Math-for-industry
Education & Research Hub

ISSN 1881-4042

博多ワークショップ「組み合わせとその応用」

−Hakata workshop, combinatrics and its applications−

編集：溝口佳寛・脇隼人・平坂貢・谷口哲至・島袋修

九州大学グローバルCOEプログラム

COE Lecture Note Vol.48 : Kyushu University

博多ワークショップ「組み合わせとその応用」
－Hakata workshop, combinatrics and its applications－

編集：溝口佳寛，脇隼人，平坂貢，谷口哲至，島袋修

About MI Lecture Note Series

The Math-for-Industry (MI) Lecture Note Series is a successor to the COE Lecture Notes, published for the 21st COE Program “Development of Dynamic Mathematics with High Functionality”, sponsored by Ministry of Education, Culture, Sports, Science and technology-Japan (MEXT) (From 2003 to 2007).

The MI series reports lectures given by scholars invited under the following two programs: “Training Program of Ph.D. and new Master’s in Mathematics as Required by Industry”, adopted as a Support Program for Improving Gradute School Education by MEXT (from 2007 to 2009); and Education-and-Research Hub for Mathematics-for-Industry”, newly adopted as a Global COE Program by MEXT (from 2008 to 2012).

July 2008
Masato Wakayama
Global COE Program “Education-and-Research Hub for Mathematics-for-Industry”
Program Leader

序文

博多ワークショップ「組合せ数学とその応用」(Hakata workshop, combinatorics and its applications)は第11回代数・組合せ数学日韓合同ワークショップのサテライトワークショップとして開催されました．第1回代数・組合せ数学日韓合同ワークショップは2006年10月に九州大学において開催されました．その後，年1回ないし2回，釜山大学校，浦項工科大学校(POSTECH)，九州大学，東北大学で開催され，今回で11回目を迎えました．組合せ数学は「組合せ論」とも呼ばれ，グラフ理論をはじめとする離散的な対象の構造を解析する新しい数学分野のひとつです．組合せ論の対象は代数学，確率論，位相数学，幾何学を始め多くの純粋数学の分野に存在します．そして，符号理論，最適化理論，離散力学系など多くの応用理論が含まれます．

今回，若手数学研究者の交流を組合せ数学の講演とポスター発表による数学ソフトウェア紹介を通じて行うために，文部科学省グローバルCOEプログラム「マス・フォア・インダストリ教育研究拠点(拠点リーダー: 九州大学若山正人教授)」の企画として本ワークショップが計画されました．3名の海外研究者の講演を含む6件の講演と14件の数学ソフトウェア紹介が行われ，約40名の参加者により2013年1月26日リファレンス駅東ビル(福岡市博多区)にて実施されました．本レクチャーノートは，その講義録やポスター発表内容を集めた資料集です．

数学で発見された定理に基づく手続きはプログラム言語で実装され，仮説の検証のために計算機実験が行われ，新たな数学の理論や定理の発見につながることがあります．平面グラフの彩色問題である「４色定理」は1976年AppleとHakenらにより計算機を用いた演算結果をもとに証明されています．3次元ユークリッド空間への最高密度の球充填配置に関わる「ケプラー予想」と呼ばれる問題は1611年に提唱され，1998年Halesは計算機を用いて全ての場合を調べ尽くした証明を与えました．近年の計算機ハードウェアの進歩により，より多くの計算実験が行えるようになり，さらには，データや理論の可視化が行えるようになり，数学理論の発展のために計算機は利用され，そのためのさまざまな数学ソフトウェアが開発され利用されています．新しい数学理論の計算機言語での実現（プログラム）は数学理論の発展のためだけでなく，最適化理論，数値解析学，統計学，システム理論等を応用した様々な分野における応用ソフトウェアの中で利用されます．

本ワークショップの目的として組合せ数学に関する最近の話題について討論すること，そして，数学ソフトウェア紹介により，数学理論発展のためのソフトウェア開発手法，および，数学理論の計算機応用について討論することを考えていました．ポスター発表形式で行った数学ソフトウェア紹介は初めての試みであり，開催前は少し不安もありました．完成されたソフトウェアの紹介というよりは数学研究者が自ら実装したプログラムを理論とともに紹介する企画とし，対象も組合せ数学分野に限らず募集しました．その結果，いろいろなバリエーションの数学ソフトウェアとその開発者が集まり，背景の数学理論の討論はもちろん，ソフトウェア開発や今後の発展に関する討論も行われ，非常に好評でした．

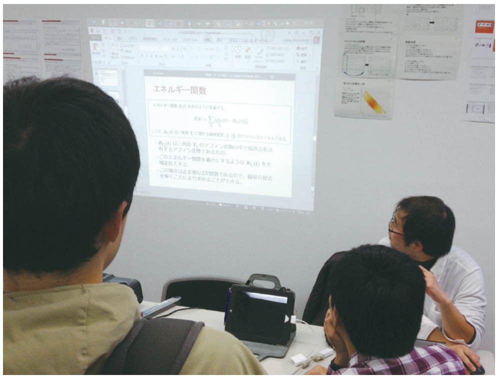
講演者をはじめ参加者の方々には，このワークショップを通じて新たな知見や交流が生まれ，今後の数学理論や数学ソフトウェアの発展につながることを期待しています．

2013年 3月

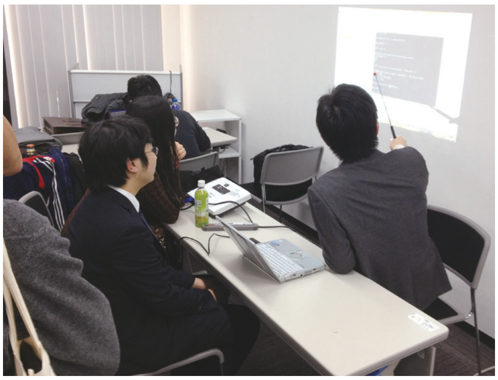
溝口 佳寛 （九州大学）
脇 隼人 （九州大学）
平坂 貢 （釜山大学）
谷口 哲至 （松江高専）
島袋 修 （崇城大学）


B2


B4
A4


エネルギー関数


B2
A3
B3



目　次

第１部　講　演

第２部　数学ソフトウェア紹介

HAKATA WORKSHOP

～ Combinatrics and its Applications ～

Organizers

Yoshihiro Mizoguchi (Kyushu University)

Hayato Waki (Kyushu University)

Mitsugu Hirasaka (Pusan National University)

Tetsuji Taniguchi (Matsue College of Technology)

Osamu Shimabukuro (Sojo University)

Laboratory of Advanced Software in Mathematics, Institute of Mathematics for Industry, Kyushu University

Supported by

Global COE Program "Education and Research Hub for Mathematics-for-Industry"

Date: January 26, 2013. 9:30-18:00

Location: Seminar Room I (2F) in Reference Eki Higashi Building. 1-16-14 Hakata-Eki-Higashi, Hakata-Ku, Fukuoka City, 812-0013

Program

9:25 - 9:30	Opening (Tetsuji Taniguchi)
9:30 - 10:05	Akihiro Munemasa (Tohoku University) Complex Hadamard matrices contained in a Bose-Mesner algebra
10:20 - 10:55	Michael Dobbins (GAIA, Postech) Reducing combinatorial to projective equivalence in realizability problems for polytopes
11:10 - 11:45	Aleksandar Jurišić (University of Ljubljana) TOWER GRAPHS AND EXTENDED GENERALIZED UADRANGLES
12:00 - 13:00	break
13:00 - 15:00	Poster Session
15:15 - 16:15	Xiao-Dong Zhang (Shanghai Jiao Tong University) The algebraic connectivity of graphs
16:30 - 17:05	Katsuhiro Ota (Keio University) Clique minors, chromatic numbers for degree sequences
17:10 - 17:45	Yota Otachi (JAIST) The path-distance-width of hypercubes
17:45 - 18:00	Closing (Yoshihiro Mizoguchi)
18:00 -	Post-meeting party

第 1 部　講　演

Complex Hadamard matrices contained in a Bose–Mesner algebra

Akihiro Munemasa[1]

[1]Graduate School of Information Sciences
Tohoku University
(joint work with Takuya Ikuta)

January 26, 2013
Combinatorics Seminar Hakata Workshop

- A Hadamard matrix of order n is an $n \times n$ matrix H with entries ± 1, satisfying $HH^\top = nI$.

Conjecture

$\exists$ Hadamard matrix of order n whenever $n \equiv 0 \pmod 4$.

- A complex Hadamard matrix of order n is an $n \times n$ matrix H with entries in $\{\zeta \in \mathbb{C} \mid |\zeta| = 1\}$, satisfying $HH^* = nI$.
- A type II (inverse orthogonal) matrix of order n is an $n \times n$ matrix H with entries in $\{\zeta \in \mathbb{C} \mid \zeta \neq 0\}$, satisfying $H{H^{(-)}}^\top = nI$, where $H^{(-)}$ is the entrywise inverse.

In this talk, a Hadamard matrix will mean a complex Hadamard matrix.

Circulant Hadamard matrix:

$$H = \sum_{i=0}^{n-1} a_i A^i = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \\ & \ddots & \ddots & \ddots \\ a_1 & & & a_0 \end{bmatrix}, \quad A = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & \ddots \\ 1 & & & \end{bmatrix}$$

$$\zeta = \exp\frac{2\pi\sqrt{-1}}{n}, \quad F = \frac{1}{\sqrt{n}}(\zeta^{ij}) : \text{unitary},$$

$$F^* A F = D = \operatorname{diag}(\zeta^j; j = 0, 1, \ldots, n-1).$$

$$\begin{aligned} HH^* = nI &\iff (F^*HF)(F^*HF)^* = nI \\ &\iff (\sum_i a_i D^i)(\sum_i a_i D^i)^* = nI \\ &\iff |\sum_i a_i \zeta^{ij}|^2 = n \qquad (\forall j). \end{aligned}$$

Björck–Fröberg (1991–1992) circulant Hadamard, $n \leq 8$

Faugère (2001), (2004) circulant Hadamard, $n = 9, 10$

de la Harpe–Jones (1990) SRG $n \equiv 1 \pmod 4) \to$
symmetric circulant complex Hadamard

Munemasa–Watatani (1992) DRT $n \equiv 3 \pmod 4 \to$
non-symmetric circulant complex Hadamard

$$H = a_0 A_0 + a_1 A_1 + a_2 A_2, \quad A_0 = I$$

$$A_1^\top = A_1,\ A_2^\top = A_2 \quad \text{(de la H.–J.)}$$

$$A_1^\top = A_2, \quad \text{(M.–W.)}$$

Unifying principle: association schemes.
(strongly regular graphs is a special case)

Godsil–Chan (2010): strongly regular graphs.

Godsil–Chan (2010) classified type II (inverse orthogonal) matrices of the form:

$$H = a_0 I + a_1 A_1 + a_2 A_2 \quad \text{(we may assume } a_0 = 1\text{)}$$

where $a_1, a_2 \in \mathbb{C}^\times$, and

$$A_1 = \text{adjacency matrix of a SRG } \Gamma,$$
$$A_2 = \text{adjacency matrix of } \overline{\Gamma}.$$

Chan (`arXiv:1102.5601v1`) classified Hadamard matrices of the above form (i.e., $|a_1| = |a_2| = 1$), also found $H = a_0 I + a_1 A_1 + a_2 A_2 + a_3 A_3$ of order 15 from the line graph $L(O_3)$ of the Petersen graph O_3.

E_{jj}:matrix unit with (j,j) entry 1

$$H = \sum_{i=0}^{n-1} a_i A^i, \quad F^* A F = D = \sum_j \zeta^j E_{jj}, \quad E_{jj} E_{kk} = \delta_{jk} E_{jj}$$

$$F^* H F = \sum_i a_i D^i = \sum_i a_i (\sum_j \zeta^j E_{jj})^i = \sum_j (\sum_i a_i \zeta^{ij}) E_{jj}$$

$$HH^* = nI \iff (F^* H F)(F^* H F)^* = nI$$
$$\iff |\sum_i a_i \zeta^{ij}|^2 = n \quad (\forall j)$$

$$H = \sum_{i=0}^{d} a_i A_i, \quad F^* A_i F = D_i = \sum_{j=0}^{d'} p_{ji} P_j, \quad P_j P_k = \delta_{jk} P_j$$

$$F^* H F = \sum_i a_i D_i = \sum_i a_i \sum_j p_{ji} P_j = \sum_j (\sum_i a_i p_{ji}) P_j$$

$$HH^* = nI \iff |\sum_{i=0}^{d} a_i p_{ji}|^2 = n \quad (0 \le \forall j \le d') \quad (d \le d')$$

Construction analogous to circulant Hadamard matrix works if

$$H = \sum_{i=0}^{d} a_i A_i, \quad F^* A_i F = D_i = \sum_{j=0}^{d'} p_{ji} P_j, \quad P_j P_k = \delta_{jk} P_j$$

- $A_0 = I, A_1, \dots, A_d$:$(0,1)$-matrices, $\sum_i A_i = J$,
- F:unitary, $F^* A_i F = D_i$:diagonal
- $A_i = \sum_{j=0}^{d'} p_{ji} E_j$, $E_j = F P_j F^*$, $P_j P_k = \delta_{jk} P_j$.

Clearly $d \le d'$. Equality holds iff

$$\langle A_0, A_1, \dots, A_d \rangle = \langle E_0, E_1, \dots, E_d \rangle$$

$\Longrightarrow$ closed under $\cdot, \circ$ (Bose–Mesner algebra).
This is essentially the definition of a (commutative) association scheme.
Symmetric $\iff$ $A_i^\top = A_i$ $(\forall i)$ $\Longrightarrow$ $p_{ji} \in \mathbb{R}$.

The Bose–Mesner algebra of a symmetric association scheme

$$\langle A_0, A_1, \dots, A_d \rangle = \langle E_0, E_1, \dots, E_d \rangle$$

$$A_i = \sum_{j=0}^{d} p_{ji} E_j, \quad E_j E_k = \delta_{jk} E_j$$

$$A_i^\top = A_i \quad (\forall i), \implies p_{ji} \in \mathbb{R}.$$

$$H = \sum_{i=0}^{d} a_i A_i, \quad |a_i| = 1$$

$$HH^* = nI \iff |\sum_i a_i p_{ji}|^2 = n \quad (\forall j)$$

$$\iff (\sum_i a_i p_{ji})(\sum_h \frac{1}{a_h} p_{jh}) = n \quad (\forall j)$$

$$(\sum_i a_i p_{ji})(\sum_h \frac{1}{a_h} p_{jh}) = n$$
$$\iff \sum_{i,h} \alpha_{ih} p_{ji} p_{jh} = n$$

where

$$\alpha_{ih} = \frac{a_i}{a_h} + \frac{a_h}{a_i}. \qquad (*)$$

1. Solve the system of linear equations in $\{\alpha_{ih}\}$
2. Find $\{a_i\}$ from $\{\alpha_{ih}\}$ by (*).
3. Given $\{\alpha_{ih}\}$, when $\exists\{a_i\}$ satisfying (*).

$$\begin{array}{rcl} f : (\mathbb{C}^\times)^m & \to & \mathbb{C}^M, \quad M = \binom{m}{2}, \\ \{x_i\}_{i=1}^m & \mapsto & \{\frac{x_i}{x_h} + \frac{x_h}{x_i}\}_{1 \le i < h < m}. \end{array}$$

Describe (image of f) $\cap [-2, 2]^M$

$$\begin{array}{rcl} f : (\mathbb{C}^\times)^2 & \to & \mathbb{C}, \\ (x, y) & \mapsto & \frac{x}{y} + \frac{y}{x} \end{array} \quad \text{surjective.}$$

$$\begin{array}{rcl} f : (\mathbb{C}^\times)^3 & \to & \mathbb{C}^3, \\ (x, y, z) & \mapsto & (\frac{x}{y} + \frac{y}{x}, \frac{x}{z} + \frac{z}{x}, \frac{y}{z} + \frac{z}{y}) \end{array}$$

Not surjective. $g(X, Y, Z) = X^2 + Y^2 + Z^2 - XYZ - 4$.

$$g(\frac{x}{y} + \frac{y}{x}, \frac{x}{z} + \frac{z}{x}, \frac{y}{z} + \frac{z}{y}) = 0$$

Indeed, image of f=zeros of g.

$$\begin{array}{rcl} f : (\mathbb{C}^\times)^4 & \to & \mathbb{C}^6, \\ (x_1, x_2, x_3, x_4) & \mapsto & (\frac{x_i}{x_j} + \frac{x_j}{x_i})_{1 \le i < j \le 4} \end{array}$$

$g(X, Y, Z) = X^2 + Y^2 + Z^2 - XYZ - 4.$

$$g_{i,j,k} = g(\frac{x_i}{x_j} + \frac{x_j}{x_i}, \frac{x_i}{x_k} + \frac{x_k}{x_i}, \frac{x_j}{x_k} + \frac{x_k}{x_j}) = 0$$

image of $f \stackrel{?}{=}$zeros of $\{g_{i,j,k}\}$. Need

$$\begin{aligned} h &= (X_{14}^2 - 4)X_{23} - X_{14}(X_{12}X_{34} + X_{13}X_{24}) \\ &\quad + 2(X_{12}X_{13} + X_{24}X_{34}) \qquad (X_{ij} = \frac{x_i}{x_j} + \frac{x_j}{x_i}) \end{aligned}$$

(and similar polynomials obtained by permuting indices)
I had to rely on MAGMA to find algebraic dependency. image of f= zeros of $g_{i,j,k}, h$.
The same is true for $\forall m \ge 4$.

Given a symmetric association scheme
$\to (p_{ji})$ $A_i = \sum_j p_{ji} E_j \to$ linear constraints on $\{\alpha_{ih}\}$,
$\sum_{i,h} \alpha_{ih} p_{ji} p_{jh} = n$ which must satisfy also

$$g_{i,j,k} = 0 \text{ and } h = 0$$

If $\exists \{\alpha_{ih}\}_{i,h} \subset (-2, 2)$, then

$$\exists \{a_i\}_i \text{ such that } \alpha_{ij} = \frac{a_i}{a_j} + \frac{a_j}{a_i}.$$

We may assume $a_0 = 1$. Then

$$a_i^2 - \alpha_{0i} a_i + 1 = 0$$

$-2 < \alpha_{0i} < 2 \implies |a_i| = 1.$

- q: a power of 2, $q \geq 4 \rightarrow q = 4$,
- $\Omega = \mathrm{PG}(2, q)$: the projective plane over $\mathbb{F}_q$,
- $Q = \{[a_0, a_1, a_2] \in \Omega \mid a_0^2 + a_1 a_2 = 0\}$: quadric,
- $X = \{[a_0, a_1, a_2] \in \Omega \setminus Q \mid [a_0, a_1, a_2] \neq [1, 0, 0]\}$,
- $|X| = q^2 - 1 \rightarrow 15$.

$$(A_i)_{xy} = \begin{cases} 1 & i = 1,\ |(x+y) \cap Q| = 2, \\ 1 & i = 2,\ |(x+y) \cap Q| = 0, \\ 1 & i = 3,\ |(x+y) \cap Q| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$\exists$ a complex Hadamard matrix in its Bose–Mesner algebra $L(O_3)$.

Reducing combinatorial to projective equivalence in realizability problems for polytopes

Michael Dobbins (GAIA, Postech)

Determining if there is a polytope of any combinatorial type that satisfies some given property is made difficult by the fact that there are polytopes with realization spaces that are homotopic to any primary semialgebraic set. I will show how, for certain properties, this can be reduced to finding such realizations among projective equivalence classes of polytopes, which are much nicer spaces. An application of this answers a question posed by Bernt Lindstrom in 1971, that there does exist a polytope without an antiprism.

TOWER GRAPHS AND EXTENDED GENERALIZED QUADRANGLES

Aleksandar Jurisic (University of Ljubljana)

Let Γ be a complete multipartite graph with at least two parts and each part of size at least 2. For example, $K_{t\times n}$, i.e., the complement of t copies of K_n, and $t, n \geq 2$. Then the local graphs and the μ-graphs (that is the graphs induced on the common neighbours of two vertices at distance 2) are again complete multipartite. They are actually the same graphs, in our example $K_{(t-1)\times n}$. If $t \geq 3$, then the geodesic closure of any μ-graph is the graph we started with.

Let now Γ be the point graph of a generalized quadrangle $\mathrm{GQ}(s,t)$. Then Γ is strongly regular like the complete multipartite graph (its valency is $s(t+1)$, while the number of triangles on an edge in (a) Γ is $s-1$, and (b) the complement of Γ is $t+1$). Its μ-graphs are $K_{1\times(t+1)}$ and when the generalized quadrangle is regular, the convex closures of μ-graphs are $K_{2\times(t+1)}$.

We study a family of graphs, denoted by $\mathcal{F}$, with the following properties

(i) their μ-graphs are complete multipartite,

(ii) there exist adjacent vertices x, y and a vertex z at distance 2 from both x, y in Γ, and the number $\alpha_\Gamma = \alpha(x,y,z)$ of common neighbours of these vertices does not depend on a choice of such a triple.

This is a generalization of the study of extended generalized quadrangles in it is intimately connected with even more general study of locally strongly regular graphs. We report on our progress of the classification of the family $\mathcal{F}$.

Outline Introduction Two Main techniques Bounds Extremal Random Reference

The Algebraic Connectivity of Graphs

Xiao-Dong ZHANG 张晓东）

Department of Mathematics

Shanghai Jiao Tong University(上海交通大学）

Shanghai 200240, P. R. China

xiaodong@sjtu.edu.cn

January 26th, 2013

Hakata Workshop 2013

Outline Introduction Two Main techniques Bounds Extremal Random Reference

Outline

- Introduction
- Two main techniques for algebraic connectivity
- Bounds for the algebraic connectivity
- Extremal graphs with maximum (minimum) algebraic connectivity
- The algebraic connectivity of random graphs

Definition and Notation

- $G = (V(G),\ E(G))$ a simple graph,
  vertex set $V(G) = \{v_1, \cdots, v_n\}$
  edge set $E(G) = \{e_1, \cdots, e_m\}$.
- $D(G) = diag(d_1, \cdots, d_n)$: degree diagonal matrix
  d_i : degree of vertex v_i (the number of edges incident to v_i).
- There are several matrices associated with a graph
- $A(G) = (a_{ij})$: Adjacency matrix of G,
  $a_{ij} = 1$ if $v_i \sim v_j$ and $a_{ij} = 0$ otherwise.
- $A(G)$ is a nonnegative symmetric $(0,1)$ matrix with the zeros on the main diagonal.

Definition and Notation

- Laplacian Matrix of a graph

$$L(G) = D(G) - A(G)$$

- The smallest eigenvalue of $L(G)$ is 0
- Fiedler (1973) The second smallest eigenvalue $\alpha(G)$ of $L(G)$ is called the algebraic connectivity of G.

Theorem 1

(Fiedler 1973) $\alpha(G) > 0$ if and only if G is connected.

上海交通大学 Shanghai Jiao Tong University

Background

Theorem 2

(Fiedler 1973) Let G be a graph with vertex connectivity $\nu(G)$, edge connectivity $\nu'(G)$ and the minimum degree $\delta(G)$. Then

$$\alpha(G) \leq \nu(G) \leq \nu'(G) \leq \delta(G).$$

- $\alpha(G)$ serves as a measure of connectivity of a graph.

上海交通大学 Shanghai Jiao Tong University

Background

- On combinatorial optimization problems: the problem of certain flowing process, the maximum cut problem and the traveling salesman problem.
- Fiedler vectors are used in algorithms for distributed memory parallel processors.
- The algebraic connectivity is a measure of the robustness in complex networks.
- Application to Continuous or Digital Space.
- Algebraic connectivity may explain the evolution of gene regulatory networks.

Basic properties

- $$\alpha(G) = \min_{f \perp e, f \neq 0} \frac{< L(G)f, f >}{< f, f >},$$
  $$= \min_{f \neq 0, \sum_{u \in V} f(u)=0} \frac{\sum_{uv \in E(G)} (f(u) - f(v))^2}{\sum_{u \in V} f(u)^2}$$
  where e is all one vector.
- If G is a graph of order n. Then
  $$\alpha(G) = n - \lambda_{max}(\overline{G}),$$
  where $\overline{G}$ is the complement of G.
- $\alpha(G + e) \geq \alpha(G)$.

