, *Bowen Parameter and Hausdorff Dimension for Expanding Rational Semigroups*, Discrete and Continuous Dynamical Systems Ser. A, Vol. 32, 2012, 2591-2606.

[SU13] H. Sumi and M. Urbański, Transversality family of expanding rational semigroups, Advances in Mathematics 234 (2013) 697–734.

18

[YHK] M. Yamaguti, M. Hata, and J. Kigami, Mathematics of fractals. Translated from the 1993 Japanese original by Kiki Hudson. Translations of Mathematical Monographs, 167. American Mathematical Society, Providence, RI, 1997.

MI レクチャーノートシリーズ刊行にあたり

本レクチャーノートシリーズは、文部科学省 21 世紀 COE プログラム「機能数理学の構築と展開」（H.15-19 年度）において作成した COE Lecture Notes の続刊であり、文部科学省大学院教育改革支援プログラム「産業界が求める数学博士と新修士養成」（H19-21 年度）および、同グローバル COE プログラム「マス・フォア・インダストリ教育研究拠点」（H.20-24 年度）において行われた講義の講義録として出版されてきた。平成 23 年 4 月のマス・フォア・インダストリ研究所（IMI）設立と平成 25 年 4 月の IMI の文部科学省共同利用・共同研究拠点として「産業数学の先進的・基礎的共同研究拠点」の認定を受け、今後、レクチャーノートは、マス・フォア・インダストリに関わる国内外の研究者による講義の講義録、会議録等として出版し、マス・フォア・インダストリの本格的な展開に資するものとする。

平成 26 年 10 月
マス・フォア・インダストリ研究所
所長　福本康秀

Workshop on “β-transformation and related topics”

発　行　　2015年3月10日
編　集　　白井朋之
発　行　　九州大学マス・フォア・インダストリ研究所
　　　　　九州大学大学院数理学府
　　　　　〒819-0395 福岡市西区元岡744
　　　　　九州大学数理・IMI 事務室
　　　　　TEL 092-802-4402　FAX 092-802-4405
　　　　　URL http://www.imi.kyushu-u.ac.jp/

印　刷　　城島印刷株式会社
　　　　　〒810-0012 福岡市中央区白金 2 丁目 9 番 6 号
　　　　　TEL 092-531-7102　FAX 092-524-4411

シリーズ既刊

Issue	Author／Editor	Title	Published
COE Lecture Note	Mitsuhiro T. NAKAO Kazuhiro YOKOYAMA	Computer Assisted Proofs - Numeric and Symbolic Approaches - 199pages	August 22, 2006
COE Lecture Note	M.J.Shai HARAN	Arithmetical Investigations - Representation theory, Orthogonal polynomials and Quantum interpolations- 174pages	August 22, 2006
COE Lecture Note Vol.3	Michal BENES Masato KIMURA Tatsuyuki NAKAKI	Proceedings of Czech-Japanese Seminar in Applied Mathematics 2005 155pages	October 13, 2006
COE Lecture Note Vol.4	宮田　健治	辺要素有限要素法による磁界解析 - 機能数理学特別講義 21pages	May 15, 2007
COE Lecture Note Vol.5	Francois APERY	Univariate Elimination Subresultants - Bezout formula, Laurent series and vanishing conditions - 89pages	September 25, 2007
COE Lecture Note Vol.6	Michal BENES Masato KIMURA Tatsuyuki NAKAKI	Proceedings of Czech-Japanese Seminar in Applied Mathematics 2006 209pages	October 12, 2007
COE Lecture Note Vol.7	若山　正人 中尾　充宏	九州大学産業技術数理研究センター キックオフミーティング 138pages	October 15, 2007
COE Lecture Note Vol.8	Alberto PARMEGGIANI	Introduction to the Spectral Theory of Non-Commutative Harmonic Oscillators 233pages	January 31, 2008
COE Lecture Note Vol.9	Michael I.TRIBELSKY	Introduction to Mathematical modeling 23pages	February 15, 2008
COE Lecture Note Vol.10	Jacques FARAUT	Infinite Dimensional Spherical Analysis 74pages	March 14, 2008
COE Lecture Note Vol.11	Gerrit van DIJK	Gelfand Pairs And Beyond 60pages	August 25, 2008
COE Lecture Note Vol.12	Faculty of Mathematics, Kyushu University	Consortium “MATH for INDUSTRY” First Forum 87pages	September 16, 2008
COE Lecture Note Vol.13	九州大学大学院 数理学研究院	プロシーディング「損保数理に現れる確率モデル」 — 日新火災・九州大学 共同研究 2008 年 11 月 研究会 — 82pages	February 6, 2009