Outline Introduction Two Main techniques Bounds Extremal Random Reference

上海交通大学 Shanghai Jiao Tong University

Basic properties

- $\alpha(K_n) = n$.
- $\alpha(P_n) = 2(1 - \cos \frac{\pi}{n})$.
- $\alpha(C_n) = 2(1 - \cos \frac{2\pi}{n})$.
- $\alpha(K_{p,q}) = \min\{p, q\}$.
- $\alpha(Persen\ graph) = 2$.

Two Main techniques

- : 1. Nonnegative matrix theory
- For a tree T with a vertex v, a branch of T at v is one of the connected components of T which results from removing vertex v and all edges incident with it.
- Bottleneck matrix of a branch at vertex k: the diagonal block of L_k^{-1}, where L_k is the principal submatrix of $L(G)$ by deleting the $k-$th row and column of $L(G)$.
- Perron value of a branch of T at vertex k is the Perron value (the spectral radius) of the corresponding bottleneck matrix.
- A branch at k is a Perron branch if the Perron value of that branch is the same as the spectral radius of L_k^{-1}.

1.Nonnegative matrix theory

Theorem 3

(Kirkland, Neumann and Shader 1996) Let T be a tree on n verteices$\{1, \cdots, n\}$ If $i \sim j$, then T is type II (no component of an eigenvector of $L(T)$ corresponding to $\alpha(G)$ is 0) if and only if there exists a $0 < \varepsilon < 1$ such that $\rho(M_1 - \varepsilon J) = \rho(M_2 - (1-\varepsilon)J)$, where M_1 is the bottleneck matrix for the branch at j containing i, and M_2 is the bottleneck matrix for the branch at i containing j. Moreover,

$$\alpha(T) = \frac{1}{\rho(M_1 - \varepsilon J)} = \frac{1}{\rho(M_2 - (1-\varepsilon)J)}.$$

1.Nonnegative matrix theory

Theorem 4

(Kirkland, Neumann and Shader 1996) Let T be a tree on n verteices$\{1, \cdots, n\}$ If $i \sim j$, then T is type I (the k-th component of an eigenvector of $L(T)$ corresponding to $\alpha(G)$ is 0) if and only if there exist two or more Perror branch of T at k. Moreover,

$$\alpha(T) = \frac{1}{\rho(L_k^{-1})}.$$

Outline Introduction Two Main techniques Bounds Extremal Random Reference

上海交通大學 Shanghai Jiao Tong University

1. Nonnegative matrix theory

Theorem 5

(Kirkland and Neumann 1997) Let T be a tree on n vertices $\{1, \cdots, n\}$ and M be the bottleneck matrix of a branch B of T at k, which does not contain all of the characteristic vertices of T. Let $\widetilde{T}$ be a tree from T by replacing the branch at k by some other branch $\widetilde{B}$ at k whose bottleneck matrix is $\widetilde{M}$. If $M \ll \widetilde{M}$ (there exist two permutation matrices P, Q such that PMP^T is entrywise dominated by a principal submatrix of $Q\widetilde{M}Q^T$), then $\alpha(\widetilde{T}) \leq \alpha(T)$.

上海交通大学 Shanghai Jiao Tong University

2. Graph Transformation

2. Use Graph perturbation.

Theorem 6

(Guo 2010) Let G_1 and G_2 be two graphs with at least two vertices, respectively. If G' is a graph by joining an edge from a vertex u of G_1 and a vertex v of G_2, and G'' is a graph by identifying u of G_1 and v of G_2 and adding a pendent edge uw, then

$$\alpha(G') \leq \alpha(G'').$$

Outline Introduction Two Main techniques Bounds Extremal Random Reference

上海交通大学 Shanghai Jiao Tong University

2.Graph Transformation

Theorem 7

(Guo 2010) Let G be a connected graph with at least two vertices. Let $G_{k,l}$ be a graph from G by attaching two paths of lengths k, l respectively, at vertex u of G; and let $G_{k+1,l-1}$ be a graph from G by attaching two paths of lengths $k+1, l-1$ respectively, at vertex u of G. If $k \geq l \geq 1$, then

$$\alpha(G_{k,l}) \geq \alpha(G_{k+1,l-1}).$$

2.Graph Transformation

Theorem 8

(Shao, Guo, Shan 2008)Let $vv_1, \cdots, vv_p$ be pendant edges of a connected graph G on n vertices. Let G' be a graph from G by adding any $0 \le t \le \frac{p(p-1)}{2}$ edges among $v_1, \cdots, v_p$. If $\alpha(G) \ne 1$, then

$$\alpha(G) = \alpha(G').$$

Outline Introduction Two Main techniques Bounds Extremal Random Reference

2.Graph Transformation

Theorem 9

(Kirkland, Oliveira and Justel 2011) Let G be a graph on vertices $1, \cdots, n$, and suppose that vertex 1 of G has degree d. Select $p-1 \ge 1$ vertices of G, say $u_1, \cdots, u_{p-1}$ none of which is adjacent to vertex 1 in G. Let H be the graph on vertices $1, \cdots, n$ whose only edges are those between vertex 1 and each of vertices $u_1, \cdots, u_{p-1}$. If $G \bigcup H \ne K_n$, then $\alpha(G \bigcup H) - \alpha(G) \le p - \varepsilon_0$, where ε_0 is the smallest positive root of the polynomial

$$d\varepsilon(p-\varepsilon) - (1-\varepsilon)^2(p-1-\varepsilon)^2.$$

Shanghai Jiao Tong University

Bounds for the algebraic connectivity

Theorem 10

(Fiedler 1973) Let G be a connected graph of order n. Then

$$2(1-2\cos\frac{\pi}{n}) \leq \alpha(G) \leq n$$

with the left equality if and only if G is a path,the right equality if and only if $G = K_n$

Theorem 11

(Kirkland, Molitierno, Neumann and Shader 2002) Let G be a connected graph of order n with vertex connectivity $\nu(G)$. Then $\alpha(G) \leq \nu(G)$ with equality if and only if $G = G_1 \bigvee G_2$, where G_1 is a disconnected graph of order $n-\nu(G)$ and G_2 is a graph of order $\nu(G)$ with $\alpha(G_2) \geq 2\nu(G) - n$.

Outline Introduction Two Main techniques Bounds Extremal Random Reference

Shanghai Jiao Tong University

Bounds for the algebraic connectivity

Theorem 12

(Belhaiza, Abreu, Hansen and Oliveira 2005) Let G be a simple graph of order n and size m. If $G \neq K_n$, then

$$\alpha(G) \leq \lfloor -1 + 2\sqrt{1+2m} \rfloor.$$

Theorem 13

(Mohar 1992) Let G be a graph of order n with diameter $diam(G)$. Then

$$\alpha(G) \geq \frac{4}{n diam(G)}.$$

Bounds for the algebraic connectivity

Theorem 14

(Grone, Merris and Sunder 1990) Let T be a tree with diameter $diam(T)$. Then

$$\alpha(T) \leq 2(1 - \cos \frac{\pi}{diam(T) + 1}).$$

Theorem 15

(Molitierno 2006) Let T be a planar graph. Then

$$\alpha(G) \leq 4$$

with equality if and only if $G = K_4$ or $G = K_{2,2,2}$.

The dominating number

The dominating number $\gamma(G)$: The smallest number of $|S|$ such that for each vertex in $G - S$ is adjacent to one vertex in $S \subseteq V$.

Theorem 16

(Lu, Liu and Tian 2005) Let G be a connected graph with the dominating number $\gamma(G)$. Then

$$\alpha(G) \leq \frac{n(n - 2\gamma(G) + 1)}{n - \gamma(G)}$$

with equality if and only if $G = K_{2,2}$.

Theorem 17

(Nikiforov 2007) Let G be a connected graph of order n with the dominating number $\gamma(G) > 1$. Then $\alpha(G) \leq n - \gamma(G)$.

上海交通大学 Shanghai Jiao Tong University

The dominating number

Theorem 18

(Aouchiche, Hansen and Stevanovic 2010) Let G be a connected graph of order n with the dominating number $\gamma(G) \geq 3$.
(1) If $n = 2k \geq 6$, then $\alpha(G) \leq 2k - 2\gamma(G) + \frac{k+2-\sqrt{k^2+4}}{2}$.
(2) If $n = 2k + 1 \geq 9$ with the minimum degree $\delta(G) \in \{1, 3, 5\}$ or δ is even and $G \notin \{F_6, F_7, F_8\}$, then
$\alpha(G) \leq 2k - 2\gamma(G) + \frac{k+3-\sqrt{(k+1)^2+4}}{2}$.

上海交通大学 Shanghai Jiao Tong University

The dominating number

Conjecture 19

(Aouchiche, Hansen and Stevanovic 2010) Let G be a connected graph of order $n = 2k + 1$ with the dominating number $\gamma(G) \geq 3$. If $G \notin \{A_3, A_4, F_6, F_7, F_8\}$, then

$$\alpha(G) \leq 2k - 2\gamma(G) + \frac{k+3-\sqrt{(k+1)^2+4}}{2}.$$

The cut-vertex

Theorem 20

(Kirkland 2000, 2001) Let G be a connected graph of order n with k cut-vertices.
(1) If $2 \le k \le \frac{n}{2}$, then $\alpha(G) \le \frac{2(n-k)}{n-k+2+\sqrt{(n-k)^2+4}}$. with equality if and only if G is obtained from K_{n-k} by attaching a pendant vertex at each vertex in k vertices of K_{n-k}.
(2) If $k > \frac{n}{2}$ and there exist positive integer q and nonnegative integer l such that $k = \frac{qn+l}{q+1}$. Then $\alpha(G) \le \alpha(E_l(q,m))$, where $E_l(q,m)$ is defined as follows: starting with a graph H with m vertices which has at least r vertices of degree $m-1$ for $m \ge r \ge l$; select r such vertices at each attached a path of $q+1$ vertices, at each remaining vertex i of H attaching a path of j_i vertices subject to the condition $r + \sum_{i=1}^{m-r}(j_i - q) = l$.

Outline Introduction Two Main techniques Bounds Extremal Random Reference

Tree

Theorem 21

(Z 2004) Let T be a tree on n vertices with independence number β.
(i) If $\beta = n-1$, then $\alpha(T) \le 1$ with equality if and only if T is $T_{n,n-1}$, i.e., T is the star $K_{1,n-1}$.
(ii) If $\beta = n-2$, then $\alpha(T) \le \alpha(T_{n,n-2})$, where $\alpha(T_{n,n-2})$ is the smallest root of the following equation $\lambda^3 - (n+2)\lambda^2 + (3n-2)\lambda - n = 0$. Moreover, equality holds if and only if T is $T_{n,n-2}$.
(iii) If $\beta < n-2$, then $\alpha(T) \le \frac{3-\sqrt{5}}{2}$ with equality if and only if T is $T_{n,\beta}$.

Shanghai Jiao Tong University

Extremal Graphs with Algebraic Connectivity

- Extremal graph theory is a branch of graph theory. One is interested in relations between the various graph invariants, such as order, size, connectivity, chromatic number, diameter and eigenvalues, and also in the values of these invariants which ensure that the graph has certain properties.
- Given a property $\mathfrak{P}$ and an invariant ψ for a class $\mathcal{H}$ of graphs, how to determine the smallest value m for which every graph G in $\mathcal{H}$ with $\psi(G) > m$ has property $\mathfrak{P}$ and with $\psi(G) = m$ are called the extremal graphs for the problem.
- In other words, for given an invariant ψ for a class $\mathcal{H}$ of graphs, determine all graphs with the maximum (minimum) values in $\mathcal{H}$.

Shanghai Jiao Tong University

Extremal Graphs with Algebraic Connectivity

Theorem 22

(Fallat and Kirkland 1998) Let $\mathcal{T}_{n,d}$ be the set of all trees of order n with diamter d. The extremal trees that has the minimum (maximum) algebraic connectivity in $\mathcal{T}_{n,d}$ are unique. Moreover, this tree is obtained by identifying one pendant vertex of a path P_{d-1} of order $d-1$ and the center of $K_{1,\lfloor\frac{n-d+1}{2}\rfloor}$, and the other pendant vertex of P_{d-1} and the center of $K_{1,\lceil\frac{n-d+1}{2}\rceil}$ (by identifying one pendant vertex of a path P_{d-1} of order $d-1$ and the center of $K_{1,n-d+1}$).

Extremal Graphs with Algebraic Connectivity

- The girth of a graph G is the length (number of vertices, or edges) of the shortest cycle in G.
- Let $\mathcal{H}_{n,g}$ be the set of all connected graphs of order n and girth g.
- How to determine extremal graphs with the maximum (minimum) algebraic connectivity in $\mathcal{H}_{n,g}$.

Extremal Graphs with Algebraic Connectivity

Conjecture 23

(Fallat and Kirkland 1998) The extremal graphs with the minimum algebraic connectivity in $\mathcal{H}_{n,g}$ are only lollipop graph $G_{n,g}$ that is obtained from g-cycle with a path of length $n-g$ joined at exactly one vertex on the cycle.

They prove a part result of this conjecture.

Theorem 24

(Fallat and Kirkland 1998) The extremal graphs with the minimum algebraic connectivity in $\mathcal{H}_{n,3}$ are only graph G that is obtained from 3-cycle with a path of length $n-3$ joined at exactly one vertex on the cycle.

上海交通大学 Shanghai Jiao Tong University

Extremal Graphs with Algebraic Connectivity

Guo 2008 proves this conjecture.

Theorem 25

(Guo 2008) The extremal graphs with the minimum algebraic connectivity in $\mathcal{H}_{n,g}$ are only graph G that is obtained from g-cycle with a path of length $n-g$ joined at exactly one vertex on the cycle.

Fallat and Kirkland (1998) pointed out that determination of the graph on n vertices with fixed girth g that maximizes the algebraic connectivity appears to be more difficult.
Now there are part results.

上海交通大学 Shanghai Jiao Tong University

Extremal Graphs with Algebraic Connectivity

Let $\mathcal{U}_{n,g}$ be the set of all unicyclic graphs of order n and girth g.

Theorem 26

(Fallat and Kirkland 1998) The extremal graphs with the maximum algebraic connectivity in $\mathcal{U}_{n,3}$ are the graph $G_{n,3}$ that is obtained by taking a 3-cycle and appending $n-3$ pendant vertices to a single vertex on the cycle.

Theorem 27

(Fallat, Kirkland and Pati 2003)Fixed a girth g, there exists an N such that if $n > N$, then the extremal graphs with the maximum algebraic connectivity in $\mathcal{U}_{n,g}$ is $G_{n,g}$. In particular, for $g = 4$, the conjecture holds.

Extremal Graphs with Algebraic Connectivity

- Let $\mathcal{M}_{n,m}$ be all connected graphs of given order n and size m.
- (Biyikoglu and Leydold 2012) How to determine the extremal graphs with the maximum (minimum) algebraic connectivity in $\mathcal{M}_{n,m}$?
- The problem seems to be more difficult.
- For $n-1 \le m \le \frac{n(n-1)}{2} - 2$, there exists a $1 \le t \le n-2$ such that
$$\frac{(n-t)(n-t-1)}{2} + t \le \frac{(n-t)(n-t-1)}{2} + n - 2.$$
- Then $m = \frac{(n-t)(n-t-1)}{2} + t + p$, where $1 \le t \le n-2$, $1 \le p \le n-t-1$.

Outline Introduction Two Main techniques Bounds Extremal Random Reference

Extremal Graphs with Algebraic Connectivity

- A graph of order n with size m such that
$$\frac{(n-t)(n-t-1)}{2} + t \le m \le \frac{(n-t)(n-t-1)}{2} + n - 2$$
is called (n,p,t) path-complete graph, denoted $PC_{n,p,t}$ if and only if
(1) the maximal clique of $PC_{n,p,t}$ is K_{n-t}.
(2) has a path of order $P_{t+1} = \{v_0, v_1, v_2, \cdots, v_t\}$ such that $v_0 \in K_{n-t} \bigcap P_{t+1}$ and v_1 is joined to K_{n-t} by p edges;
(3) there are no other edges.

Shanghai Jiao Tong University

Extremal Graphs with Algebraic Connectivity

Conjecture 28

(Belhaiza, Abreu, Hansen and Oliveira 2005) The extremal graphs with the minimum algebraic connectivity in $\mathcal{M}_{n,m}$ for $n-1 \leq m \leq \frac{n(n-1)}{2} - 1$ are all path-complete graphs.

Conjecture 29

(Belhaiza, Abreu, Hansen and Oliveira 2005) For each $n > 3$, the minimum algebraic connectivity of a graph G with n vertices and m edges is an increasing, piecewise concave function of m. Moreover, each concave piece corresponds to a family of path-complete graphs. Finally, for $t = 1$, $\alpha(G) = \delta(G)$, and for $t > 2, \alpha(G) \leq 1$.

Outline Introduction Two Main techniques Bounds Extremal Random Reference

Shanghai Jiao Tong University

Extremal Graphs with Algebraic Connectivity

Theorem 30

(Belhaiza, Abreu, Hansen and Oliveira 2005) For all $\frac{(n-1)(n-2)}{2} \leq m \leq \frac{n(n-1)}{2}$, the extremal graphs G with the maximum algebraic connectivity in $\mathcal{M}_{n,m}$ has the property that the complement of G is the disjoint union of triangles K_3, paths P_3, edges K_2 and isolated vertices K_1.

- Biyikoglu and Leydold (2013)investigate the structure of connected graphs of given size and order that have minimal algebraic connectivity.
- How about other value of size?

Extremal Graphs with Algebraic Connectivity

- In 1941,Turán determined the maximal number of edges of a graph G which does not contain a copy of the complete graph K_{r+1}, which started the research of the extremal theory of graphs.
- Let $T_{n,r}$, called *Turán graph*, be the complete r-partite graph of order n, and the size of every class of which is $\lceil \frac{n}{r} \rceil$ or $\lfloor \frac{n}{r} \rfloor$.

Theorem 31

(Turan 1941) Let G be a graph of order n not containing K_{r+1}. Then $e(G) \le e(T_{n,r})$ with equality holding if and only if $G = T_{n,r}$, where $e(G)$ is the number of edges in G.

Outline Introduction Two Main techniques Bounds Extremal Random Reference

Extremal Graphs with Algebraic Connectivity

- Erdős and Stone (1946), and Erdős and Simonovits (1966) expanded the above results.
- Let $\mathcal{H}$ be the set of graphs and $\chi(H)$ be the chromatic number of H, and let $\psi(\mathcal{H}) = min\{\chi(H) | H \in \mathcal{H}\} - 1$

Theorem 32

(Erdős-Stone-Simonovits theorem) Let $ex(n, \mathcal{H})$ be the maximum number of edges of a graph with order n not containing a copy of any graph in $\mathcal{H}$. If $\psi(\mathcal{H}) > 1$, then

$$\lim_{n\to\infty} \frac{ex(n, \mathcal{H})}{\binom{n}{2}} = 1 - \frac{1}{\psi(\mathcal{H})}.$$

上海交通大学 Shanghai Jiao Tong University

Extremal Graphs with Algebraic Connectivity

- Are there similar results for the algebraic connectivity?
- Yes. There is an analogy for Erdős-Stone-Simonovits theorem in spectral graph theory.
- Characterize all graphs of order n not containing a complete subgraph K_r which have the maximum and minimum algebraic connectivity.

Outline Introduction Two Main techniques Bounds Extremal Random Reference

上海交通大学 Shanghai Jiao Tong University

Extremal Graphs with Algebraic Connectivity

Theorem 33

(Jin and Z 2013) Let $\alpha(n, \mathcal{H})$ be the largest algebraic connectivity of graphs of order n without containing a copy of any graph H in $\mathcal{H}$. Then

$$\lim_{n\to\infty} \frac{\alpha(n,\mathcal{H})}{n} = 1 - \frac{1}{\psi(\mathcal{H})},$$

where $\psi(\mathcal{H}) = min\{\ \chi(H) |\ \ H \in \mathcal{H}\ \} - 1.$

上海交通大学
Shanghai Jiao Tong University

Extremal Graphs with Algebraic Connectivity

Theorem 34

(Jin znad Z 2013) Let G be a non-complete graph of order n not containing K_{r+1}. Then

$$\alpha(G) \leq n - \lceil \frac{n}{r} \rceil = \alpha(T_{n,r}), \tag{1}$$

where $\lceil a \rceil$ is the least integer no less than a. Moreover, if $n = kr$ or $n = kr + r - 1$, then equality (1) holds if and only if G is Turán graph $T_{n,r}$. If $n = kr + t, 0 < t < r - 1$, then equality (1) holds if and only if there exist graphs $H_1, \dots, H_t$ of order $k + 1$ with no edges and H of order $n - (k+1)t$ not containing K_{r+1-t} such that $G = H_1 \vee H_2 \cdots \vee H_t \vee H$ and $\alpha(H) \geq n - (k+1)(t+1)$.

Outline Introduction Two Main techniques Bounds Extremal Random Reference

上海交通大学
Shanghai Jiao Tong University

Extremal Graphs with Algebraic Connectivity

Theorem 35

(Jin and Z 2013) Let G be a connected graph with the clique number $r \geq 2$. Then

$$\alpha(G) \geq \alpha(Ki_{n,r}), \tag{2}$$

where $Ki_{n,r}$ is a kite graph of order n which is obtained by adding a pendant path of length $n - r$ to a vertex of K_r. Moreover, equality (2) holds if and only if $G = Ki_{n,r}$.

上海交通大学 Shanghai Jiao Tong University

Random graphs

- G be an ER random graph: For labeling n vertices $\{v_1, \cdots, v_n\}$, the probability that two vertices v_i and v_j are adjacent is p and each edge is independent.

Theorem 36

(Juhasz 1991) Let G be an ER random graph of order n with the probability p. For any $\varepsilon > 0$, we have

$$\alpha(G) = pn + o(n^{1/2+\varepsilon}), \text{ in probability.}$$

Outline Introduction Two Main techniques Bounds Extremal Random Reference

上海交通大学 Shanghai Jiao Tong University

Random graphs

Theorem 37

(Gu, Z and Zhou 2010) Let $\mathcal{S}(n, c, k)$ be the small-world network with n nodes, which is a union of an Erdös-Réyni random graph $\mathcal{G}(n, \frac{c}{n})$ and a $2k$ regular cycle. Then the algebraic connectivity of $\mathcal{S}(n, c, k)$ is almost surely bounded below by

$$\frac{k^2c^2 \log\log n}{2(k+1)^2 \log^3 n}. \tag{3}$$

Shanghai Jiao Tong University

Random graphs

Olfati and Saber (2008) defined

$$\gamma_2(n,c,k) = \frac{\lambda_2(n,c,k)}{\lambda_2(n,0,k)} \tag{4}$$

to be the *algebraic connectivity gain* of $\mathcal{S}(n,c,k)$.

Theorem 38

(Gu, Z and Zhou 2010) The algebraic connectivity gain of the small-world network $\mathcal{S}(n,c,k)$ follows almost surely inequality

$$\gamma_2(\mathcal{S}(n,c,k)) \geqslant \frac{3kc^2n^2 \log\log n}{2(k+1)^3(2k+1)\pi^2 \log^3 n}. \tag{5}$$

Outline Introduction Two Main techniques Bounds Extremal Random Reference

Shanghai Jiao Tong University

Random graphs

- The above results give a mathematical rigorous estimation of the lower bound for the algebraic connectivity of the small-world networks, which is much larger than the algebraic connectivity of the regular circle.
- This result explains why the consensus problems on the small-world network have a ultrafast convergence rate and how much it can be improved.
- It also characterizes quantitatively what kind of the small-world networks can be synchronized.

上海交通大學
Shanghai Jiao Tong University

N. M. M. de Abreu
Old and new results on algebraic connectivity of graphs,
Linear Algebraic and its Applications, **423**(2007) 53-73.

S. Belhaiza, N. M. M. de Abreu, P. Hansen, C. S. Oliveira
Variable neighborhood search for extremal graphs 11. Bounds on algebraic connectivity,
in: O. Marcotte, D. Avis, A. Hertz (Eds.), Graph Theory and Combinatorial Optimization, Springer, 2005, pp. 1 16.

S. Fallat and S. Kirkland,
Extremizing algebraic connectivity subject to graph theoretic constraints,
The Electronic Journal of Linear Algebra, **3**(1998), 48-74.

Outline Introduction Two Main techniques Bounds Extremal Random Reference

上海交通大學
Shanghai Jiao Tong University

M.Fiedler
Algebraic connectivity of graph,
Czechoslovake Mathematical Journnal, **23(98)**(1973), 298-305.

J. M. Guo,
A conjecture on the algebraic connectivity of connected graphs with fixed girth,
Discrete Math., 308 (2008) 5702 5711.

J. M. Guo,
The algebraic connectivity of graphs under perturbation,
Linear Algebra Appl., 433 (2010) 1148 1153.

上海交通大学 Shanghai Jiao Tong University

- L. Gu, X. D. Zhang and Q. Zhou,
  Consensus and synchronization problems on small-world networks,
  Journal of Mathematical Physics, 51 (2010) 082701.
- S. Kirkland, M. Neumann and B. Shader,
  Characteristic vertices of weighted trees via Perron values,
  Linear and Multilinear Algebra, 40 (1996), 311-325.
- S. Kirkland and M. Neumann,
  Algebraic connectivity of weighted trees under perturbation,
  Linear and Multilinear Algebra, **42**(1997), 187-203.

Clique minors, chromatic numbers for degree sequence

Katsuhiro Ota

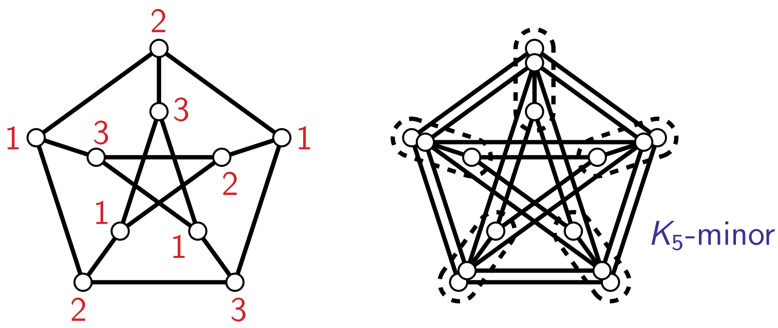
Department of Mathematics, Keio University

Joint work with G. Chen and R. Hazama

Chromatic number and Hadwiger number

Let G be a graph. (We only consider *simple* graphs.)

- $\chi(G)$: the chromatic number of G.
- $h(G)$: maximum size of clique minors in G, called the Hadwiger number of G.



$\chi(G) = 3$ $h(G) = 5$

Hadwiger's conjecture

Hadwiger's Conjecture (1943)

Every graph with chromatic number k has a K_k-minor.
(Equivalently, $\forall G,\ h(G) \geq \chi(G)$.)

Hadwiger's conjecture

- was proved for $k = 4$ by Dirac (1952);
- for $k = 5$ implies Four Color Theorem (FCT);
- is affirmative for $k = 5$ by FCT and Wagner (1937);
- is affirmative for $k = 6$ by Robertson, Seymour and Thomas (1993) using FCT;
- is open for $k \geq 7$.

Hadwiger's conjecture for degree sequences

Let $D = (d_1, d_2, \ldots, d_n)$ be a degree sequence of a graph.

- $\chi(D) := \max\{\chi(G) \ :\ G \text{ has deg. seq. } D\}$.
- $h(D) := \max\{h(G) \ :\ G \text{ has deg. seq. } D\}$.

Robertson and Song (2009) posed:

Hadwiger's Conjecture for Degree Sequences

For every degree sequence D, $h(D) \geq \chi(D)$ holds.

- If Hadwiger's conjecture is true, then Hadwiger's conjecture for degree sequences is also true.

Hadwiger's conjecture for degree sequences

Hadwiger's Conjecture for Degree Sequences

For every degree sequence D, $h(D) \geq \chi(D)$ holds.

Theorem (Robertson, Song 2009)

Hadwiger's conjecture for degree sequences is true for all near regular degree sequences.

A degree sequence $D = (d_1, d_2, \ldots, d_n)$ is said to be near regular if $\max_i\{d_i\} - \min_i\{d_i\} \leq 1$.

Dvořák and Mohar have proved!

Recently, Hadwiger's Conjecture for Degree Sequences was confirmed by showing a stronger statement.

Theorem (Dvořák, Mohar 2012+)

For every degree sequence D, $h'(D) \geq \chi(D)$ holds.

- $h'(D) := \max\{h'(G) : G \text{ has deg. seq. } D\}$.
- $h'(G)$: maximum k such that G has a *topological K_k-minor*.
- A topological K_k-minor of a graph is a subgraph isomorphic to a subdivision of K_k.
- Note: $h(G) \geq h'(G)$, and hence $h(D) \geq h'(D)$.

Note on $h'(G)$: Hajós' number

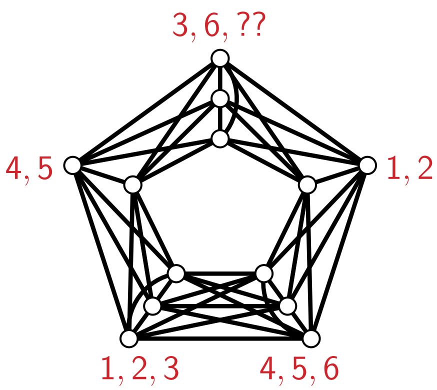
(known as) Hajós' Conjecture

$\forall G,\ h'(G) \geq \chi(G)$.
(Every graph with $\chi = k$ has a topological K_k-minor.)

Hajós' conjecture

- implies Hadwiger's conjecture, since $h(G) \geq h'(G)$;
- is true for $k \leq 4$ by Dirac (1952);
- for $k = 5$ implies Four Color Theorem (FCT);
- is false for $k \geq 7$ by Catlin (1979);
- is false for almost all graphs, by Erdős and Fajtlowicz (1981);
- is open for $k = 5, 6$.

Counterexample to Hajós conjecture



$$\chi(G) = 7 \quad > \quad h'(G) = 6.$$

Hajós' conjecture

(known as) Hajós' Conjecture

$\forall G$, $h'(G) \geq \chi(G)$.
(Every graph with $\chi = k$ has a topological K_k-minor.)

Hajós' conjecture

- implies Hadwiger's conjecture, since $h(G) \geq h'(G)$;
- is true for $k \leq 4$ by Dirac (1952);
- for $k = 5$ implies Four Color Theorem (FCT);
- is false for $k \geq 7$ by Catlin (1979);
- is false for almost all graphs, by Erdős and Fajtlowicz (1981);
- is open for $k = 5, 6$.

2013.1.26 Hakata

Katsuhiro Ota | Clique minors, chromatic numbers for degree sequence

Dvořák and Mohar's Result

Theorem (Dvořák, Mohar 2013+)

For every degree sequence D, $h'(D) \geq \chi(D)$ holds.

- Their proof involves a lot, and is complicated.
- They did not determine the exact values of $h'(D)$ or $\chi(D)$.

We shall give

1. an alternative and very short proof of $h(D) \geq \chi(D)$; (Unfortunately, our argument does not work for proving $h'(D) \geq \chi(D)$ so far.)
2. the exact values of $h'(D)$ for near regular case;
3. a good bound for $\chi(D)$ for (near) regular case;

2013.1.26 Hakata

Katsuhiro Ota | Clique minors, chromatic numbers for degree sequence

Some Remarks on Hajós' Numbers and Chromatic Numbers for Degree Sequences

Observations

Suppose $D = (d_1, d_2, \ldots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n$.

- If $h'(D) = k$, then we have $d_k \geq k-1$. This means:

$$h'(D) \;\leq\; \max\{k \mid d_k \geq k-1\}.$$

  - Note that, $h(D)$ can be as large as $\sqrt{n}$ even when $d_1 = \cdots = d_n = 3$.

- We can greedily color the graph with the degree sequence D using at most $\max\{k \mid d_k \geq k-1\}$ colors.
- So if the equality $h'(D) = \max\{k \mid d_k \geq k-1\}$ holds, then we conclude $h'(D) \geq \chi(D)$ as required.
- However, this is *not true* in general.

Results for regular degree sequences

- $D = (d, d, \ldots, d) = (d^n)$, $(0 \le d \le n-1,\ dn : \text{even})$.
- $\overline{d} := n - 1 - d$.

Theorem 1

$$h'(D) = \begin{cases} d+1 & \text{if } d \le (n-1)/2; \\ \left\lfloor \left(\frac{1}{2} + \frac{1}{2\overline{d}+2}\right) n \right\rfloor & \text{if } d > (n-1)/2. \end{cases}$$

Theorem 2

$$\chi(D) \le \begin{cases} d+1 & (\text{if } d \le (n-1)/2); \\ \left\lfloor \left(\frac{1}{2} + \frac{1}{4\overline{d}+2}\right) n \right\rfloor & \text{if } d > (n-1)/2. \end{cases}$$

Proof (the upper bound for $h'(D)$)

Show that: $h'(D) \le \left(\frac{1}{2} + \frac{1}{2\overline{d}+2}\right) n$.

- Let G be a d-regular n-vertex graph with $h'(G) = k$.
- Let X be the set of branch vertices of a top. K_k-minor. $Y := V(G) - X$.
- Let r be the number of nonadjacent pairs in X.
- $e_{\overline{G}}(X, Y) = \sum_{x \in X} d_{\overline{G}}(x) - 2r = \overline{d}|X| - 2r = \overline{d}k - 2r$.
- $e_{\overline{G}}(X, Y) \le \sum_{y \in Y} d_{\overline{G}}(y) = \overline{d}|Y| = \overline{d}(n-k)$.
- There are at least r subdividing vertices in Y, hence $r \le |Y| = n - k$.

$$\overline{d}(n-k) \ge \overline{d}k - 2r \ge \overline{d}k - 2(n-k),$$
$$(2\overline{d} + 2)k \le (\overline{d} + 2)n. \qquad \square$$

Exact value of $h'(D)$ for near regular case

- $D = ((d+1)^p, d^{n-p})$
  $(0 \leq d \leq n-1,\ 0 \leq p \leq n-1,\ dn + p :$ even$)$.
- $\overline{d} := n - 1 - d$.

Theorem 3

$$h'(D) = \begin{cases} d+2 & \text{if } d \leq \frac{n-2}{2} \text{ and } p \geq d+2; \\ d+1 & \text{if } d \leq \frac{n-2}{2} \text{ and } p \leq d+1; \\ \left\lfloor \frac{(\overline{d}+2)n+p}{2\overline{d}+2} \right\rfloor & \text{if } d \geq \frac{n-1}{2} \text{ and } p \leq \frac{(\overline{d}+2)n}{2\overline{d}+1}; \\ \left\lfloor \frac{(\overline{d}+2)n-p}{2\overline{d}} \right\rfloor & \text{if } d \geq \frac{n-1}{2} \text{ and } p > \frac{(\overline{d}+2)n}{2\overline{d}+1}. \end{cases}$$

A Short Proof of Hadwiger's Conjecture for Degree Sequences

Definition: k-CDP

- Let $(V_1, V_2, \ldots, V_k)$ be a partition of $V(G)$.
- $(V_1, V_2, \ldots, V_k)$ is said to be a connected dominating partition of size k (k-CDP for short)

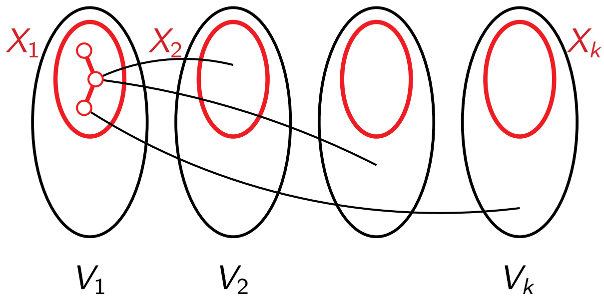
  if for $1 \leq \forall i \leq k$, $\exists X_i$: a connected component of $G[V_i]$ such that $E(X_i, V_j) \neq \emptyset$ for every $j \neq i$.
- The CDP number of G:

$$\rho(G) := \max\{k \mid G \text{ has a } k\text{-CDP}\}.$$

2013.1.26 Hakata

Katsuhiro Ota Clique minors, chromatic numbers for degree sequence

Definition: k-CDP



2013.1.26 Hakata

Katsuhiro Ota Clique minors, chromatic numbers for degree sequence

Observations on CDP number

Proposition 1

$\forall G,\ \chi(G) \leq \rho(G)$.

Proof:

- Let $k = \chi(G)$, and let $V_1, \ldots, V_k$ be the color classes.
- Then, for each i, $\exists x_i \in V_i$ s.t. $E(x_i, V_j) \neq \emptyset$ for $\forall j$, for otherwise we can recolor all vertices of V_i without using color i.
- Put $X_i = \{x_i\}$, then we obtain a k-CDP $(V_1, \ldots, V_k)$. $\square$

Observations on CDP number

Proposition 1

$\forall G,\ \chi(G) \leq \rho(G)$.

Proposition 2

$\forall G,\ h(G) \leq \rho(G)$.

Proof:

- Let $k = h(G)$, and let $X_1, \ldots, X_k$ be disjoint sets of vertices such that the contraction of X_i into v_i $(1 \leq i \leq k)$ yields a complete graph on $\{v_1, \ldots, v_k\}$.
- Expand each X_i into V_i to obtain a partition $(V_1, \ldots, V_k)$ of $V(G)$, which is a k-CDP of G. $\square$

Observations on CDP number

Proposition 1

$\forall G,\ \chi(G) \le \rho(G)$.

Proposition 2

$\forall G,\ h(G) \le \rho(G)$.

- $\rho(D) := \max\{\rho(G) : G \text{ has deg. seq. } D\}$.

Corollary

$\forall D,\ \chi(D) \le \rho(D)$ and $h(D) \le \rho(D)$.

Main Theorem

Theorem 4

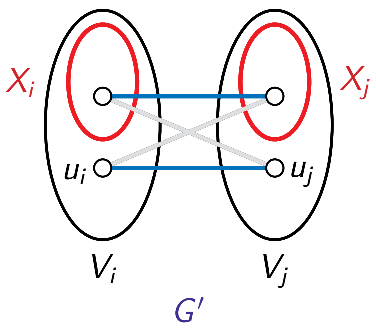
$\forall D,\ h(D) = \rho(D)$. Consequently, $\chi(D) \le h(D)$.

Proof (1/2)

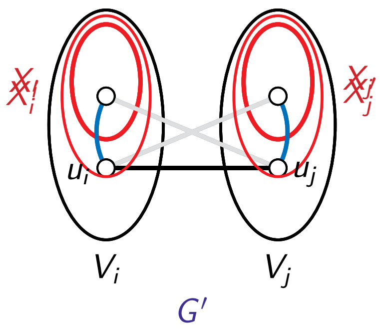
- We need to prove that $h(D) \geq \rho(D)$.
- Let $k = \rho(D) = \rho(G)$.
- Let $(V_1, \dots, V_k)$ be a k-CDP, with a conn. cpt. X_i in V_i.
- If $E(X_i, X_j) \neq \emptyset$ for all pairs i, j, then by contracting each X_i into a single vertex, we obtain a K_k. Thus,
$$h(D) \geq h(G) \geq k = \rho(D).$$
- Otherwise, $E(X_i, X_j) = \emptyset$ for some i, j.

Proof (2/2)

Case 1: $u_i u_j \notin E(G)$



Case 2: $u_i u_j \in E(G)$



Open Problems

- Give a short proof of $h'(D) \geq \chi(D)$.
- Determine $h'(D)$ for all degree sequences D, or give an algorithm determining $h'(D)$ for given D.
- Give a better upper bound for $\chi(D)$ for (near) regular degree sequences D.
  Our bound $\chi(D) \leq \left\lfloor \left(\frac{1}{2} + \frac{1}{4\bar{d}+2}\right) n \right\rfloor$ for regular degree sequences is sharp for $d \in \{n-1, n-3, n/2\}$.
- Consider $\min\{h(G)\}$, $\min\{h'(G)\}$ and $\min\{\chi(G)\}$ of the graphs with a given degree sequence.

The path-distance-width of hypercubes

Yota Otachi

Japan Advanced Institute of Science and Technology

Hakata Workshop 2013. January 26

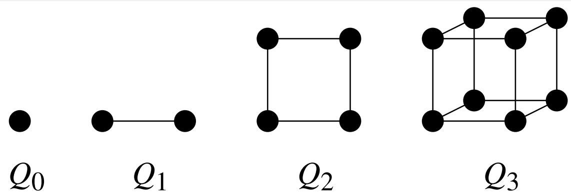
Outline

Hypercube

Definition (Hyercube)

The *d-dimensional hypercube* Q_d, or d-cube, is the graph with

- $V(Q_d)$ = the set of all binary strings of length d,
- $E(Q_d)$ = the pairs of strings of Hamming distance 1.



Properties of hypercubes

- $|V(Q_d)| = 2^d$, $|E(Q_d)| = d \cdot 2^{d-1}$ $\quad$ ($\because$ Q_d is d-regular)
- diameter of Q_d is d $\quad$ (attained by 0^d and 1^d)

We study a width parameter of hypercubes.

Width parameters of graphs

Several width parameters are studied in algorithmic graph theory

Parameters from industrial applications

- Band-width: minimizing maximum dilation in a linear circuit
- Cut-width: minimizing maximum congestion in a linear circuit

Graph minor related parameters

- Tree-width: measures how close a graph is to a tree
- Path-width: measures how close a graph is to a path
- Branch-width, Carving-width, etc.

Many intractable graph problems become easy for graphs of bounded width parameters.

Our width parameter

- Path-distance-width: minimizing the width of BFS

Width parameters of hypercubes

Width parameters of hypercubes

Studies on width parameters of hypercubes give better understanding on these parameters. This is because:

- Usually, hypercubes have large width parameters;
- But basic tools (for l.b.) do not work for hypercubes (eg. degree);
- Thus wee need a nice tool for a good lower bound;

Known results

- Cut-width (e) by [Harper (1964)]
- Band-width (e) by [Harper (1966)]
- Path-width (e) and Treewidth (a) by [Chandran & Kavitha (2006)]
- Carving-width (e) by [Chandran & Kavitha (2006)]

e: exact, a: asymptotic

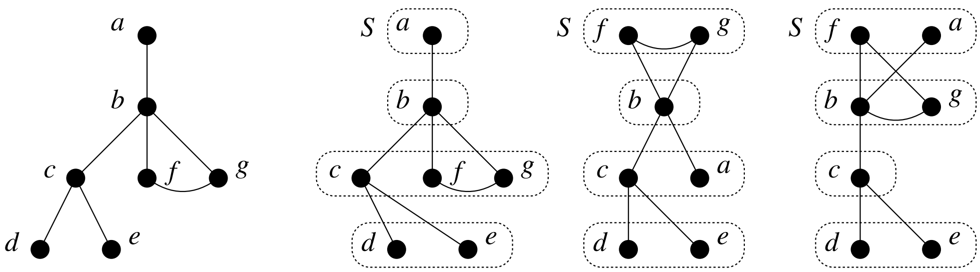
Definition: Path-distance-width (1 of 2)

Definition (Distance structure)

$D(S) = (L_0, \ldots, L_t)$ is a distance structure of G rooted at S if

- $\bigcup_{0 \le i \le t} L_i = V(G)$, and
- $L_i = \{v \in V(G) \mid d(S, v) = i\}$,

where $d(S, v) = \min_{u \in S} d(u, v)$.



Definition: Path-distance-width (2 of 2)

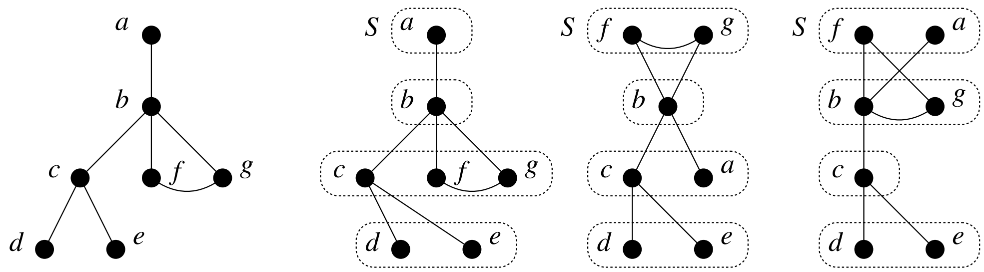
Definition (Path-distance-width)

The *path-distance-width* of G with initial set S, denoted $\mathrm{pdw}_S(G)$, is

$$\mathrm{pdw}_S(G) = \max_{L_i \in D_G(S)} |L_i|.$$

The *path-distance-width* of G is defined as

$$\mathrm{pdw}(G) = \min_{S \subseteq V(G)} \mathrm{pdw}_S(G).$$



$*$ pdw is defined for connected graphs only.

History of path-distance-width

Some algorithmic results are known.

- Introduced to study the graph isomorphism problem [Yamazaki, Bodlaender, de Fluiter, Thilikos (1997)]
  - Determining the pdw of a graph is NP-hard
  - For pdw bounded graphs, GI can be solved in poly time.
- Approximation hardness for trees [Yamazaki (2001)]
- Constant-factor approximation algorithms for graphs with path-like structures [O. *et al.* (2011)]
- An improved algorithm for the graph isomorphism problem [O. (2012)]
  - For a subclass of the class of pdw bounded graphs, GI can be solved in FPT time.

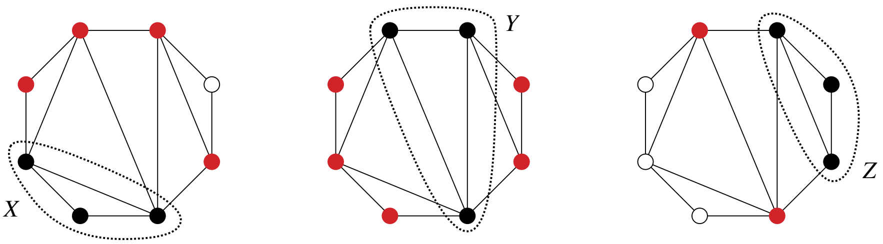
Vertex- and edge-boundaries

Definition (Vertex-boundary)

Let G be a graph, $X \subseteq V(G)$, and $s \in \{1, \dots, |V(G)|\}$.

- Vertex boundary: $\partial_V(X) = \{v \in V(G) \setminus X \mid \exists u \in X, \{u, v\} \in E(G)\}$
- Vertex isoperimetric value: $\partial_V(s) = \min_{|X|=s} |\partial_V(X)|$

$*$ Edge boundry ∂_E is defined analogously.



$|\partial_V(X)| = 4$ $|\partial_V(Y)| = 5$ $|\partial_V(Z)| = 2$

In this example, $\partial_V(3) = 2$.

Isoperimetric value based lower bounds

The following lower bounds are developed to determine the width parameters of hypercubes, but hold for any graphs.

Theorem (Chandran & Subramanian (2005))

tree-width$(G) \geq \min_{s/2 \leq i \leq s} \partial_V(i)$ *for any* $s \leq |V(G)|$.

Theorem (Chandran & Kavitha (2006))

carving-width$(G) \geq \min_{s/2 \leq i \leq s} \partial_E(i)$ *for any* $s \leq |V(G)|$.

We present a lower bound in a similar form:

Theorem

For any w *and* s *with* $w \leq s \leq |V(G)|$,

$$\mathrm{pdw}(G) \geq \min \left\{ w, \min_{s-w \leq i \leq s} \partial_V(i) \right\}.$$

Our reulst

We present a general lower bound on path-distance-width of graphs, and determine the path-distance-width of hypcubes by applying the bound.

Theorem (General lower bound)

For any w and s with $w \le s \le |V(G)|$,

$$\mathrm{pdw}(G) \ge \min\left\{w, \min_{s-w \le i \le s} \partial_v(i)\right\}.$$

Theorem (The path-distance-width of hypercubes)

For any d,

$$\mathrm{pdw}(Q_d) = \binom{d}{\lfloor d/2 \rfloor}.$$

Upper bound

The upper bound can be achieved by taking only one vertex.