シリーズ既刊

Issue	Author／Editor	Title	Published
COE Lecture Note Vol.14	Michal Beneš, Tohru Tsujikawa Shigetoshi Yazaki	Proceedings of Czech-Japanese Seminar in Applied Mathematics 2008 77pages	February 12, 2009
COE Lecture Note Vol.15	Faculty of Mathematics, Kyushu University	International Workshop on Verified Computations and Related Topics 129pages	February 23, 2009
COE Lecture Note Vol.16	Alexander Samokhin	Volume Integral Equation Method in Problems of Mathematical Physics 50pages	February 24, 2009
COE Lecture Note Vol.17	矢嶋　　徹 及川　正行 梶原　健司 辻　　英一 福本　康秀	非線形波動の数理と物理　66pages	February 27, 2009
COE Lecture Note Vol.18	Tim Hoffmann	Discrete Differential Geometry of Curves and Surfaces　75pages	April 21, 2009
COE Lecture Note Vol.19	Ichiro Suzuki	The Pattern Formation Problem for Autonomous Mobile Robots —Special Lecture in Functional Mathematics—　23pages	April 30, 2009
COE Lecture Note Vol.20	Yasuhide Fukumoto Yasunori Maekawa	Math-for-Industry Tutorial: Spectral theories of non-Hermitian operators and their application　184pages	June 19, 2009
COE Lecture Note Vol.21	Faculty of Mathematics, Kyushu University	Forum "Math-for-Industry" Casimir Force, Casimir Operators and the Riemann Hypothesis 95pages	November 9, 2009
COE Lecture Note Vol.22	Masakazu Suzuki Hoon Hong Hirokazu Anai Chee Yap Yousuke Sato Hiroshi Yoshida	The Joint Conference of ASCM 2009 and MACIS 2009: Asian Symposium on Computer Mathematics Mathematical Aspects of Computer and Information Sciences　436pages	December 14, 2009
COE Lecture Note Vol.23	荒川　恒男 金子　昌信	多重ゼータ値入門　111pages	February 15, 2010
COE Lecture Note Vol.24	Fulton B.Gonzalez	Notes on Integral Geometry and Harmonic Analysis　125pages	March 12, 2010
COE Lecture Note Vol.25	Wayne Rossman	Discrete Constant Mean Curvature Surfaces via Conserved Quantities 130pages	May 31, 2010
COE Lecture Note Vol.26	Mihai Ciucu	Perfect Matchings and Applications　66pages	July 2, 2010