Lemma

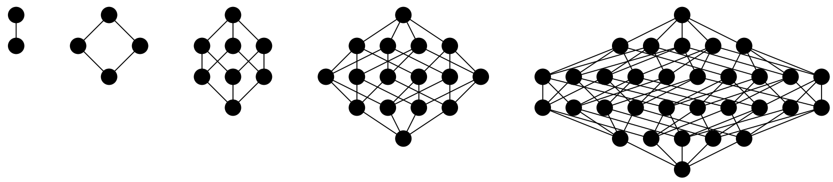
$$\mathrm{pdw}(Q_d) \le \binom{d}{\lfloor d/2 \rfloor}.$$

Proof.

Let $D(\{0^d\}) = (L_0, \ldots, L_t)$ is the distance structure rooted at $\{0^d\}$. Then

$$L_i = \{u \in Q_d \mid i \text{ has exactly } i \text{ non-zero entries}\}.$$

Thus $\max_i |L_i| = \max_i \binom{d}{i} = \binom{d}{\lfloor d/2 \rfloor}$. □



$\mathrm{pdw}(Q_4) \le \binom{4}{\lfloor 4/2 \rfloor} = 6$ $\mathrm{pdw}(Q_5) \le \binom{5}{\lfloor 5/2 \rfloor} = 10$

Naïve lower bounds

We have two general lower bounds. They are too weak for hypecubes.

Lemma

For any connected graph G with minimum degree $\delta(G)$,

$$\mathrm{pdw}(G) \geq (\delta(G)+1)/2.$$

$\implies \mathrm{pdw}(Q_d) \geq (d+1)/2.$

Lemma

For any connected graph G with diameter $\mathrm{diam}(G)$,

$$\mathrm{pdw}(G) \geq |V(G)|/(\mathrm{diam}(G)+1).$$

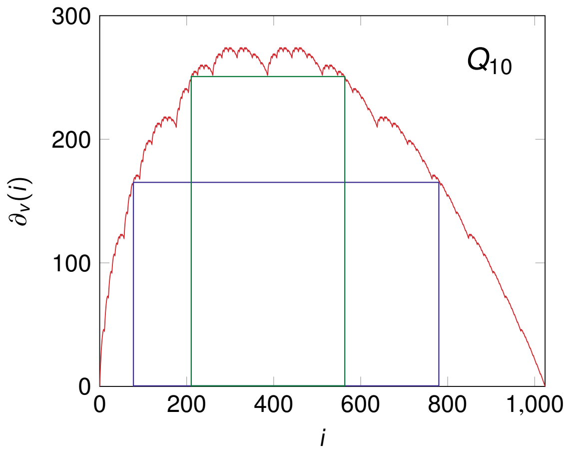
$\implies \mathrm{pdw}(Q_d) \geq 2^d/(d+1).$ $\binom{d}{d/2} \sim 2^d/\sqrt{\pi d/2}$

Lower bound

Theorem

For any w and s with $w \leq s \leq |V(G)|$,

$$\textit{path-distance-width}(G) \geq \min\left\{w, \min_{s-w \leq i \leq s} \partial_V(i)\right\}.$$



Finding a fat rectangle below the red line.

- width $= w$
- height $= \min_{s-w \leq i \leq s} \partial_V(i)$

min{width, height} gives a l.b.

Blue rectangle: 700×160

Green rectangle: 350×250

Proof of the lower bound

Remark

Let $(L_0, \dots, L_t)$ be a distance structure of G. Then, for any $j \le t - 1$,

$$\partial(\bigcup_{i=0}^{j} L_i) = L_{j+1}.$$

Theorem

For any w and s with $w \le s \le |V(G)|$,

$$\textit{path-distance-width}(G) \ge \min\left\{w, \min_{s-w \le i \le s} \partial_v(i)\right\}.$$

Proof.

Let $(L_0, \dots, L_t)$ be a distance structure of G.

- $|\bigcup_{i=0}^{j} L_i| \in \{s - w, \dots, s\}$ for some $j \implies |L_{j+1}| \ge \min_{s-w \le i \le s} \partial_v(i)$.
- Otherwise, $\exists j$ s.t. $|\bigcup_{i=0}^{j} L_i| < s - w$ and $|\bigcup_{i=0}^{j} L_i \cup L_{j+1}| > s$.

□

Isoperimetric ordering for hypercubes

For a binary string u, let pop(u) denote the number of 1's in u.

Definition (Simplicial ordering)

The *simplicial ordering* $\prec$ on the binary strings of length d is an odering such that $u_1 u_2 \dots u_d \prec v_1 v_2 \dots v_d \iff \mathrm{pop}(u) < \mathrm{pop}(v)$, or $\mathrm{pop}(u) = \mathrm{pop}(v)$ and there exists j s.t. $u_j > v_j$ and $u_i = v_i$ for $i < j$.

eg: $000 \prec 100 \prec 010 \prec 001 \prec 110 \prec 101 \prec 011 \prec 111$

Theorem (Harper 1966)

Let X_s be the set of the first s vertices of Q_d in the simplicial order. Then

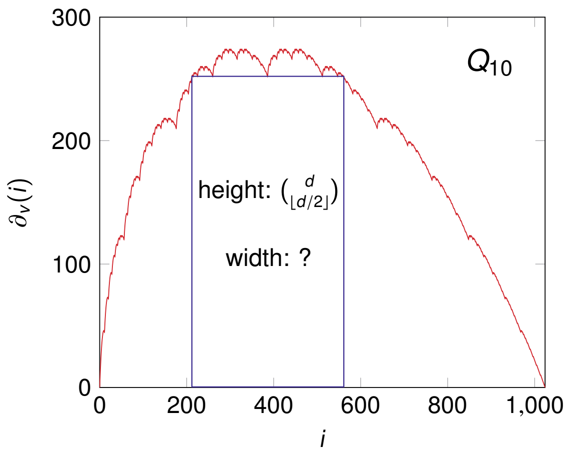
$$\partial_v(s) = |\partial_v(X_s)|.$$

The above theorem is developed to determine the band-width of Q_d

Theorem (Harper 1966 and Wang, Wu, Dumitrescu 2009)

$$\textit{band-width}(Q_d) = \sum_{i=0}^{d-1} \binom{i}{\lfloor i/2 \rfloor}.$$

Ploting isoperimetric values of Q_d



We know: $\mathrm{pdw}(Q_d) \le \binom{d}{\lfloor d/2 \rfloor}$

Task: finding the widest rectangle with height $\binom{d}{\lfloor d/2 \rfloor}$ below the red line.

Proof of the lower bound

Let $s = \sum_{i=0}^{\lfloor d/2 \rfloor} \binom{i}{\lfloor i/2 \rfloor}$ and $w = \binom{d}{\lfloor d/2 \rfloor}$. We can show that, for $s - w \le i \le s$, $\partial_v(i) \ge \binom{d}{\lfloor d/2 \rfloor}$ by exploiting the structure of the simplicial ordering.

There is a shortcut. Let $f(m) = \sum_{i=1}^{m/2} \binom{2i-1}{i}$ for even m.

Theorem (Kleitman 1986)

If d is even and $s \in \{2^{d-1} - \frac{1}{2}\binom{d}{d/2} - f(d) + 1, \ldots, 2^{d-1} + f(d-2)\}$ or d is odd and $s \in \{2^{d-1} - f(d+1) + 1, \ldots, 2^{d-1} + f(d-1)\}$, then

$$\partial_v(s) \ge \binom{d}{d/2}.$$

It suffices to show that these ranges are wide enough
$\Longleftarrow$ just a routine task.

Conclusion

Theorem (General lower bound)

For any w and s with $w \le s \le |V(G)|$, $\mathrm{pdw}(G) \ge \min\left\{w, \min_{s-w \le i \le s} \partial_V(i)\right\}$.

Theorem (The path-distance-width of hypercubes)

For any d, $\mathrm{pdw}(Q_d) = \binom{d}{\lfloor d/2 \rfloor}$.

More applications of the lower bound?

- For grids (Cartesian products of paths), the generalized simplicial ordering gives isoperimetric values.
- For even tori (Cartesian products of cycles of even length), the generalized simplicial ordering gives isoperimetric values.
- For Hamming graphs (Cartesian products of complete graphs), no such ordering is known.

Other (more applicable) lower bounds?

Thank you!

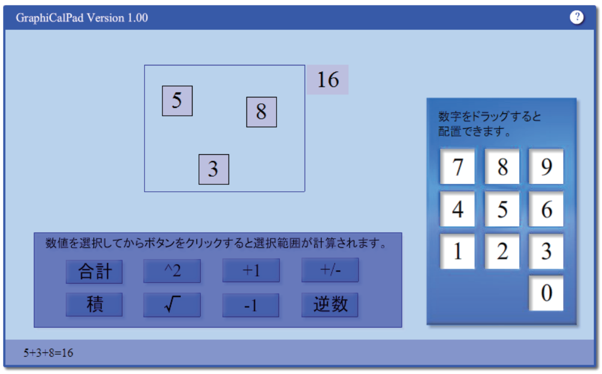
第２部　数学ソフトウェア紹介

GraphiCalPad

岩淵勇樹
http://butchi.jp/

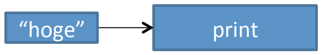
GraphiCalPad Version 1.00

- http://butchi.jp/documents/graphicalpad

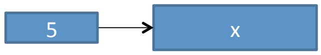


展望

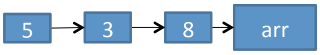
- 矢印による操作
  - 代入
  - ベクトル化
  - 関数入力
  - 配列アクセス



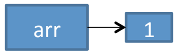
print関数の引数を矢印で渡す



変数xに5を代入する



配列変数arrに[5, 3, 8]を代入する



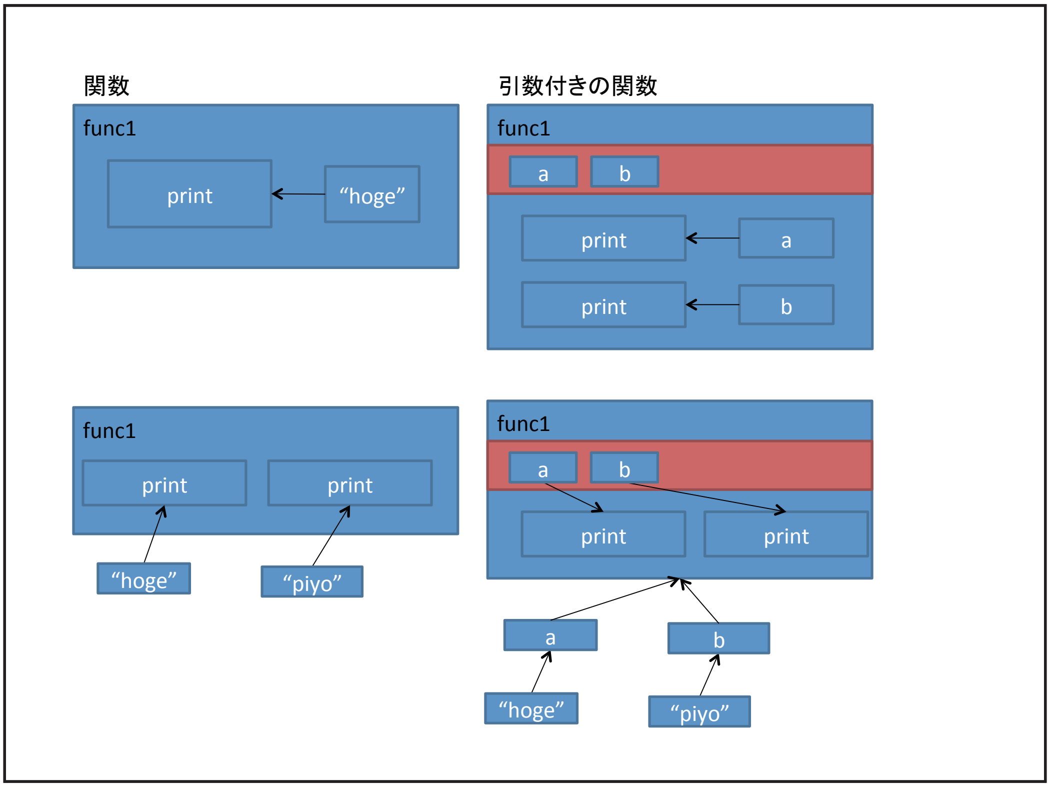
配列変数arrの
1番目の要素にアクセスする

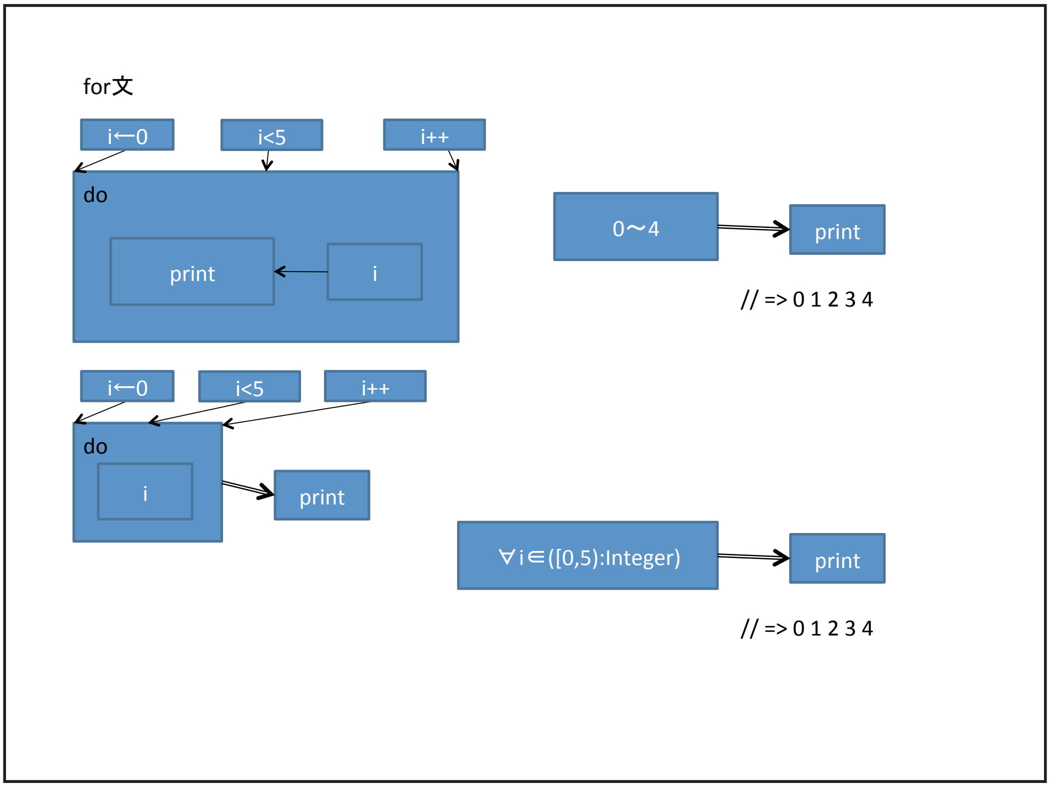
展望

- ボックスの入れ子による関数入力

- プレーンテキストでの入力

- すべて同一パッド内で併用可能


関数
引数付きの関数
func1
print
“hoge”
func1
a
b
print
a
print
b
func1
print
print
“hoge”
“piyo”
func1
a
b
print
print
a
b
“hoge”
“piyo”


for文
i←0
i<5
i++
do
print
i
0～4
print
// => 0 1 2 3 4
i←0
i<5
i++
do
i
print
∀i∈([0,5):Integer)
print
// => 0 1 2 3 4

正規圧縮距離を用いたクラスタリング

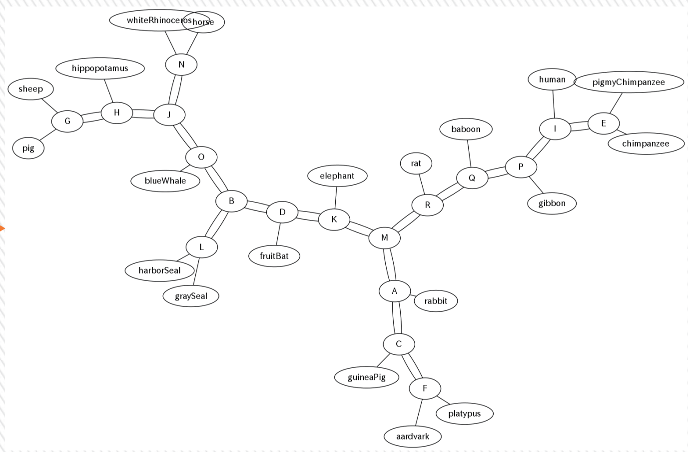
九州大学 理学部 物理学科 情報理学コース
久保 浩平

2013年1月26日 13:00~14:00

	hippopotamus	Pigmy Chimpanzee	sheep	gibbon
hippopotamus	0.475728	0.964223	0.946837	0.958763
Pigmy Chimpanzee	0.964223	0.471247	0.963565	0.928932
sheep	0.946837	0.963565	0.459139	0.965563
gibbon	0.958763	0.928932	0.965563	0.468304

….

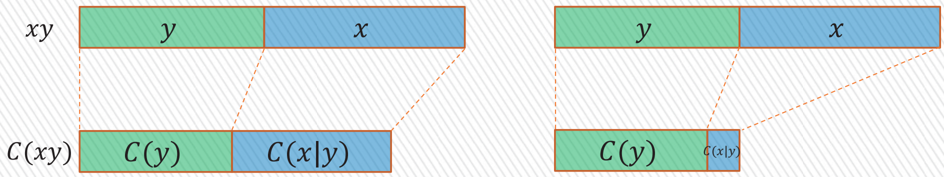
⋮



$$\boldsymbol{NCD(x,y)} = \frac{\max\{C(x|y), C(y|x)\}}{\max\{C(x), C(y)\}} = \frac{C(yx) - \min\{C(x),\ C(y)\}}{\max\{C(x), C(y)\}}$$

もしxとyが似ていなかったら...

もしxとyが似ていたら...



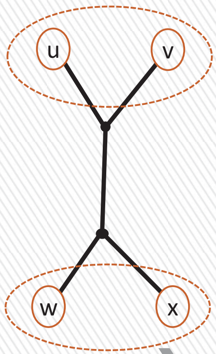
NCD

3

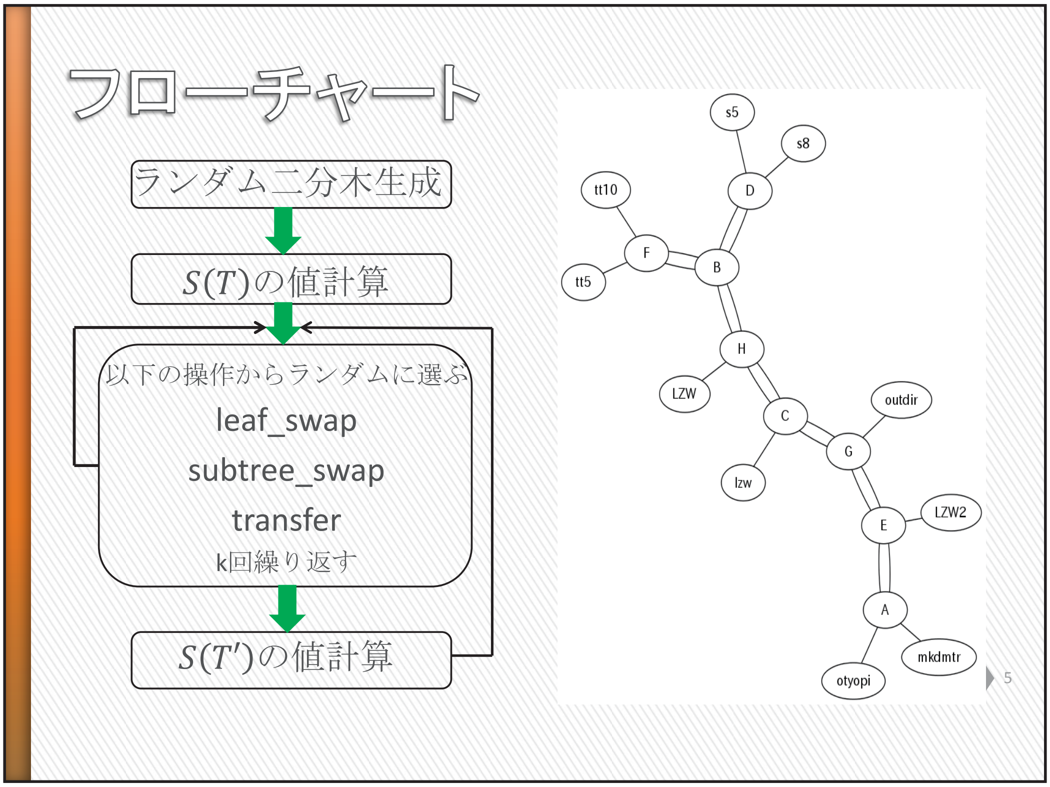
$$minimize\ C_T = \sum_{\{u,v,w,x\} \subseteq N} \{C_{uv|wx} : uv|wx\text{が}consistent\}$$

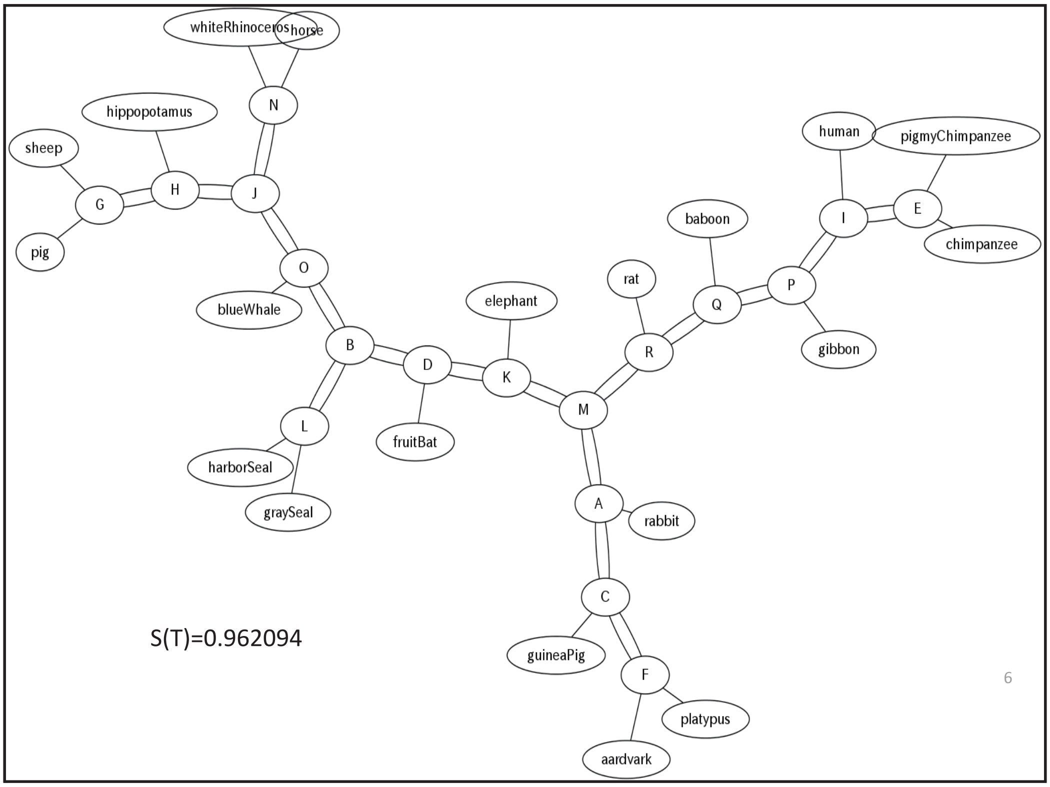
> consistentと呼ばれる木の部分構造のコストを全て足し合わせたもの。

- $C_{uv|wx} = NCD(u,v) + NCD(w,x)$



目的関数


フローチャート
ランダム二分木生成
S(T)の値計算
以下の操作からランダムに選ぶ
leaf_swap
subtree_swap
transfer
k回繰り返す
S(T')の値計算
s5
s8
tt10
D
F
B
tt5
H
LZW
C
outdir
G
lzw
E
LZW2
A
mkdmtr
otyopi
5


whiteRhinoceros
horse
N
hippopotamus
sheep
G
H
J
pig
O
blueWhale
B
D
K
M
elephant
fruitBat
L
harborSeal
graySeal
A
rabbit
C
guineaPig
F
platypus
aardvark
R
rat
Q
baboon
P
gibbon
I
human
pigmyChimpanzee
E
chimpanzee
S(T)=0.962094
6

MAGMA

Osamu Shimabukuro

Sojo University

2013.01.26

崇城大学

http://www.ed.sojo-u.ac.jp/~osamu/magma/

1.Association Scheme

Let X be a finite set, $\{R_i\}_{0\leq i\leq d}$ a collection of binary relations of X. R_i is a subset of $X\times X$. Assume that $\{R_i\}_{0\leq i\leq d}$ is a partition of $X\times X$. We say that $\mathfrak{X}=(X,\{R_i\}_{0\leq i\leq d})$ is an *association scheme, if the following conditions are satisfied.*

1. *(i)* $R_0 := \{(x,x)|x\in X\}$,
2. *(ii)* $X\times X = R_0\cup R_1\cup\cdots\cup R_d$ *and* $R_i\cap R_j=\phi$ $(i\neq j)$,
3. *(iii) For* $i\in\{0,1,\ldots,d\}$, *there exists* $i'\in\{0,1,\ldots,d\}$ *such that* ${}^tR_i=R_{i'}$ *where* ${}^tR_i=\{(y,x)|(x,y)\in R_i\}$,
4. *(iv) For* $f,g,h\in\{0,1,\ldots,d\}$, $\exists p_{f,g}^h\in\mathbb{Z}_{\geq 0}$ *such that* $p_{f,g}^h=|\{z\in X|(x,z)\in R_f,(z,y)\in R_g\}|$ *(intersection number) whenever* $(x,y)\in R_h$.

Example. Johnson Scheme

Let M be a finite set $|M| = m$ For n $(0 < n \leq m/2)$, define a set $\binom{M}{n} = \{N \subset M \mid |N| = n\}$ For any $N_1, N_2 \in \binom{M}{n}$, we can define a Johnson distance $\rho(N_1, N_2) = n - |N_1 \cap N_2|$ Relations are defined $R_i = \{(N_1, N_2) \in \binom{M}{n}^2 \mid \rho(N_1, N_2) = i\}$. $\mathfrak{X} = (\binom{M}{n}, \{R_i\}_{0 \leq i \leq n})$ is an association scheme We write $J(m, n)$

1
http://www.ed.sojo-u.ac.jp/~osamu/magma/

Adjacency matrices

For $i \in \{0, 1, \ldots, d\}$, we define the *adjacency matrix* A_i *of the* i*-th relation as follows.*

$$(A_i)_{x,y} = \begin{cases} 1 & (x, y) \in R_i, \\ 0 & \textit{other wise.} \end{cases}$$

1. *(i),* A_0 *is the identity matrix,*
2. *(ii),* $\sum_{i=0..d} A_i = J$,
3. *(iii),* $A_{i'} =^t A_i \in \{A_i\}_{0 \leq i \leq d}$,
4. *(iv), For* $f, g, h \in \{0, 1, \ldots, d\}$, $\exists p^h_{f,g} \in \mathbb{Z}_{\geq 0}$ *such that* $A_f A_g = \sum_{0 \leq h \leq d} p^h_{f,g} A_h$.

$A_f A_g = A_g A_f$ *for any* f, g $in \{R_i\}_{0 \leq i \leq d}$, $\mathfrak{X}$ *is called a commutative association scheme.*

Relation matrix

We use relation matrices for inputting association schemes.

$$R = \sum_{i=0}^{d} iA_i$$

example: Johnson scheme

$$(R)_{x,y} = \rho(x,y), x, y \in \binom{M}{n}.$$

2
http://www.ed.sojo-u.ac.jp/˜osamu/magma/

Character Table

Example: Johnson Scheme $J(11,4)$

$$\begin{pmatrix} 1 & 28 & 126 & 140 & 35 \\ 1 & 17 & 27 & -25 & -20 \\ 1 & 8 & -9 & -10 & 10 \\ 1 & 1 & -9 & 11 & -4 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}.$$

3
http://www.ed.sojo-u.ac.jp/˜osamu/magma/

Fusion schemes (subscheme)

Let $\mathfrak{X} := (X, \{R_i\})$, $\mathfrak{Y} := (X, \{S_j\})$ be two association schemes. The scheme $\mathfrak{Y}$ is called a fusion scheme of $\mathfrak{X}$, iff each relations of $\mathfrak{Y}$ is a union of some relations from $\mathfrak{X}$.

Example : Fusion scheme of J(11,4)

Character Table of $J(11,4)$

$$\begin{pmatrix} 1 & 28 & 126 & 140 & 35 \\ 1 & 17 & 27 & -25 & -20 \\ 1 & 8 & -9 & -10 & 10 \\ 1 & 1 & -9 & 11 & -4 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}.$$

Fusion scheme of $J(11,4)$ is
$< A_0, A_1 + A_4, A_2 + A_3 >=< E_0, E_1 + E_3 + E_4, E_2 >$.

Known Results

Example

1. $J(2n, n)$: class 2 and class $\frac{n}{2} + 1$(n: even), $\frac{n+1}{2}$(n: odd),
2. $J(2n+1, n)$: class $\frac{n}{2}$(n:even), $\frac{n+1}{2}$(n:odd),
3. sporadic: $J(10, 3)$, $J(11, 4)$, $J(12, 4)$,$J(13, 6)$ class 2.
4. $J(14, 7)$ class 3(new?).

Theorem.[Muzychuk]
If $m \geq 3n + 4 \geq 13$, then $J(m, n)$ has no non-trivial fusion scheme.

4
http://www.ed.sojo-u.ac.jp/˜osamu/magma/

Improved program

Fact1. Johnson Scheme is P-polynomial, so $R_1 \in \{S_j\}_{0 \leq j \leq d'}$ implies $\mathfrak{Y} = \mathfrak{X}$.

Fact2. If a subscheme $\mathfrak{Y} = (X, \{S_j\}_{0 \leq j \leq d'})$ of $\mathfrak{X} := (X, \{R_i\})$ exists, for any S_{j_1},S_{j_2},$S_{j_3} \in \{S_j\}$,

$$\sum_{i \in R_i \subset S_{j_1}, j \in R_j \subset S_{j_2}} p_{i,j}^{k_1} = \sum_{i \in R_i \subset S_{j_1}, j \in R_i \subset S_{j_2}} p_{i,j}^{k_2},$$

$\forall k_1, k_2 \in S_{j_3}$.

Classification of maximal 2-distance sets by Magma

Hiroshi Nozaki

Aichi University of Education

Hakata Workshop 2013
Reference Eki Higashi Building
January 26, 2013

Definition and Purpose

Definition 1

- $X \subset \mathbb{R}^d$ is a 2-distance set $\Leftrightarrow$ $|\{\sqrt{\sum (x_i - y_i)^2} \mid x, y \in X\}| = 2$
- A 2-distance set in $\mathbb{R}^d$ is maximal $\Leftrightarrow$ $\nexists x \in \mathbb{R}^d \setminus X$ such that $X \cup \{x\}$ is still a 2-distance set in $\mathbb{R}^d$

Purpose

We classify maximal 2-distance set X in $\mathbb{R}^d$ with $|X| \geq d+3$ for $d \leq 6$ by using Magma.

The algorithm is mainly based on Lisonek's (JCTA 1997):

Simple graph $\rightarrow$ 2-distance set

Theorems

Theorem 2 (Menger)

Let d be a positive integer. A metric space M of cardinality $|M| \geq d+3$ is isometrically embeddable in $\mathbb{R}^d$ if and only if any subspace of M of cardinality exactly $d+3$ is.

Corollary 3

Let G_i be the induced subgraph removing a vertex v_i of a simple graph G. Then G is isometrically embeddable in $\mathbb{R}^d$ if and only if $G_1, \ldots, G_{d+4}$ is isometrically embeddable in $\mathbb{R}^d$

Theorem 4 (Einhorn-Schoenberg(1966))

A simple graph G is isometrically embeddable in $\mathbb{R}^{n-2}$, where $|G| = n$.

3 / 5

Algorithm for classification of maximal 2-distance set in $\mathbb{R}^d$

Let L_n $(n \geq d+2)$ denote the set of graphs of size n which is isometrically embeddable in $\mathbb{R}^d$.
Rough sketch:

1. Obtain graphs of size $d+3$ from the database of Magma. Pick out all graphs which are isometrically embeddable in $\mathbb{R}^d$ from the graphs of size $d+3$. It is L_{d+3}.
2. Construct L_{j+1} from L_j as follows.
   Fix $G = (V, E) \in L_j$. Take a new vertex v_0 such that for $v_1 \in V$, we have $G_1 = ((V \setminus \{v_1\}) \cup \{v_0\}, E_1\}) \in L_j$. Let $G_i := (V_i = (V \setminus \{v_i\}) \cup \{v_0\}, (E \cup E_1)|_{V_i}\})$. By Corollary 3, $G_2, \ldots, G_{d+2} \in L_j$, if and only if $G' := (V \cup \{v_0\}, E \cup E_1) \in L_{j+1}$
   - If there does not exist such v_0, or $G' \notin L_{j+1}$, then the embedding of (V, E) is maximal.

4 / 5

Results

$d/\lvert X\rvert$	7	8	9	10	11	12	13	15	16	17	21	27
4	5	2	1	1								
5		26	2	13	1	1		1	1			
6			90	21	17	5	4	1	35	1	1	1

Number of maximal spherical two-distance sets

$d/\lvert X\rvert$	8	9	10	11	12	14	16	17
4	6	3						
5	4	1	1					
6		3	10	1	10	1	1	1

Number of maximal non-spherical two-distance sets

Refer the database of maximal 2-distance sets:
http://auemath.aichi-edu.ac.jp/~hnozaki/data_English.html

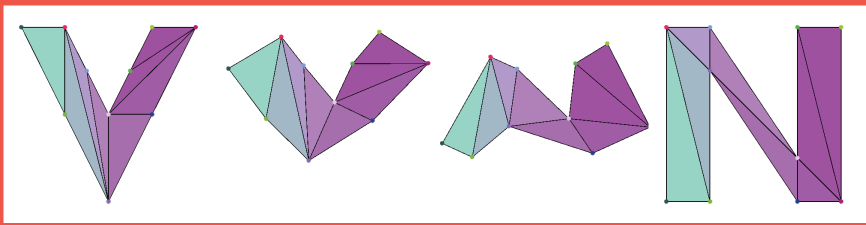
5 / 5

アフィン写像を用いた補間による2次元アニメーション作成ソフトウェア

松下昂平（九州大学大学院数理学府）
「数学ソフトウェア紹介」in Hakata Workshop 2013

概要

- モーフィングと呼ばれる2次元アニメーションの自動生成において，ARAP と呼ばれるアフィン写像を用いた補間手法が存在する．
- ARAP はオブジェクトの形状をなるべく保つことを目的とする補間手法である．
- ARAP を用いた補間手法を解析したり，実際に生成されたアニメーション結果を確認するために，Python 言語を用いてソフトウェアを開発した．

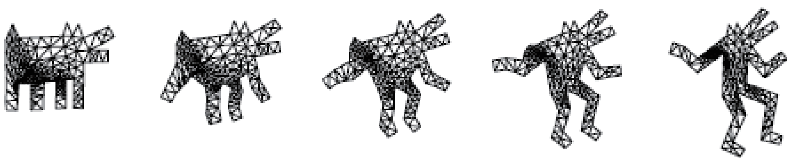


イントロダクション

モーフィング

モーフィング

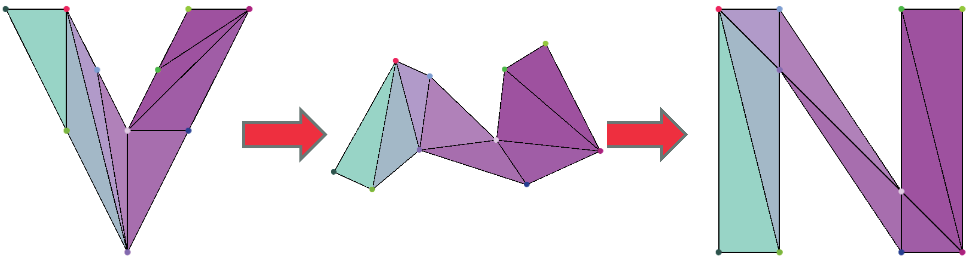
- キーフレームと呼ばれる少数のデータから，フレーム間を滑らかに補間してアニメーションを自動生成するCG技術．
- 特殊な映像効果を生み出したり，フレーム間の補間作業量の削減に有用．

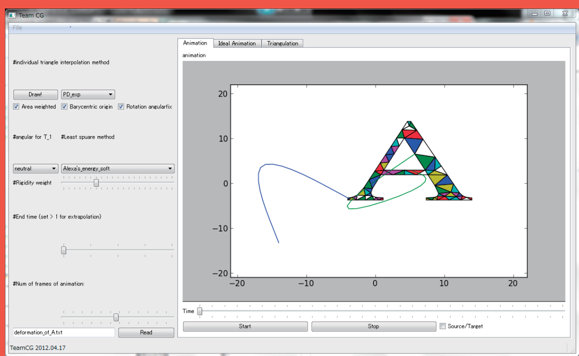


出典：Alexa 2000 p.157

モーフィング例

ソース　　　　　　　　　　　　ターゲット





ソフトウェア概要

- アルゴリズム
- 実装した機能

2013/1/26 「数学ソフトウェア紹介」in Hakata Workshop 2013 7

アルゴリズム

- **As-rigid-as-possible shape interpolation (ARAP)**
  - 2000年に Alexa らによって始められる.
  - オブジェクトの幾何をなるべく保つような変形を構成する.
  - キャラクターアニメーションに適している.

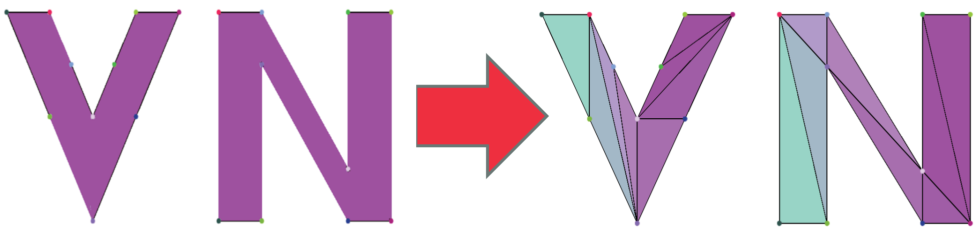
- 具体的には，以下の3ステップに分けたアルゴリズムで補間を実現する枠組み.

1. 両立三角形分割
2. 局所変形
3. 大域変形

2013/1/26 「数学ソフトウェア紹介」in Hakata Workshop 2013 8

両立三角形分割

- 入力された2つの多角形を，組合せ同型になるように三角形分割する.
- 図形の特徴を崩すことなく補間を作成するために“良い”三角形分割を与える.



- 本ソフトウェアでは，既存のアルゴリズムで両立三角形分割されたデータを用いる.

2013/1/26 「数学ソフトウェア紹介」in Hakata Workshop 2013 9

局所変形

与えられた三角形の頂点の対応からアフィン変換Aがただ一つ定まる.

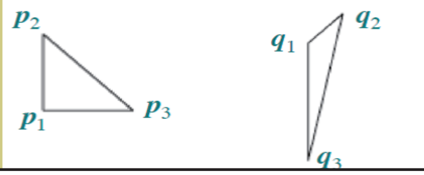
定義 1. 与えられた 2 つの三角形 $P = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3\}$, $Q = \{\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3\}$ に対して, $\sum_{i=1}^{3} \boldsymbol{p}_i = \sum_{i=1}^{3} \boldsymbol{q}_i = 0$ と仮定する. このとき, P から Q への変換行列を A としたとき,

$$A(0)\boldsymbol{p}_i = \boldsymbol{p}_i,\ A(1)\boldsymbol{p}_i = A\boldsymbol{p}_i = \boldsymbol{q}_i, \quad i = 1, 2, 3$$

を満たすような時間 $t \in [0, 1]$ でパラメータづけされた変換の族 $A(t)$ を P から Q への変形と呼ぶ.

$A(t)$ 内の t を $0 \leq t \leq 1$ で連続的に変化させることで, P から Q への滑らかな変形を生成できる.

$$\{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3\} \overset{A(t)}{\Rightarrow} \{\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3\}$$



2013/1/26 「数学ソフトウェア紹介」in Hakata Workshop 2013 10

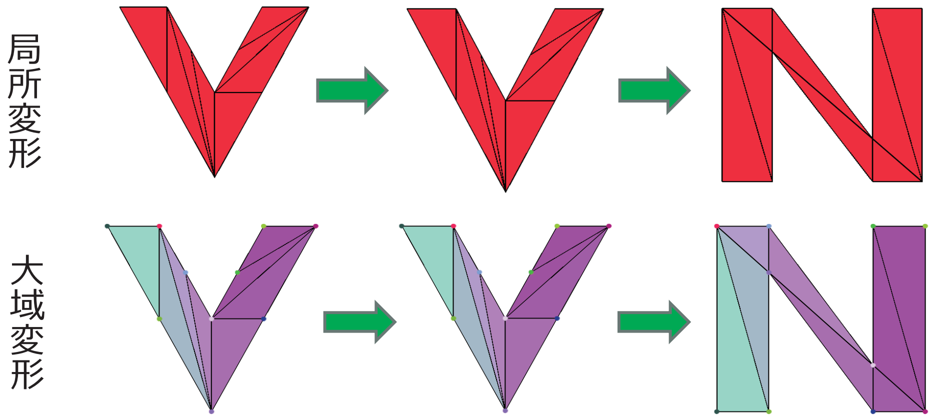
$A(t)$ の構成法

1. 線形補間
$$A(t) = (1-t)I + tA$$
2. 特異値分解を用いた補間
$$A(t) = R_{t\alpha}((1-t)I + tD)R_{t\beta}$$
3. 極分解線形補間 [Alexa 2000]
$$A(t) = R_{t\gamma}((1-t)I + tS)$$
4. 極分解指数補間 [Kaji 2012]
$$A(t) = R_{t\gamma}S^t$$

今回開発したソフトウェアには上の４つの局所変形を選択できるように実装している.

大域変形

- 局所変形を元に隣接する三角形が分離したり，重なり合わないように，エネルギー関数を導入して大域変形を構成する．



エネルギー関数

エネルギー関数 $E(t)$ を次のように定義する．

$$E(t) := \sum_{T_i} ||A_{T_i}(t) - B_{T_i}(t)||_F^2$$

ここで，$A_{T_i}(t)$ は三角形 T_i に関する局所変形，$||\cdot||_F^2$ はフロベニウスノルムである．

- $\boldsymbol{B_{T_i}(t)}$ は三角形 $\boldsymbol{T_i}$ のアフィン変換の中で境界点を共有するアフィン変換であるもの．
- このエネルギー関数を最小にするような $\boldsymbol{B_{T_i}(t)}$ を大域変形とする．
- この場合は正定値な2次関数であるので，線形方程式を解くことにより求めることができる．

エネルギー関数の構成法

1. フロベニウスノルム（変換距離）[Alexa 2000]
$$E(t) = \sum_{T_i} \|A_{T_i}(t) - B_{T_i}(t)\|_F^2$$
2. 点距離
$$E(t) = \sum_{T_i} \sum_{j=1,2,3} \|A_{T_i}(t)p_{i,j} - B_{T_i}(t)p_{i,j}\|_F^2$$
3. 相似・回転不変ノルム（変換距離）[Kaji 2012]
$$E(t) = \sum_{T_i} \left(\min_{s_i,\delta_i \in \mathbb{R}} \|s_i R_{\delta_i} A_{T_i}(t) - B_{T_i}(t)\|_F^2 \right)$$
4. 相似・回転不変ノルム（点距離）[Igarashi 2005]
$$E(t) = \sum_{T_i} \left(\min_{s_i,\delta_i \in \mathbb{R}} \sum_{j=1,2,3} \|s_i R_{\delta_i} A_{T_i}(t)p_{i,j} - B_{T_i}(t)p_{i,j}\|_F^2 \right)$$

今回開発したソフトウェアには上のエネルギー関数を選択できるように実装している.

最小となるエネルギー関数の求め方1/2

- エネルギー関数 $E(t)$ が正定値な2次関数であるとき，次のように書くことができる.
$$E(t) = v(t)^T G(t) v(t) + v(t)^T u(t) + c$$
ただし，
  - $v(t)$ はオブジェクトの各頂点の x 座標と y 座標を一列に並べたサイズ $2n$ の列ベクトル.
  - $G(t)$ は $2n \times 2n$ の対称行列.
  - $u(t)$ は サイズ $2n$ の列ベクトル.
  - c は定数項
- 目的は，$E(t)$ が最小となる $v(t)$ を求めることである.

2013/1/26　「数学ソフトウェア紹介」in Hakata Workshop 2013　15

最小となるエネルギー関数の求め方2/2

- エネルギー関数 $E(t)$ の $v(t)$ に関する偏微分 $\frac{\partial}{\partial v(t)}E(t)$ の値が 0 になるときが最小の場合となる．
- 偏微分 $\frac{\partial}{\partial v(t)}E(t)$ は次のようになる．

$$\frac{\partial}{\partial v(t)}E(t) = 2G(t)v(t) + u(t)$$

- これより，頂点に関する列ベクトル $v(t)$ が

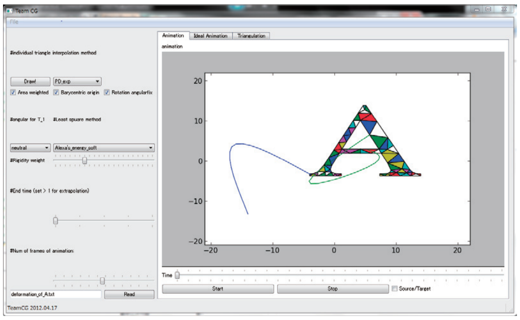
$$v(t) = \frac{1}{2}G(t)^{-1}u(t)$$

のとき，$E(T)$ は最小となる．

2013/1/26　「数学ソフトウェア紹介」in Hakata Workshop 2013　16

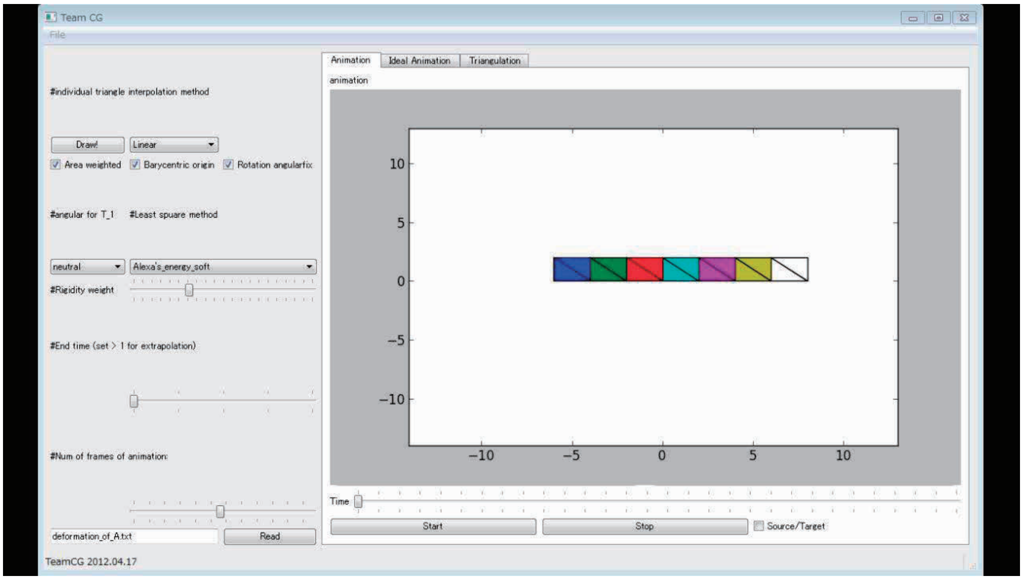
ソフトウェアの機能

- ファイルからデータを読み込み，アニメーションを作成する．
- 局所変形の補間方法の変更
- 大域変形におけるエネルギー関数の変更
- 回転の調整，エネルギー関数のウェイトの変更



2013/1/26 「数学ソフトウェア紹介」in Hakata Workshop 2013 17

ソフトウェアのデモ



2013/1/26 「数学ソフトウェア紹介」in Hakata Workshop 2013 18

まとめ今後の課題

- ARAP を用いたアニメーション作成ソフトウェアを Python 言語で開発した.
- 既存の補間手法の特徴を解析するために，GUI 上で各パラメータを逐次変更できるように実装した.
  - 局所変形
    - 線形補間，特異値分解線形補間，極分解線形補間，極分解指数補間
  - 大域変形（エネルギー関数）
    - フロベニウスノルム，点距離ノルム，相似・回転不変ノルム（変換距離，点距離）
- 同じ入力データから異なった補間手法による結果を比較することができる.