シリーズ既刊

Issue	Author／Editor	Title	Published
COE Lecture Note Vol.27	九州大学大学院 数理学研究院	Forum “Math-for-Industry” and Study Group Workshop Information security, visualization, and inverse problems, on the basis of optimization techniques 100pages	October 21, 2010
COE Lecture Note Vol.28	ANDREAS LANGER	MODULAR FORMS, ELLIPTIC AND MODULAR CURVES LECTURES AT KYUSHU UNIVERSITY 2010 62pages	November 26, 2010
COE Lecture Note Vol.29	木田　雅成 原田　昌晃 横山　俊一	Magma で広がる数学の世界 157pages	December 27, 2010
COE Lecture Note Vol.30	原　　隆 松井　　卓 廣島　文生	Mathematical Quantum Field Theory and Renormalization Theory 201pages	January 31, 2011
COE Lecture Note Vol.31	若山　正人 福本　康秀 高木　　剛 山本　昌宏	Study Group Workshop 2010 Lecture & Report 128pages	February 8, 2011
COE Lecture Note Vol.32	Institute of Mathematics for Industry, Kyushu University	Forum “Math-for-Industry” 2011 “TSUNAMI-Mathematical Modelling” Using Mathematics for Natural Disaster Prediction, Recovery and Provision for the Future 90pages	September 30, 2011
COE Lecture Note Vol.33	若山　正人 福本　康秀 高木　　剛 山本　昌宏	Study Group Workshop 2011 Lecture & Report 140pages	October 27, 2011
COE Lecture Note Vol.34	Adrian Muntean Vladimír Chalupecký	Homogenization Method and Multiscale Modeling 72pages	October 28, 2011
COE Lecture Note Vol.35	横山　俊一 夫　　紀恵 林　　卓也	計算機代数システムの進展 210pages	November 30, 2011
COE Lecture Note Vol.36	Michal Beneš Masato Kimura Shigetoshi Yazaki	Proceedings of Czech-Japanese Seminar in Applied Mathematics 2010 107pages	January 27, 2012
COE Lecture Note Vol.37	若山　正人 高木　　剛 Kirill Morozov 平岡　裕章 木村　正人 白井　朋之 西井　龍映 栄　伸一郎 穴井　宏和 福本　康秀	平成 23 年度 数学・数理科学と諸科学・産業との連携研究ワークショップ　拡がっていく数学　～期待される “見えない力” ～ 154pages	February 20, 2012

シリーズ既刊

Issue	Author／Editor	Title	Published
COE Lecture Note Vol.38	Fumio Hiroshima Itaru Sasaki Herbert Spohn Akito Suzuki	Enhanced Binding in Quantum Field Theory　204pages	March 12, 2012
COE Lecture Note Vol.39	Institute of Mathematics for Industry, Kyushu University	Multiscale Mathematics; Hierarchy of collective phenomena and interrelations between hierarchical structures　180pages	March 13, 2012
COE Lecture Note Vol.40	井ノ口順一 太田　泰広 筧　　三郎 梶原　健司 松浦　　望	離散可積分系・離散微分幾何チュートリアル 2012　152pages	March 15, 2012
COE Lecture Note Vol.41	Institute of Mathematics for Industry, Kyushu University	Forum “Math-for-Industry” 2012 “Information Recovery and Discovery”　91pages	October 22, 2012
COE Lecture Note Vol.42	佐伯　　修 若山　正人 山本　昌宏	Study Group Workshop 2012 Abstract, Lecture & Report　178pages	November 19, 2012
COE Lecture Note Vol.43	Institute of Mathematics for Industry, Kyushu University	Combinatorics and Numerical Analysis Joint Workshop　103pages	December 27, 2012
COE Lecture Note Vol.44	萩原　　学	モダン符号理論からポストモダン符号理論への展望　107pages	January 30, 2013
COE Lecture Note Vol.45	金山　　寛	Joint Research Workshop of Institute of Mathematics for Industry (IMI), Kyushu University “Propagation of Ultra-large-scale Computation by the Domain-decomposition-method for Industrial Problems (PUCDIP 2012)” 121pages	February 19, 2013
COE Lecture Note Vol.46	西井　龍映 栄　伸一郎 岡田　勘三 落合　啓之 小磯　深幸 斎藤　新悟 白井　朋之	科学・技術の研究課題への数学アプローチ ―数学モデリングの基礎と展開―　325pages	February 28, 2013
COE Lecture Note Vol.47	SOO TECK LEE	BRANCHING RULES AND BRANCHING ALGEBRAS FOR THE COMPLEX CLASSICAL GROUPS　40pages	March 8, 2013
COE Lecture Note Vol.48	溝口　佳寛 脇　　隼人 平坂　　貢 谷口　哲至 島袋　　修	博多ワークショップ「組み合わせとその応用」　124pages	March 28, 2013