今後の課題

- プログラムの高速化
- 3次元上のアニメーションへの拡張
- Maya ソフトウェア（3D アニメーションソフトウェア）のプラグインとして組み込むことで，容易な補間アニメーションが作成できる環境を構築する．

共同開発者

- 井慶喜（九州大学理学部）
- 池田有希（九州大学大学院数理学府）
- 松田元輝（九州大学大学院数理学府）
- 濱田裕康（九州大学大学院数理学研究院）
- 溝口佳寛（九州大学マス・フォア・インダストリ研究所）

参考文献

1. M. Alexa, D. Cohen-Or, and D. Levin, As-rigid-as-possible shape interpolation, Proceedings of AMC SIGGRAPH 2000, pp.157-164, 2000.
2. S. Kaji, S. Hirose, S. Sakata, Y. Mizoguchi, and K. Anjyo, Mathematical Analysis on Affine Maps for 2D Shape Interpolation, Proceedings of ACM SIGGRAPH 2012 (SCA2012) , pp.71-76, 2012. http://dl.acm.org/citation.cfm?id=2422368
3. T. Igarashi, T. Moscovich, and J.F. Hughes, As-rigid-as-possible shape manipulation, ACM Transactions on Graphics 2005, pp.1134-1141, 2005.

トーリックイデアルの二次生成判定法の実装

鹿間 章宏

大阪大学大学院 情報科学研究科 M2

January 26, 2013

鹿間 章宏 (Osaka Univ.) トーリックイデアルの二次生成判定法の実装 January 26, 2013 1 / 6

有限グラフのトーリックイデアル, グレブナー基底

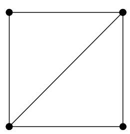
定義 1 (トーリックイデアル)

有限グラフ G から生起するトーリックイデアルとは，G の隣接行列 A の核 $\mathrm{Ker}(A)$ で定義される，多項式環の部分集合である．

定義 2 (グレブナー基底)

イデアル $I(\neq 0)$ について，I に属する多項式の有限集合 $\mathcal{G} = \{g_1, g_2, \dots, g_s\}$ が，$<$ に関する I のグレブナー基底であるとは，$\mathrm{in}_<(I) = (\mathrm{in}_<(g_1), \mathrm{in}_<(g_2), \dots, \mathrm{in}_<(g_s))$ となるときに言う．

例 3



$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

この図では，トーリックイデアルの生成系とグレブナー基底はいずれも $\{x_1x_4 - x_2x_5\}$ となる．

鹿間 章宏 (Osaka Univ.) トーリックイデアルの二次生成判定法の実装 January 26, 2013 2 / 6

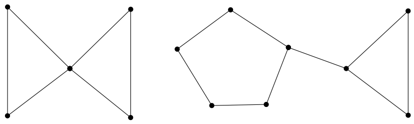
二次生成トーリックイデアルを持つグラフ

定理 4 (Ohsugi-Hibi (1999))

トーリックイデアルの生成系が二次式のみで構成できるのは，以下の 3 つの条件をすべて満たすときで，かつそのときに限る

- 長さ 6 以上の偶サイクル C について，「C には偶弦が存在」または「C には 3 本の奇弦が存在して，少なくとも 2 本 は C の中で交差」を満たす．
- C_1 と C_2 が極小奇サイクルで，唯一つの共通頂点を持つとき，辺 $\{i,j\} \notin E(C_1) \cup E(C_2)$ で，$i \in V(C_1)$ と $j \in V(C_2)$ を満たすものが存在する．
- C_1 と C_2 が極小奇サイクルで，共通の頂点を持たないとき，C_1 と C_2 を結ぶ橋が少なくとも 2 本存在する．

例 5



この図はそれぞれ定理 4 の条件のうち下二つにあてはまらない例．

今回使った数学ソフトウェアの紹介

- NAUTY (B.D. McKay)
  n 頂点のグラフを隣接行列や接続リストの形で列挙する．今回は，連結なグラフに限った出力を使った．
- cypath (宇野毅明)
  グラフに含まれるサイクルを全列挙したり，特定の頂点から別の頂点へのパスを見つけることができる．
- Macaulay2 (D.R. Grayson, M.E. Stillman)
  汎用の計算代数システム．今回は，グレブナー基底計算に用いた．
- 4ti2 (4ti2 team)
  隣接行列から，トーリックイデアルを計算できる．
- TiGERS (B. Huber)
  隣接行列を入力とし，そのグラフから生起するトーリックイデアルのグレブナー基底を全列挙する．

Ruby における実装

頂点数が小さいグラフから順に, 付随するトーリックイデアルが二次生成であるが二次のグレブナー基底 (GB) を持たないものを探索した.
定理 4 を用いた二次生成判定プログラム, TiGERS が出力したグレブナー基底の中に二次のものが含まれるかを判定するプログラムを Ruby で実装した.
Ruby は各ソフトウェア間の入出力の受け渡しも担う.

(頂点数)	→	NAUTY	→
(隣接リスト)	→	cypath	→
(サイクルリスト)	→	トーリックの二次判定	→
(二次生成グラフ)	→	TiGERS	→
(GB の全列挙)	→	GB の二次判定	→
(条件を満たすグラフ)			

頂点数	連結なグラフ	二次生成
3	2	2
4	6	6
5	21	20
6	112	95
7	853	568

7 頂点完全グラフでベンチマーク
トーリックの二次判定:1.89 sec
(CPU:Core i7 870 2.93GHz)
(メモリ:6.1GB)

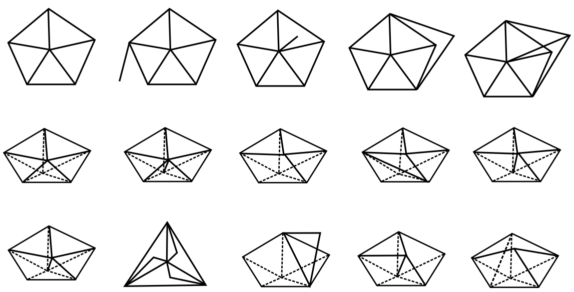
実験結果

定理 6

7 頂点以下の有限連結グラフで,

性質「グラフから生起するトーリックイデアルが
二次生成だが二次グレブナー基底を持たない」

を持つものは, 以下の図の 15 種類であり, かつそれらに限る.



q-determinantの最小化

九州大学　数理学府修士課程1年
照本直敏
2013年1月26日　組合せ数学セミナー

定義(q-determinant)

$q \in \mathbb{R},\ A = \left(a_{ij}\right)$: n次正方行列に対し，次のqの多項式を考える

$$q\text{-}\det A \coloneqq \sum_{\sigma \in S_n} q^{\iota(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)}$$

ただし，$\iota(\sigma) \coloneqq \#\{1 \leq i < j \leq n; \sigma(i) > \sigma(j)\}$
転倒数 (inversion number) という

- $q = -1$のとき　$q\text{-}detA = detA$
- $q = 0$のとき　$q\text{-}detA = \prod_{i=1}^{n} a_{ii}$
- $q = 1$のとき　$q\text{-}detA = perA = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$

例(n=3)

- $\iota(\sigma)=\#\{1 \le i < j \le 3; \sigma(i) > \sigma(j)\}$

σ	e	(12)	(13)	(23)	(123)	(132)
$\iota(\sigma)$	0	1	3	1	2	2

したがって，

$$
\begin{aligned}
q\text{-}\det A &= \sum_{\sigma\in S_3} q^{\iota(\sigma)} \prod_{i=1}^{3} a_{i\sigma(i)} \\
&= a_{11}a_{22}a_{33} + q a_{12}a_{21}a_{33} + q^3 a_{13}a_{22}a_{31} \\
&\quad + q a_{11}a_{23}a_{32} + q^2 a_{12}a_{23}a_{31} + q^2 a_{13}a_{21}a_{32} \\
&= a_{11}a_{22}a_{33} + q(a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31}) \\
&\quad + q^2(a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) + q^3 a_{13}a_{22}a_{31}
\end{aligned}
$$

問題設定

q-det は次の性質をもっている．

① Bozejko & Speicher (1991)
$-1 \le q \le 1$ のとき，A が半正定値なら $q\text{-}\det A \ge 0$

研究の目的

$q \in \mathbb{R}$ に対して，
$\lambda_n(q) := \min\{q\text{-}\det A \mid A : n$次半正定値，$A_{ii} = 1\}$
を求めたい．

研究の結果

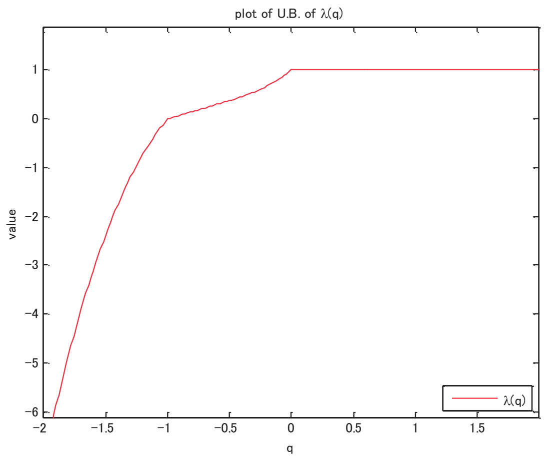
結果

$$\lambda_3(q) \leq \begin{cases} \dfrac{1}{(1+q)(1+q+q^2)} & 0 \leq q \\ \dfrac{3}{4}(1+q) & -\dfrac{1}{2} \leq q \leq 0 \\ 1+q^3 & -1 \leq q \leq -\dfrac{1}{2} \\ & q \leq -1 \end{cases}$$

研究の結果

予想

$\lambda_n(q)$は前頁の関係が等号で成立することが予想される．
また，グラフから連続性及び単調性が期待できる．



研究の手法

- $\lambda_n(q)$ をq 毎に求めたい.
  ⇒ 最小化問題は難しい（非凸のため）

- 最適化手法を使って，求めたい $\lambda_n(q)$の下界を与えるような最適化関数 $\mu_n(q)$ を構成し計算する.

- $\mu_n(q) \leq \lambda_n(q)$ の gap が 0 あるいは小さくなるような最適化問題を作りたい.
  ⇒ 半正定値計画緩和（SDP緩和）

SDP緩和

SDP

$$\min_{y}\left\{ \mathrm{b}^T \mathrm{y} \mid \mathrm{C} + \sum_{j=1}^{m} A_j\, y_j \ : N\text{次半正定値} \right\}$$

ただし，$b\,, y \in \mathbb{R}^m$,　$\mathrm{C}\,, A_j$: N次実対称行列

- SDPは凸性を有している
  ⇒比較的容易に解けるうえ，ソフトウェアも充実している.
- $\lambda_n(q)$ からSDPを作る.（SDP緩和）
- この手法を用いると　“（SDPの最適値）$\leq \lambda_n(q)$”が保証される

SDP緩和の例(n=2,r=2)

- $A_2[x] = \begin{pmatrix}1 & x\\ x & 1\end{pmatrix}$, $q\text{-}\det A_2[x] = 1 + qx^2$, $M_1[x] = \begin{pmatrix}1 & x\\ x & x^2\end{pmatrix}$

$$\lambda_2(q) = \{1 + qx^2 \mid \begin{pmatrix}1 & x\\ x & 1\end{pmatrix} \geq O\} = \{1 + qx^2 \mid \begin{pmatrix}1 & x\\ x & x^2\end{pmatrix} \otimes \begin{pmatrix}1 & x\\ x & 1\end{pmatrix} \geq O\}$$

$$= \{1 + qx^2 \mid \begin{pmatrix}1 & x & x & x^2\\ x & 1 & x^2 & x\\ x & x^2 & x^2 & x^3\\ x^2 & x & x^3 & x^2\end{pmatrix} \geq O\}$$

$$\leftarrow \begin{cases} x \to y_1\\ x^2 \to y_2\\ x^3 \to y_3\end{cases}$$

$$\geq \{1 + qy_2 \mid \begin{pmatrix}1 & y_1 & y_1 & y_2\\ y_1 & 1 & y_2 & y_1\\ y_1 & y_2 & y_2 & y_3\\ y_2 & y_1 & y_3 & y_2\end{pmatrix} \geq O\} \; (= \mu_{2,2}(q)) \longleftarrow \text{SDP}$$

たとえば，本来 $x^2 = x \cdot x$ $(y_2 = {y_1}^2)$ であるが，半正定値性条件 $y_2 \geq {y_1}^2$ に制約が緩和されている．

SDPのつくり方 I

- $u_{r-1}[\mathrm{x}] := (1, x_1, x_2, x_3, {x_1}^2, x_1x_2, \ldots, {x_3}^{r-1})^T$
- $M_{r-1}[\mathrm{x}] := u_{r-1}[\mathrm{x}]u_{r-1}[\mathrm{x}]^T$ (rank 1 行列)

性質　$A_n[\mathrm{x}]$: 半正定値　$\Leftrightarrow$　$M_{r-1}[\mathrm{x}] \otimes A_n[\mathrm{x}]$: 半正定値

$$\min \{q\text{-}\det A \mid M_{r-1}[\mathrm{x}] \otimes A_n[\mathrm{x}]: \text{半正定値}\}$$

は $\lambda_n(q) = \min\{q\text{-}\det A \mid A: n\text{次半正定値}\}$ と等価．

SDPのつくり方 II

- $\lambda_n(q) = \min\{q\text{-}\det A_n[x] \mid M_{r-1}[x] \otimes A_n[x]$: 半正定値 $\}$
  は，x に関する多項式の最小化問題である．
  （制約条件は行列の各要素が x の多項式）

ここで，$y_\alpha := x^\alpha$ とおくと，
$\min\{(y_\alpha$の線形関数$) \mid y_\alpha$に関する半正定値条件$\}$ $(\leq \lambda_n(q))$
⇒ SDPになっている

y_αの決め方から，たとえば
$y_{22} = y_{20}y_{02}$ $({x_1}^2{x_2}^2 = {x_1}^2 \cdot {x_2}^2)$は成立しなくてもよい
⇒ 条件が緩くなっている
⇒ $\lambda_n(q)$の下界値が得られる．

SDP緩和の注意

- r を大きくすると良い下界値が得られる．すなわち，
  $$\mu_{n,r}(q) \leq \mu_{n,r+1}(q) \leq \lambda_n(q) \qquad ({}^\forall r, {}^\forall q)$$
  ⇒ できるだけ大きい r でSDPを作って解きたい．

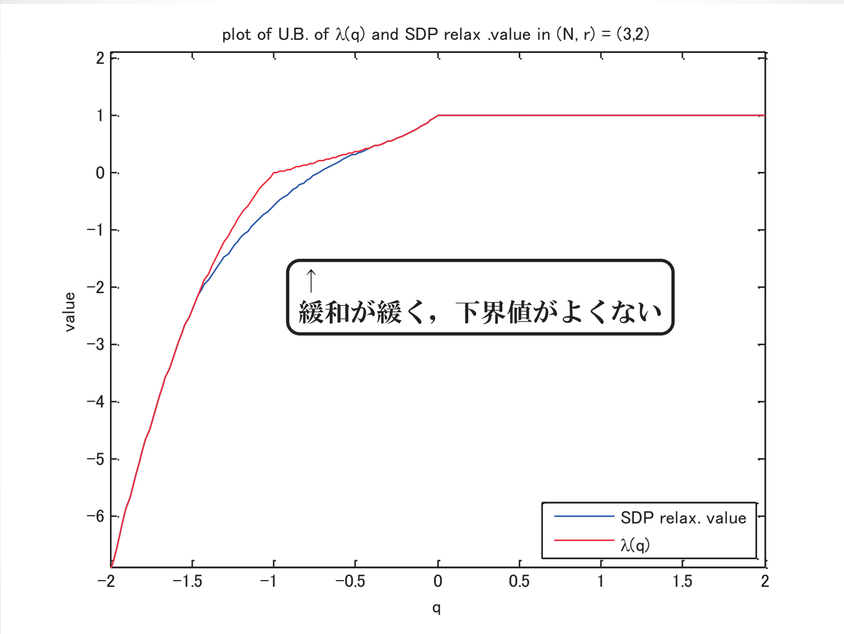
- しかし，r が大きいとSDPがスパコンでも解けない大規模になる．

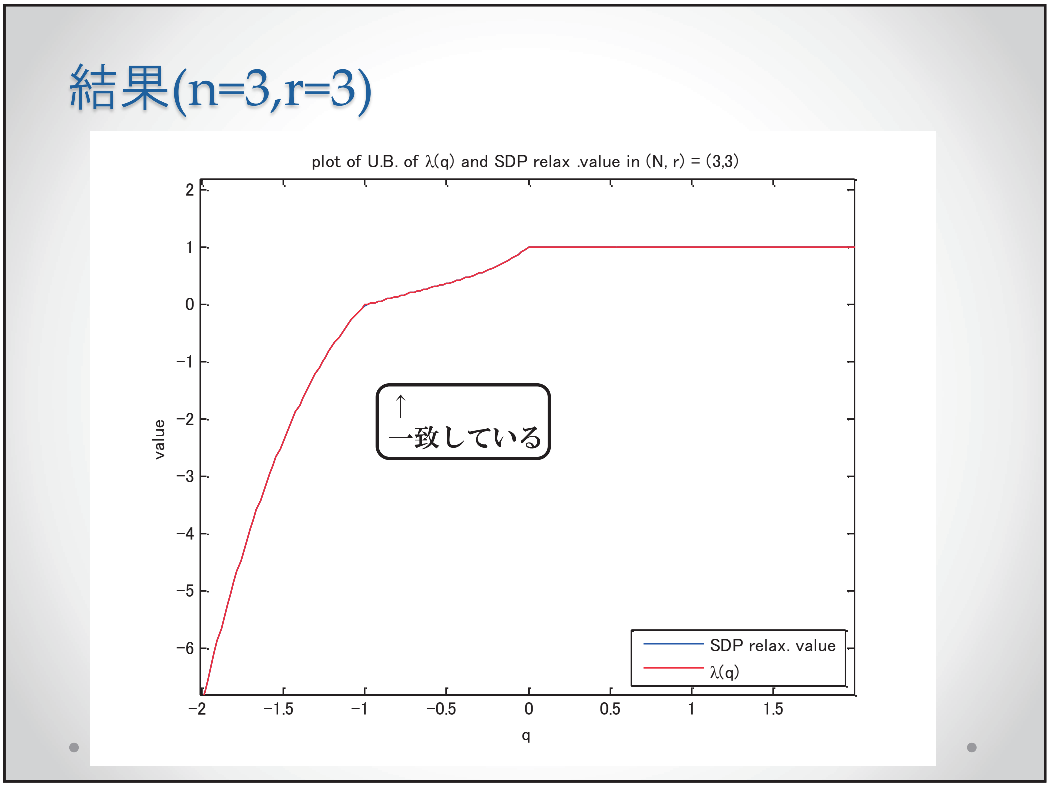
- 実用上は r = 2 , 3 で充分よい下界値が得られる．

プログラムの概要

- 入力　n：行列$A_n[x]$のサイズ
  r：relaxOrder（u_{r-1}の次数）
- 出力　$-2 \le q \le 2$における $\mu_{n,r}(q)$ のグラフ

結果(n=3,r=2)





結果(n=3,r=3)
plot of U.B. of λ(q) and SDP relax .value in (N, r) = (3,3)
value
q
SDP relax. value
λ(q)
↑
一致している

A generation method of a certain subset of E_8-lattice using PARI/GP

Tomoya Kiyuna

Graduate School of Mathematics, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka, 819-0395, Japan

Hakata Workshop 2013, Saturday, January 26, 2013

1 / 5

Theorem

We define the inner product $\langle x, y\rangle = \sum_{i=1}^{d} x_i y_i$ and take $a, b \in \mathbb{C}^d$ s.t.$\langle a, a\rangle = \langle a, b\rangle = \langle b, b\rangle = 0$. Let L be an even unimodular lattice in $\mathbb{R}^d$.

For $X = {}^t(x, y) \in M_{2,d}(\mathbb{C})$, we put

$$P_v(X) = \binom{j}{v} \langle x, a\rangle^{j-v} \langle y, a\rangle^{v} \begin{vmatrix} \langle x, a\rangle & \langle x, b\rangle \\ \langle y, a\rangle & \langle y, b\rangle \end{vmatrix}^{k} (0 \le v \le j)$$

and $P(X) = {}^t(P_v(X))$. Then

$$\Theta_{L,a,b,(k,j)}(Z) = \sum_{X \in L^2} P(X) \exp(\pi\sqrt{-1}\mathrm{Tr}({}^tXZX)) \ (Z \in H_2)$$

is a vector-valued Siegel modular form of weight $\det^{\frac{d}{2}+k} \otimes \mathrm{Sym}(j)$ with respect to $Sp(2, \mathbb{Z})$.

2 / 5

For $Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in H_2$, we put $q_1 = e^{2\pi\sqrt{-1}\tau}$, $\zeta = e^{2\pi\sqrt{-1}z}$ and $q_2 = e^{2\pi\sqrt{-1}\omega}$. We see from the definition of theta functions that

$$\Theta_{L,a,b,(k,j)}(Z) = \sum_{n,m=0}^{\infty} \Big(\sum_{\substack{x,y\in L \\ \langle x,x\rangle=2n \\ \langle y,y\rangle=2m}} P(X)\zeta^{\langle x,y\rangle}\Big) q_1^n q_2^m.$$

Therefore the $v+1$-st component of $\Theta_{L,a,b,(k,j)}(Z)$ is

$$\sum_{n,m=0}^{\infty} \Big(\sum_{\substack{x,y\in L \\ \langle x,x\rangle=2n \\ \langle y,y\rangle=2m}} P_v(X)\zeta^{\langle x,y\rangle}\Big) q_1^n q_2^m.$$

3 / 5

The case $L = E_8$

We take $L = E_8$, the well-known 8-dimensional root lattice

$$E_8 = \Big\{x = (x_i) \in \mathbb{Z}^8 \cup (\mathbb{Z} + \frac{1}{2})^8 \Big| \textstyle\sum_{i=1}^8 x_i \in 2\mathbb{Z}\Big\}.$$

We consider the subset of E_8

$$E_{8,N} := \Big\{x = (x_i) \in E_8 \Big| \langle x,x\rangle = N\Big\} \ (N \in 2\mathbb{Z}_{\geq 0}).$$

For $x \in E_{8,N}$,

$$x \in \mathbb{Z}^8 \Longrightarrow x_i^2 \leq \langle x,x\rangle = N. \quad \therefore |x_i| \leq \sqrt{N}.$$

$$x = y + (\frac{1}{2}, \ldots, \frac{1}{2}) \in (\mathbb{Z} + \frac{1}{2})^8 \Longrightarrow (y_i + \frac{1}{2})^2 \leq N.$$

$$\therefore -\sqrt{N} - \frac{1}{2} \leq y_i \leq \sqrt{N} - \frac{1}{2}, \quad x_i = \frac{2y_i + 1}{2}.$$

4 / 5

The source code

```
Lat(N)={
 local(L=[],n=floor(sqrt(N)),m=floor(sqrt(N)-1/2),l=ceil(-sqrt(N)-1/2));
 for(x1=-n,n,for(x2=-n,n,for(x3=-n,n,for(x4=-n,n,
 for(x5=-n,n,for(x6=-n,n,for(x7=-n,n,for(x8=-n,n,
     if(lift(Mod(x1+x2+x3+x4+x5+x6+x7+x8,2))==0,
        if(x1^2+x2^2+x3^2+x4^2+x5^2+x6^2+x7^2+x8^2==N,
           L=concat(L,[[x1,x2,x3,x4,x5,x6,x7,x8]]);
           );
        );
     ))))))));
 for(y1=l,m,for(y2=l,m,for(y3=l,m,for(y4=l,m,
 for(y5=l,m,for(y6=l,m,for(y7=l,m,for(y8=l,m,
     x1=(2*y1+1)/2;x2=(2*y2+1)/2;x3=(2*y3+1)/2;x4=(2*y4+1)/2;
     x5=(2*y5+1)/2;x6=(2*y6+1)/2;x7=(2*y7+1)/2;x8=(2*y8+1)/2;
     if(lift(Mod(y1+y2+y3+y4+y5+y6+y7+y8,2))==0,
        if(x1^2+x2^2+x3^2+x4^2+x5^2+x6^2+x7^2+x8^2==N,
           L=concat(L,[[x1,x2,x3,x4,x5,x6,x7,x8]]);
           );
        );
     ))))))));
 return(L);
};
```

5 / 5

谷口 哲至

スターコンプリメントテクニックと最小固有値が-2以上のグラフの生成

2012/1/26

概 要

最小固有値が -2 以上のグラフの生成には、ルート系の部分集合（ノルム 2 のベクトルからなるユークリッド空間の部分集合）が用いられる。その様なグラフの隣接行列はそれらベクトルの部分集合のなす行列 D によるグラム行列で表される。グラフの最小固有値研究の場面では、こようなグラフを生成する必要に（頻繁に）駆られる。D. Cvetković, P. Rowlinson, S.K. Simić 氏達によって、スターコンプリメントテクニックを用いた極大例外型グラフの分類が成された (cf. [1])。このテクニックを紹介する。

[1] Star complements and exceptional graphs. Linear Algebra and its Applications (2007), 423(1), 146–154.

1 Star complemet technique

ある特定の固有値を持つグラフが（重複度も含め）自由に生成できたら嬉しい場面が、あるのではないでしょうか。ここではその様な場面で有用な **star complement tecnique** を紹介します。

Definition: Let Γ be a graph with $V(\Gamma) = \{1, \dots, n\}$. Let P be the orthognal projection of $\mathbb{R}^n$ onto $\mathscr{E}(\mu)$, where $\mathscr{E}(\mu)$ is the eigenspace of $A(\Gamma)$ for the eigenvalue μ of $A(\Gamma)$. Then a subset X of $V(\Gamma)$ satisfying the following condition is called a **star set** for μ of Γ:
the vectors $P\mathbf{e}_j$ $(j \in X)$ form a basis for $\mathscr{E}(\mu)$, where $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$.

Definition: Let Γ be a graph with $V(\Gamma) = \{1, \ldots, n\}$ and an eigenvalue μ. Let X be a star set for μ of Γ. Then the subgraph $\Gamma - X$ of Γ is called the **star complement** for μ corresponding to X.

Let Γ be a graph with adjacency matrix $\begin{pmatrix} A_X & B^T \\ B & C \end{pmatrix}$, where X is a star set for an eigenvalue μ of Γ. Then we define a bilinear form on $\mathbb{R}^{n-|X|}$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle_X = \mathbf{x}^T (\mu I - C)^{-1} \mathbf{y},$$

and denote the columns of B by $\mathbf{b}_v$ $(v \in X)$.

Theorem 1: Suppose that μ is not an eigenvalue of the graph Γ'. Then there exists a graph Γ with a star set X for μ such that $\Gamma - X = \Gamma'$ if and only if the characteristic vectors $\mathbf{b}_v$ $(v \in X)$ satisfy

(i) $\langle \mathbf{b}_v, \mathbf{b}_v \rangle_X = \mu$ for all $v \in X$,

(ii) $\langle \mathbf{b}_u, \mathbf{b}_v \rangle_X \in \{-1, 0\}$ for all pairs u, v of distinct vertices in X.

2 Star complemet technique の流れ

例えば、λ という固有値を持つグラフを作りたいとする。このとき、次の流れで、固有値 λ（重複度 m）を持つグラフをつくる：

(i) Let Γ be a graph without eigenvalue λ.

(ii) Compute $S := \{\mathbf{b} \in \{0,1\}^{|V(\Gamma)|} \mid \mathbf{b}^T(\lambda I - A(\Gamma))^{-1}\mathbf{b} = \lambda\}$.
※ ここで $S = \emptyset$ となれば、Γ に頂点を加えることで、固有値 λ をもつグラフは生成出来ない事を意味する。

(iii) Let T be a subset of S such that $\mathbf{u}^T(\lambda I - A(\Gamma))^{-1}\mathbf{v} \in \{0, -1\}$ for all pairs $\mathbf{u}, \mathbf{v}$ of distinct vectors in T.

(iv) Let G be a graph with adjacency matrix

$$\begin{pmatrix} \lambda I - B^T(\lambda I - A(\Gamma))^{-1}B & B^T \\ B & A(\Gamma) \end{pmatrix}.$$

※ Theorem 1 から、G が固有値 λ（重複度 m）を持つことがわかる。

以下は magma による、固有値 $-1-\sqrt{2}$（重複度 2）を持つグラフを生成するプログラムである。

MAGMA:

```
//準備
n:=7;
Q:=Rationals(); P<x>:=PolynomialRing(Q);
//グラフの定義
G:=Graph<{@i: i in [1..7]@}|
    {@ {1, 5}, {1, 6}, {2, 3}, {2, 5}, {2, 7}, {3, 4}, {3, 5},
    {3, 6}, {3, 7}, {4, 5}, {4, 6}, {5, 6}, {6, 7} @}
    >;
//G:=PathGraph(n);
Cp:=CartesianPower([0,1],n);
Vecs:=[Matrix(n,[cc : cc in c]) : c in Cp];
//隣接行列
M:=AdjacencyMatrix(G);
//Zero に-1-√2 を持つ多項式
p:=x^2+2*x-1;
f:=P!CharacteristicPolynomial(M);

F:=SplittingField(p);
C:=ChangeRing(M,F);
B:=Basis(F);
_:=exists(r){b:b in B| b notin Q};

MR:=MatrixRing(F,Order(G));
W:=(r*MR!1-M)^-1;
V:=[ChangeRing(v,F) : v in Vecs];
X:=[v :v in V |
    (v*W*Transpose(v))[1,1] eq r
];

S2:=[Setseq(s):s in Subsets(Seqset(X),2)];
Ts:=[s: s in S2 |
    (s[1]*W*Transpose(s[2]))[1,1] in [0,-1]
];

T:=Random(Ts);
v1:=T[1];
v2:=T[2];

Bt:=VerticalJoin(ChangeRing(v1,F),ChangeRing(v2,F));
L2:=VerticalJoin(Bt,C);
```

```
B:=Transpose(Bt);
K:=-Bt*W*B;
for i in [1..2] do
        K[i,i]:=0;
end for;
Ut:=VerticalJoin(K,ChangeRing(B,F));
L2:=HorizontalJoin(Ut,L2);
f2:=P!CharacteristicPolynomial(L2);
f2/p in P;
Factorization(f2);


/*
> Spectrum(Graph<{@i:i in [1..9]@}|L2>);
[ <-2.4142135623730950488016887242096980785696718753769480731 7638, 2>,
<0.41421356237309504880168872420969807856967187537694807317638, 2>,
<0.16142518519415694875674044541390708865916260872093420786805, 1>,
<1.43108998400099700379096592141838254487761558765778855602668, 1>,
<-0.99999999999999999999999999999999999999999999999999999998961, 1>,
<-1.69629044702659275883281351396184189700248948421534986465 92, 1>,
<5.10377527783143880628510714712955226346571128783662710072160, 1> ]
*/
```

MAGMAによるRamanujan graph チェッカー

DAHAN, Xavier, (九州大学大学院数理学研究院)

Some families of Cayley graphs based on quaternions have led to famous graphs the "Ramanujan graphs". They have a small 2nd largest eigenvalue and have a large girth. Using octonions is a quite promising idea in order to improve the girth, but more challenging. A similar construction of Caylay graphs remains possible. However due to the non-associativity, whether the same remarkable properties hold or not is not easy to proof or disprove. In order to check this, a package in MAGMA that computes the girth and the 2nd largest eigenvalue was written. It was possible to check that the graphs based on octonions have a surprisingly small girth and have not a small 2nd eigenvalue (therefore are not Ramanujan).

SSD Problem

九州大学大学院数理学府
古川 貴司

1)モデルの説明
2)アルゴリズム
3)数値実験
4)結果の考察

1)モデルの説明

2)アルゴリズム

3)数値実験

4)結果の考察

Flash SSD (Flash Solid State Drive) はコンピュータに用いられる補助記憶装置の一種である。
HDD と比べるとデータ処理がスムーズに行えたり、衝撃に強いなどの利点があるが、メモリの書き換え可能回数が少ないという欠点もある。

モデルの説明

V	X	V	V	…	V	UW
V	V	V	V	…	X	UW
⋮	⋮	⋮	⋮	⋱	⋮	⋮
V	V	V	V	…	X	UW

ブロック　ページ

ページには 1) $valid(V)$: 情報として意味のあるページ,
2)$invalid(X)$: 情報として意味のないページ,
3)$unwritten(UW)$: *データを保存できるページ*
の状態がある。

ブロックのうち1つはすべてUWの$free\ block$ が存在する。

1)モデルの説明

2)アルゴリズム

3)数値実験

4)結果の考察

アルゴリズムの詳細

*Page Update*を*free block*以外に*unwritten page* が存在しなくなるまで行う

交互に行う

*Garbage collection*を行い新しく*unwritten page* を作る

アルゴリズムの詳細

・*Page Update*
*valid page*をランダムに選択し*unwritten page* に書き込む。
(元の*valid page*は *invalid page*に変わる。)

例

V	X	X	V	V	V	UW
V	V	V	X	X	X	UW
X	V	V	V	UW	X	UW
X	X	V	V	UW	X	UW

アルゴリズムの詳細

・*Page Update*
*valid page*をランダムに選択し*unwritten page* に書き込む。
(元の*valid page*は *invalid page*に変わる。)

例

V	X	X	V	V	V	UW
V	V	V	X	X	X	UW
X	X	V	V	V	X	UW
X	X	V	V	UW	X	UW

アルゴリズムの詳細

・*Garbage collection*
free block 以外で*valid page*が最も少ないブロック(複数ある場合はその中からランダムに選択する)をさがし、そのブロックの*valid page*を*free block*に移す。選ばれたブロック 全体を*unwritten*の状態にすることで新たな*free block* とする。

例

V	X	X	V	V	V	UW
V	V	V	X	X	X	UW
X	X	V	V	V	X	UW
X	X	V	X	V	X	UW
2	1	3	2	3	1	free

*valid page*の数

アルゴリズムの詳細

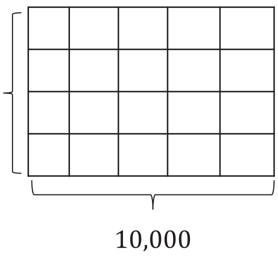
・*Garbage collection*
free block 以外で*valid page*が最も少ないブロック(複数ある場合はその中からランダムに選択する)をさがし、そのブロックの*valid page*を*free block*に移す。選ばれたブロック 全体を*unwritten*の状態にすることで新たな*free block* とする。

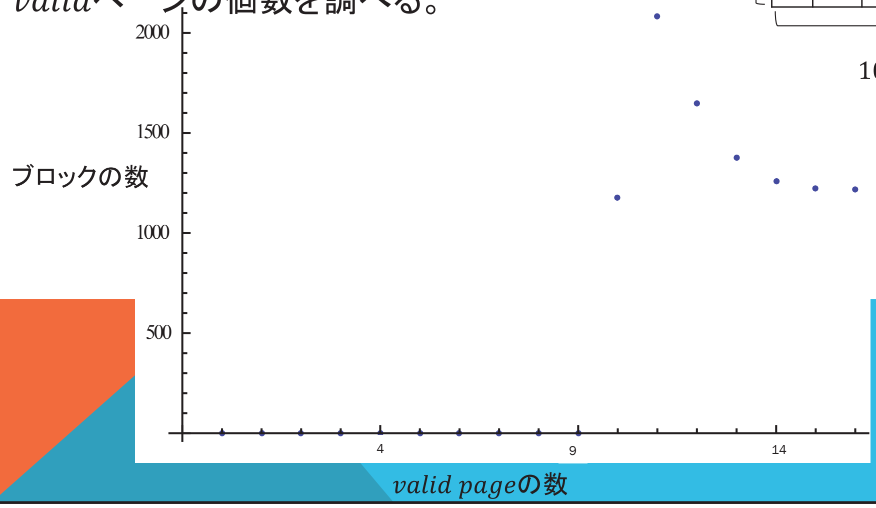
例

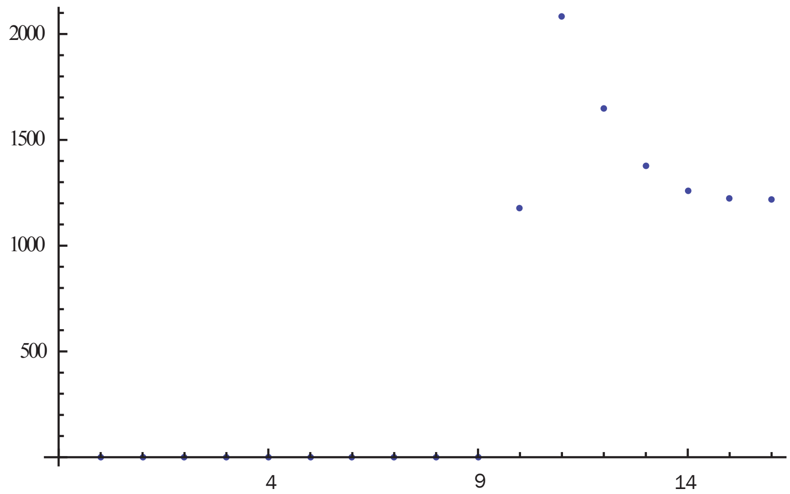
V	UW	X	V	V	V	X	
V	UW	V	X	X	X	UW	
X	UW	V	V	V	X	UW	
X	UW	V	X	V	X	UW	
2	free	3	2	3	1	1	*valid page*の数

Werner Bux, Ilias Iliadisらによる先行研究(2010)

ブロックの数:10,000、ページの数:16
valid ページの数:128,000
としたとき、定常状態での各ブロックに存在する*valid*ページの個数を調べる。







先行研究では、定常状態ではあるc^*が存在し、$Garbage\ Collection$ を行うとき高確率で$valid\ page$ の数がc^*以下のブロックは選ばれないことが示された。
この場合$c^* = 9$である。

アルゴリズムの詳細

先行研究では、$Page\ Update$は一様に行っていた。
本研究では、モデルを一般化するため$Page\ Update$される確率が異なる$valid$ ページが存在するモデルを考えた。

考えるモデル

A	X	B	B	⋯	A	UW
A	B	B	X	⋯	X	UW
⋮	⋮	⋮	⋮	⋱	⋮	⋮
X	A	B	A	⋯	X	UW

$valid\ page$が$A\ valid(A)$と$B\ vald(B)$の2種類ある以外は先行研究と同じである。

アルゴリズムの詳細

・*Page Update*

1 / (1 + μ)の確率で*A valid* を、μ/(1 + μ)の確率で*B valid*を選択し、選ばれた方の*valid page*をランダムに選択し*unwritten page* に書き込む。
(元の*valid page*は *invalid page*に変わる。)

例

X	X	A	B	A	UW
B	A	X	X	X	UW
X	A	B	UW	X	UW
X	B	A	UW	B	UW

アルゴリズムの詳細

・*Page Update*

1 / (1 + μ)の確率で*A valid* を、μ/(1 + μ)の確率で*B valid*を選択し、選ばれた方の*valid page*をランダムに選択し*unwritten page* に書き込む。
(元の*valid page*は *invalid page*に変わる。)

例

X	X	A	B	A	UW
B	A	X	X	X	UW
X	X	B	A	X	UW
X	B	A	UW	B	UW

アルゴリズムの詳細

・*Garbage collection*

free block 以外で*valid page*が最も少ないブロック(複数ある場合はその中からランダムに選択する)をさがし、そのブロックの*valid page*を*free block*に移す。選ばれたブロック 全体を*unwritten*の状態にすることで新たな*free block* とする。

例

X	X	A	B	A	UW	
B	A	X	X	X	UW	
X	X	B	A	X	UW	
X	B	A	A	B	UW	
(0,1)	(1,1)	(2,1)	(2,1)	(1,1)	free	(*A valid page*の数, *B valid page*の数)
1	3	2	3	1	free	*valid page*の数

アルゴリズムの詳細

・*Garbage collection*

free block 以外で*valid page*が最も少ないブロック(複数ある場合はその中からランダムに選択する)をさがし、そのブロックの*valid page*を*free block*に移す。選ばれたブロック 全体を*unwritten*の状態にすることで新たな*free block* とする。

例

UW	X	A	B	A	B	
UW	A	X	X	X	UW	
UW	X	B	A	X	UW	
UW	B	A	A	B	UW	
free	(1,1)	(2,1)	(2,1)	(1,1)	(0,1)	(*A valid page*の数, *B valid page*の数)
free	3	2	3	1	1	*valid page*の数

1)モデルの説明
2)アルゴリズム
3)数値実験
4)結果の考察

実験内容

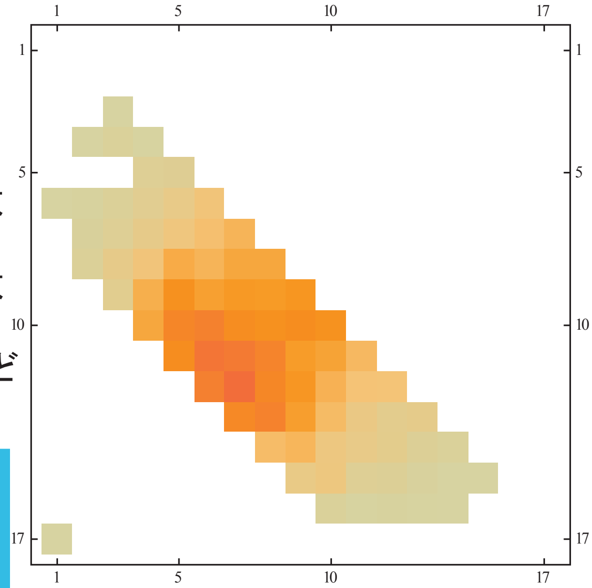
1. ブロックの数(b) : 5,000
ページの数(c) : 16
A $valid$ $page$の数(J) : $40{,}000 \times \rho$
B $valid$ $page$の数(K) : $40{,}000 \times \rho$
としたときのμとρ の値を変えることで
各ブロックの$valid$ ページの数がどのような変化するか調べる。

2. ブロックの数(b) : 5,000
ページの数(c) : 16
A $valid$ $page$の数(J) : $24{,}000 \times \rho$
B $valid$ $page$の数(K) : $56{,}000 \times \rho$
としたときのμとρ の値を変えることで
各ブロックの$valid$ ページの数がどのような変化するか調べる。

得られた結果の一例

1. $\rho = 0.8$,
   $\mu = 1$ の場合

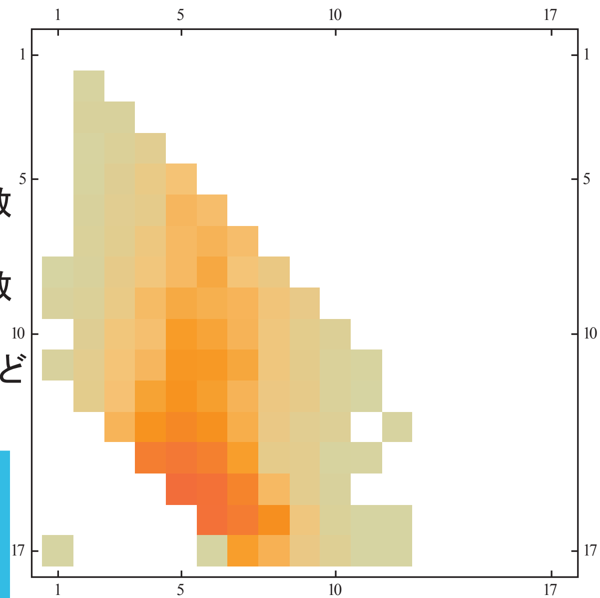
縦軸がα $valid$ $page$の数(上に行くほど多い)に、横軸がβ $valid$ $page$の数(右に行くほど多い)に対応し、色が濃いものほど数が多い。
この場合$c^* = 9$である。



得られた結果の一例

1. $\rho = 0.6$,
   $\mu = 0.2$の場合

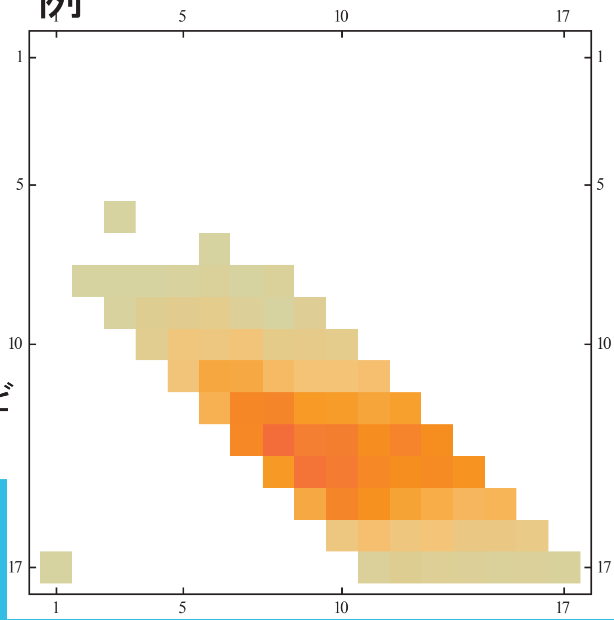
縦軸がα $valid$ $page$の数(上に行くほど多い)に、横軸がβ $valid$ $page$の数(右に行くほど多い)に対応し、色が濃いものほど数が多い。
この場合$c^* = 5$である。



得られた結果の一例

2. $\rho = 0.8$,
$\mu = 5$ の場合

縦軸がα valid pageの数(上に行くほど多い)に、横軸がβ valid pageの数(右に行くほど多い)に対応し、色が濃いものほど数が多い。
この場合$c^* = 9$である。



得られた結果のまとめ

μとρを変化させた時のc^*の値を以下の表にまとめた

1.

$\rho \diagdown \mu \times J/K$	0.1	0.2	0.5	1	2	5	10
0.6	5	5	4	4	4	5	5
0.7	7	7	6	6	6	7	7
0.8	9	9	9	9	9	9	9
0.9	12	12	12	12	12	12	12

2.