シリーズ既刊

Issue	Author／Editor	Title	Published
COE Lecture Note Vol.49	照井　　章 小原　功任 濱田　龍義 横山　俊一 穴井　宏和 横田　博史	マス・フォア・インダストリ研究所　共同利用研究集会 II 数式処理研究と産学連携の新たな発展　137pages	August 9, 2013
MI Lecture Note Vol.50	Ken Anjyo Hiroyuki Ochiai Yoshinori Dobashi Yoshihiro Mizoguchi Shizuo Kaji	Symposium MEIS2013: Mathematical Progress in Expressive Image Synthesis　154pages	October 21, 2013
MI Lecture Note Vol.51	Institute of Mathematics for Industry, Kyushu University	Forum “Math-for-Industry” 2013 “The Impact of Applications on Mathematics”　97pages	October 30, 2013
MI Lecture Note Vol.52	佐伯　　修 岡田　勘三 髙木　　剛 若山　正人 山本　昌宏	Study Group Workshop 2013 Abstract, Lecture ＆ Report 142pages	November 15, 2013
MI Lecture Note Vol.53	四方　義啓 櫻井　幸一 安田　貴徳 Xavier Dahan	平成25年度　九州大学マス・フォア・インダストリ研究所 共同利用研究集会　安全・安心社会基盤構築のための代数構造 ～サイバー社会の信頼性確保のための数理学～　158pages	December 26, 2013
MI Lecture Note Vol.54	Takashi Takiguchi Hiroshi Fujiwara	Inverse problems for practice, the present and the future　93pages	January 30, 2014
MI Lecture Note Vol.55	栄　伸一郎 溝口　佳寛 脇　　隼人 渋田　敬史	Study Group Workshop 2013 数学協働プログラム Lecture & Report 98pages	February 10, 2014
MI Lecture Note Vol.56	Yoshihiro Mizoguchi Hayato Waki Takafumi Shibuta Tetsuji Taniguchi Osamu Shimabukuro Makoto Tagami Hirotake Kurihara Shuya Chiba	Hakata Workshop 2014 ~ Discrete Mathematics and its Applications ~　141pages	March 28, 2014
MI Lecture Note Vol.57	Institute of Mathematics for Industry, Kyushu University	Forum “Math-for-Industry” 2014: “Applications + Practical Conceptualization + Mathematics = fruitful Innovation”　93pages	October 23, 2014
MI Lecture Note Vol.58	安生健一 落合啓之	Symposium MEIS2014: Mathematical Progress in Expressive Image Synthesis　135pages	November 12, 2014

シリーズ既刊

Issue	Author／Editor	Title	Published
MI Lecture Note Vol.59	西井　龍映 岡田　勘三 梶原　健司 髙木　　剛 若山　正人 脇　　隼人 山本　昌宏	Study Group Workshop 2014 数学協働プログラム Abstract, Lecture ＆ Report　196pages	November 14, 2014
MI Lecture Note Vol.60	西浦　　博	平成 26 年度九州大学 IMI 共同利用研究・研究集会（I） 感染症数理モデルの実用化と産業及び政策での活用のための新たな展開　120pages	November 28, 2014
MI Lecture Note Vol.61	溝口　佳寛 Jacques Garrigue 萩原　　学 Reynald Affeldt	研究集会 高信頼な理論と実装のための定理証明および定理証明器 Theorem proving and provers for reliable theory and implementations (TPP2014)　138pages	February 26, 2015

Institute of Mathematics for Industry
Kyushu University

九州大学マス・フォア・インダストリ研究所
九州大学大学院 数理学府

〒819-0395 福岡市西区元岡744　TEL 092-802-4402　FAX 092-802-4405
URL http://www.imi.kyushu-u.ac.jp/