$\rho \diagdown \mu \times J/K$	0.1	0.2	0.5	1	2	5	10
0.6	5	5	4	4	4	4	5
0.7	7	7	6	6	6	7	7
0.8	9	9	9	9	9	9	9
0.9	12	12	12	12	12	12	12

1)モデルの説明
2)アルゴリズム
3)数値実験
4)結果の考察

結果の考察

・c^*の値はμの値(どちらの$valid\ page$ が選ばれやすいか)にはあまり依存していない。

・c^*の値はρの値(どのくらいブロックを使用しているか)に依存している。

・c^*の値は各$valid\ page$ の個数には依存していない。

３同種写像に付随するセルマー群の高速計算アルゴリズム

2013 年 1 月 26 日（土）高妻倫太郎
立命館アジア太平洋大学
rintaro@apu.ac.jp

はじめに

代数体上の楕円曲線の階数の上限を与えるセルマー群は、その計算可能性が理論的に保障されている重要な数論的対象であり、現在、代数系処理ソフト Magma をはじめとした様々な数学ソフトウェアに計算アルゴリズムが実装されている。今回、有理数体上の位数３の有理点をもつ任意の楕円曲線に対して、論文（R.Kozuma, Rocky Mountain J. Math., Vol.40, No.4 (2010), 1227--1255.）の“公式”を応用したセルマー群の計算アルゴリズムを、コンソールアプリとして Visual C++2010 で実装した。この“公式”（図１）では、セルマー群の計算に必要となるガロアコホモロジーの局所連結準同型の像を、約４回の単純な条件分岐（数値の大小比較、剰余計算）のみで決定することが可能であり、既存のアルゴリズムを高速化すると期待される。

プログラムについて

本プログラムは、１つのソースファイル（main.cpp）および３つのヘッダファイル（ellcv.h、arith.h、prime.h）からなる。main.cpp（２頁）ではコンソールのメインルーチンを実行し、ellcv.h（３頁）では楕円曲線と悪還元をもつ素数に関する２つのクラス（EllCv、BadPrime）を定義した。また、prime.h では 49999 番目までの素数をテーブル化し、必要となる数論的関数を arith.h に記述した。全体的な処理の流れとしては、ユーザーからの係数（A・B）入力取得後、悪還元をもつ素数における小平記号および局所連結準同型の像を論文（R.Kozuma, 2010）の“公式”で決定・格納し、それを用いてセルマー群を計算・出力する。

開発環境

ＣＰＵ	Intel Core i5-2520M（モバイル）2.50GHz
メモリ	4GB
ＯＳ	Windows 7 Professional (64bit)
開発	Visual Studio 2010 (C++)

～簡易フローチャート～

位数３の有理点をもつ $\mathbb{Q}$ 上の任意の楕円曲線

$$E : y^2 + Axy + By = x^3 \quad (A,\ B \in \mathbb{Z})$$

$A,\ B$ 入力

悪還元をもつ素数を計算

$$\Phi = \{\text{悪還元をもつ素数}\} \cup \{3\}$$

局所連結準同型の像を計算

$$\hat{\delta}_p : E(\mathbb{Q}_p) \to \mathbb{Q}_p^* / \mathbb{Q}_p^{*3}$$

セルマー群の計算

$$S^{(\hat{\phi})}(\widehat{E}/\mathbb{Q}) \simeq \left\{ d \in \mathbb{Q}^*/\mathbb{Q}^{*3} \;\middle|\; \begin{array}{c} d \in \mathrm{Im}\,\hat{\delta}_p \quad ({}^{\forall}\text{prime } p \in \Phi) \\ \nu_p(d) \equiv 0 \pmod 3 \quad ({}^{\forall}\text{prime } p \notin \Phi) \end{array} \right\}$$

双対セルマー群の計算（カッセルズの公式）

$$\#S^{(\phi)}(E/\mathbb{Q}) = \#S^{(\hat{\phi})}(\widehat{E}/\mathbb{Q}) \times \frac{\#E(\mathbb{Q})[\phi]}{\#\widehat{E}(\mathbb{Q})[\hat{\phi}]} \prod_p \frac{\hat{c}_p}{c_p}$$

出力

セルマー群による上限

$$\mathrm{rank}E(\mathbb{Q}) \le \dim_{\mathbb{F}_3} S^{(\hat{\phi})}(\widehat{E}/\mathbb{Q}) + \dim_{\mathbb{F}_3} S^{(\phi)}(E/\mathbb{Q}) - 1$$

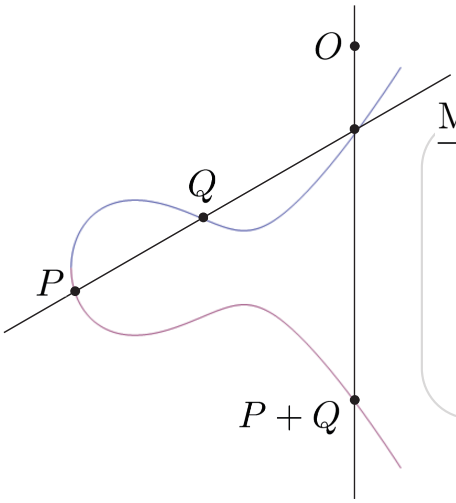
実行ファイルダウンロードはこちら

http://www.apu.ac.jp/~rintaro/program

3 同種写像に付随するセルマー群の高速計算アルゴリズム

2013年1月26日（土）
高妻倫太郎
立命館アジア太平洋大学

E : elliptic curve / a number field F $\ni$ a rational point O



Mordell-Weil Theorem (1922,-28)

$$E(F) \simeq \bigoplus_{i=1}^{t} \mathbb{Z}/d_i\mathbb{Z} \bigoplus \mathbb{Z}^{r}$$

rank$E(F)$

Selmer groups give upper bounds for rank$E(F)$
(effectively computed)

Set
- $F = \mathbb{Q}$
- $E/\mathbb{Q}$ has a rational 3-torsion point

$$E : y^2 + Axy + By = x^3 \quad (A,\ B \in \mathbb{Z})$$

Selmer groups $S^{(\phi)}(E/\mathbb{Q}),\ S^{(\hat{\phi})}(\widehat{E}/\mathbb{Q})$

$$\mathrm{rank} E(\mathbb{Q}) \leq \dim_{\mathbb{F}_3} S^{(\hat{\phi})}(\widehat{E}/\mathbb{Q}) + \dim_{\mathbb{F}_3} S^{(\phi)}(E/\mathbb{Q}) - 1$$

$\phi : E \to \hat{E}$: isogeny
$\hat{\phi} : \hat{E} \to E$: dual isogeny

Under our condition

$$\hat{\delta}_p : E(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p^* / \mathbb{Q}_p^{*3}$$

$$S^{(\hat{\phi})}(\widehat{E}/\mathbb{Q}) \simeq \left\{ d \in \mathbb{Q}^* / \mathbb{Q}^{*3} \;\middle|\; \begin{array}{ll} d \in \mathrm{Im}\, \hat{\delta}_p & ({}^\forall \text{prime } p \in \Phi) \\ \nu_p(d) \equiv 0 \pmod 3 & ({}^\forall \text{prime } p \notin \Phi) \end{array} \right\}$$

$$\Phi = \{\text{bad primes for } E/\mathbb{Q}\} \cup \{3\} \ (\text{finite})$$

This Selmer group is a subgroup of the finite group

$$R = \left\{ d \in \mathbb{Q}^* / \mathbb{Q}^{*3} \;\middle|\; \nu_p(d) \equiv 0 \pmod 3 ({}^\forall \text{prime } p \notin \Phi) \right\}$$

<u>Algorithm</u>

$\boxed{\text{step 1.}}$ Compute $S^{(\hat{\phi})}(\widehat{E}/\mathbb{Q})$ by searching all the elements in the finite group R contained in $\mathrm{Im}\,\hat{\delta}_p$ for every p

$\boxed{\text{step 2.}}$ Compute (Cassels' formula)

$$\#S^{(\phi)}(E/\mathbb{Q}) = \#S^{(\hat{\phi})}(\widehat{E}/\mathbb{Q}) \times \frac{\#E(\mathbb{Q})[\phi]}{\#\widehat{E}(\mathbb{Q})[\hat{\phi}]} \prod_p \frac{\hat{c}_p}{c_p}$$

$\bigcirc$ $\cdots$ use the 'formula' proven in

"R.Kozuma, Rocky Mountain J. Math., Vol.40, No.4 (2010), 1227–1255."

Institute of Mathematics for Industry
Kyushu University

非線形時系列に対する モデル推定と選択のソフト

九州大学
マス・フォア・インダストリ研究所
秦 攀　西井 龍映

1

ソフトの概要

Institute of Mathematics for Industry
Kyushu University

- 本ソフトは、時系列データを対象とし、統計モデルを推定と選択するものである。MATLAB を用いて作成し、時系列モデルの推定、評価および選択の三つの部分より構成する。
- 時系列解析、システム制御、信号処理などの領域に適用できる。現在、太陽風と地球磁場の相互関係に関する解析に応用されている。

2

ソフトの機能

モデルの定義

外部変数付き非線形自己回帰モデル

$$y(t)=\theta_0+\sum_{k=1}^{K}\theta_k b_k\big(\boldsymbol{x}(t)\big)+e(t)$$

- $\{y(t),u(t)|t=1,\dots,N\}$ は時系列データ
- $\boldsymbol{x}(t)=\big[y(t-1),\cdots,y(t-l_y),u(t-1),\cdots,u(t-l_u)\big]^T$
- $y(t)$ と $u(t)$ は目的変数と外部説明変数
- $e(t)\sim N(0,\sigma^2)$ はノイズ
- $b_k\big(\boldsymbol{x}(t)\big)$ は$\boldsymbol{x}(t)$ に関する基底関数
- $\theta_k\in R$ は回帰係数

モデルの推定

- 最尤法や重み付き最小二乗法などの推定法を用い、モデルを推定する

モデルの評価

- AIC や BIC、GIC などの情報量を計算し、モデルを評価する

モデルの選択

- 計算量を節約するために、増加法を用い、モデル候補から適切なモデルを選択する
- 開発者たちが提案したロバストモデル選択法を用い、モデル選択を行う (Qin, Nishii and Yang, 2012)

3

Dst 指数のモデリングへの応用 (1/2)

Dst 指数 (Disturbance Storm Time index)

磁気嵐の強さを示す指標である。磁気嵐が発電所や通信システムに損害を与えられるので、Dst 指数を予測するのは重要である。

外部変数付き自己回帰モデル

$$y(t)=\theta_0+\sum_{i=1}^{K_y}\theta_i^{Dst}y(t-i)$$
$$+\sum_{j=1}^{K_N}\theta_j^{N}N(t-j)+\sum_{k=1}^{K_V}\theta_k^{V}V(t-k)+\sum_{l=1}^{K_P}\theta_l^{P}P(t-l)+e(t)$$

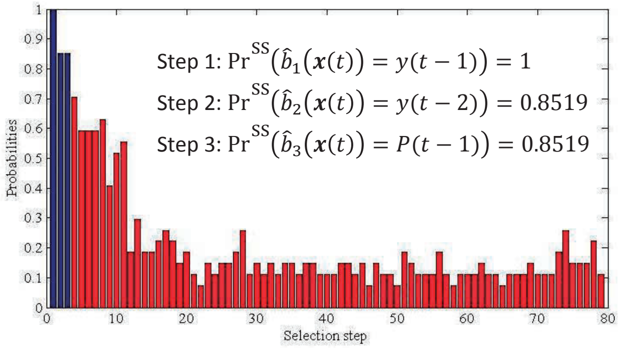
- $y(t)$ は Dst 指数
- $\{N(t),\ V(t),\ P(t)\}$ は太陽風の粒子密度、速度と圧力

The Salem nuclear plant transformer was damaged at **March 13, 1989, Superstorm, Dst = -589.**
John Kappenman, Metatech Corporation

4

Dst 指数のモデリングへの応用 (2/2)

ソフトの解析結果



増加法では、Subsamplingにより近似した各説明変数が選択された確率

モデル選択法	モデルのサイズ	R^2_{tr}	R^2_{te}
ロバスト選択法	4	0.9460	0.9486
AIC に基づいた増加法	39	0.9534	0.9469
BIC に基づいた選択法	22	0.9527	0.9478

5

将来の展開

- 非線形状態空間モデルなどのモデルに応対できるように多様化する。
- データの欠損の補足などの前処理機能を強化する。
- データおよび結果の可視化について改善する。
- **グラフィカルユーザインタフェースを追加する。**

6

ON BUTSON-TYPE HADAMARD MATRICES $H(17, 17)$

Kyoung-Tark Kim
poisonn00@hanmail.net
(with Mitsugu Hirasaka and Yoshihiro Mizoguchi)

Pusan National University

January 26, 2013
Hakata Workshop 2013

BUTSON-TYPE HADAMARD MATRICES

An $n \times n$ matrix $H = (H_{ij})$ is called an $H(p, n)$-MATRIX if

$H_{ij}^p = 1$ and $HH^{CT} = nI$ where I is the $n \times n$ identity matrix.

$\Longrightarrow$ An $H(p, n)$-matrix is called a Butson-type Hadamard matrix.

BASIC PROPERTIES FOR $H(p, n)$-MATRIX H

1. A permutation of the rows (columns) of H is permissible.
2. A multiplication of the elements of a row (column) of H by a fixed pth root of unity is permissible.

Difference Matrices

H : an $H(p,n)$-matrix

$\Longrightarrow$ A DIFFERENCE MATRIX $\mathrm{Diff}(H) \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_p)$ of H :

$$H = (H_{ij}) = \left(\exp\left(\frac{2\pi\sqrt{-1}E_{ij}}{p}\right)\right) \quad \overset{\text{def.}}{\Longleftrightarrow} \quad \mathrm{Diff}(H) = (E_{ij})$$

EXAMPLE : H is an $H(3,3)$-matrix

$$H = \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{pmatrix} \Longleftrightarrow \mathrm{Diff}(H) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

Fourier Matrix

The Fourier matrix F of order p is a $p \times p$ matrix such that

$$\mathrm{Diff}(F) = (ij)_{i,j=0}^{p-1} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 2 & \cdots & p-1 \\ 0 & 2 & 4 & \cdots & 2(p-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & p-1 & 2(p-1) & \cdots & (p-1)^2 \end{pmatrix}$$

$\Longrightarrow$ F is indeed an $H(p,p)$-matrix.

Kyoung-Tark Kim with Hirasaka and Mizoguchi Pusan National University

Orthogonality Correspondences

We will consider ONLY $H(p,p)$-matrix H where p is prime.

For two distinct rows of H

$$(\omega_1, \cdots, \omega_p) \perp (\eta_1, \cdots, \eta_p)$$
$$\Updownarrow$$
$$\omega_1 \cdot \overline{\eta_1} + \cdots + \omega_p \cdot \overline{\eta_p} = 0$$

For two distinct rows of $\mathrm{Diff}(H)$

$$(E_{i1}, \cdots, E_{ip}) \perp (E_{j1}, \cdots, E_{jp})$$
$$\Updownarrow$$
$$\{E_{i1} - E_{j1}, \cdots, E_{ip} - E_{jp}\} = \mathbb{Z}_p$$

Calculation of Candidates for Entry E_{ij}

Fix i and j. For $0 \leq a < i$ and $0 \leq b < j$,

$$\mathrm{Diff}(H) = \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & E_{ab} & \cdots & E_{aj} & \cdots \\ & \vdots & \ddots & \vdots & \\ \cdots & E_{ib} & \cdots & E_{ij} & \cdots \\ & \vdots & & \vdots & \ddots \end{pmatrix}$$

$\Longrightarrow E_{ib} - E_{ab} \neq E_{ij} - E_{aj}$.

Candidates for E_{ij}

E_{ij} satisfies $E_{ij} \neq E_{ib} + E_{aj} - E_{ab}$ for $0 \leq a < i$ and $0 \leq b < j$.

Some Redundancy for Calculation

Our INITIAL PART for Diff(H) is symmetric :

$\Uparrow$

$$\mathrm{Diff}(H) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 2 & 3 & 4 & \cdots \\ 0 & 2 & E_{22} & E_{23} & E_{24} & \cdots \\ 0 & 3 & E_{32} & E_{33} & E_{34} & \ddots \\ 0 & 4 & E_{42} & E_{43} & E_{44} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Symmetric Redundancy

We may assume that $E_{23} \leq E_{32}$.

Conclusion for $H(17, 17)$-matrices

For $p = 17$ we use a parallel algorithm on super computer with MPI (Massage Passing Interfaces) over TATARA FUJITSU PRIMERGY CX400 (in Kyushu Univ).

Computational Result

There is a unique $H(17, 17)$-matrix, namely,

the Fourier matrix of order 17.

◦ This calculation is done by super computer with 64 threads over computational complexity = 68 hours.

MI レクチャーノートシリーズ刊行にあたり

本レクチャーノートシリーズは、文部科学省 21COE プログラム「機能数理学の構築と展開」(H.15-19 年度) において作成した COE Lecture Notes の続刊である。今後、レクチャーノートは、文部科学省大学院教育改革支援プログラム「産業界が求める数学博士と新修士養成」(H19-21 年度) および、新しく採択された同グローバル COE プログラム「マス・フォア・インダストリ教育研究拠点」(H.20-24 年度) の推進において招聘する国内外の研究者による講義の講義録として出版するものである。

平成 20 年 7 月
グローバル COE プログラム
マス・フォア・インダストリ教育研究拠点
拠点リーダー　若山正人

博多ワークショップ「組み合わせとその応用」

－Hakata workshop, combinatrics and its applications－

発　行　2013年3月28日
編　集　溝口佳寛，脇隼人，平坂貢，谷口哲至，島袋修
発　行　九州大学マス・フォア・インダストリ研究所
九州大学大学院数理学研究院
グローバル COE プログラム「マス・フォア・インダストリ教育研究拠点」
〒819-0395 福岡市西区元岡744
九州大学伊都キャンパス数理学研究教育棟 GCOE 事務室
TEL 092-802-4404　FAX 092-802-4405
URL http://gcoe-mi.jp/

印　刷　城島印刷株式会社
〒810-0012 福岡市中央区白金 2 丁目 9 番 6 号
TEL 092-531-7102　FAX 092-524-4411

シリーズ既刊

Issue	Author／Editor	Title	Published
COE Lecture Note	Mitsuhiro T. NAKAO Kazuhiro YOKOYAMA	Computer Assisted Proofs - Numeric and Symbolic Approaches - 199pages	August 22, 2006
COE Lecture Note	M.J.Shai HARAN	Arithmetical Investigations - Representation theory, Orthogonal polynomials and Quantum interpolations- 174pages	August 22, 2006
COE Lecture Note Vol.3	Michal BENES Masato KIMURA Tatsuyuki NAKAKI	Proceedings of Czech-Japanese Seminar in Applied Mathematics 2005 155pages	October 13, 2006
COE Lecture Note Vol.4	宮田　健治	辺要素有限要素法による磁界解析 - 機能数理学特別講義 21pages	May 15, 2007
COE Lecture Note Vol.5	Francois APERY	Univariate Elimination Subresultants - Bezout formula, Laurent series and vanishing conditions - 89pages	September 25, 2007
COE Lecture Note Vol.6	Michal BENES Masato KIMURA Tatsuyuki NAKAKI	Proceedings of Czech-Japanese Seminar in Applied Mathematics 2006 209pages	October 12, 2007
COE Lecture Note Vol.7	若山　正人 中尾　充宏	九州大学産業技術数理研究センター キックオフミーティング 138pages	October 15, 2007
COE Lecture Note Vol.8	Alberto PARMEGGIANI	Introduction to the Spectral Theory of Non-Commutative Harmonic Oscillators 233pages	January 31, 2008
COE Lecture Note Vol.9	Michael I.TRIBELSKY	Introduction to Mathematical modeling 23pages	February 15, 2008
COE Lecture Note Vol.10	Jacques FARAUT	Infinite Dimensional Spherical Analysis 74pages	March 14, 2008
COE Lecture Note Vol.11	Gerrit van DIJK	Gelfand Pairs And Beyond 60pages	August 25, 2008
COE Lecture Note Vol.12	Faculty of Mathematics, Kyushu University	Consortium “MATH for INDUSTRY” First Forum 87pages	September 16, 2008
COE Lecture Note Vol.13	九州大学大学院 数理学研究院	プロシーディング「損保数理に現れる確率モデル」 — 日新火災・九州大学 共同研究 2008 年 11 月 研究会 — 82pages	February 6, 2009

シリーズ既刊

Issue	Author／Editor	Title	Published
COE Lecture Note Vol.14	Michal Beneš, Tohru Tsujikawa Shigetoshi Yazaki	Proceedings of Czech-Japanese Seminar in Applied Mathematics 2008 77pages	February 12, 2009
COE Lecture Note Vol.15	Faculty of Mathematics, Kyushu University	International Workshop on Verified Computations and Related Topics 129pages	February 23, 2009
COE Lecture Note Vol.16	Alexander Samokhin	Volume Integral Equation Method in Problems of Mathematical Physics 50pages	February 24, 2009
COE Lecture Note Vol.17	矢嶋　　徹 及川　正行 梶原　健司 辻　　英一 福本　康秀	非線形波動の数理と物理　66pages	February 27, 2009
COE Lecture Note Vol.18	Tim Hoffmann	Discrete Differential Geometry of Curves and Surfaces　75pages	April 21, 2009
COE Lecture Note Vol.19	Ichiro Suzuki	The Pattern Formation Problem for Autonomous Mobile Robots —Special Lecture in Functional Mathematics—　23pages	April 30, 2009
COE Lecture Note Vol.20	Yasuhide Fukumoto Yasunori Maekawa	Math-for-Industry Tutorial: Spectral theories of non-Hermitian operators and their application　184pages	June 19, 2009
COE Lecture Note Vol.21	Faculty of Mathematics, Kyushu University	Forum "Math-for-Industry" Casimir Force, Casimir Operators and the Riemann Hypothesis 95pages	November 9, 2009
COE Lecture Note Vol.22	Masakazu Suzuki Hoon Hong Hirokazu Anai Chee Yap Yousuke Sato Hiroshi Yoshida	The Joint Conference of ASCM 2009 and MACIS 2009: Asian Symposium on Computer Mathematics Mathematical Aspects of Computer and Information Sciences　436pages	December 14, 2009
COE Lecture Note Vol.23	荒川　恒男 金子　昌信	多重ゼータ値入門　111pages	February 15, 2010
COE Lecture Note Vol.24	Fulton B.Gonzalez	Notes on Integral Geometry and Harmonic Analysis　125pages	March 12, 2010
COE Lecture Note Vol.25	Wayne Rossman	Discrete Constant Mean Curvature Surfaces via Conserved Quantities 130pages	May 31, 2010
COE Lecture Note Vol.26	Mihai Ciucu	Perfect Matchings and Applications　66pages	July 2, 2010

シリーズ既刊

Issue	Author／Editor	Title	Published
COE Lecture Note Vol.27	九州大学大学院 数理学研究院	Forum "Math-for-Industry" and Study Group Workshop Information security, visualization, and inverse problems, on the basis of optimization techniques　100pages	October 21, 2010
COE Lecture Note Vol.28	ANDREAS LANGER	MODULAR FORMS, ELLIPTIC AND MODULAR CURVES LECTURES AT KYUSHU UNIVERSITY 2010　62pages	November 26, 2010
COE Lecture Note Vol.29	木田　雅成 原田　昌晃 横山　俊一	Magma で広がる数学の世界　157pages	December 27, 2010
COE Lecture Note Vol.30	原　　　隆 松井　　卓 廣島　文生	Mathematical Quantum Field Theory and Renormalization Theory 201pages	January 31, 2011
COE Lecture Note Vol.31	若山　正人 福本　康秀 高木　　剛 山本　昌宏	Study Group Workshop 2010 Lecture & Report　128pages	February 8, 2011
COE Lecture Note Vol.32	Institute of Mathematics for Industry, Kyushu University	Forum "Math-for-Industry" 2011 "TSUNAMI-Mathematical Modelling" Using Mathematics for Natural Disaster Prediction, Recovery and Provision for the Future　90pages	September 30, 2011
COE Lecture Note Vol.33	若山　正人 福本　康秀 高木　　剛 山本　昌宏	Study Group Workshop 2011 Lecture & Report　140pages	October 27, 2011
COE Lecture Note Vol.34	Adrian Muntean Vladimír Chalupecký	Homogenization Method and Multiscale Modeling　72pages	October 28, 2011
COE Lecture Note Vol.35	横山　俊一 夫　　紀恵 林　　卓也	計算機代数システムの進展　210pages	November 30, 2011
COE Lecture Note Vol.36	Michal Beneš Masato Kimura Shigetoshi Yazaki	Proceedings of Czech-Japanese Seminar in Applied Mathematics 2010 107pages	January 27, 2012
COE Lecture Note Vol.37	若山　正人 高木　　剛 Kirill Morozov 平岡　裕章 木村　正人 白井　朋之 西井　龍映 栄　伸一郎 穴井　宏和 福本　康秀	平成 23 年度 数学・数理科学と諸科学・産業との連携研究ワークショップ　拡がっていく数学　～期待される "見えない力" ～ 154pages	February 20, 2012

シリーズ既刊

Issue	Author／Editor	Title	Published
COE Lecture Note Vol.38	Fumio Hiroshima Itaru Sasaki Herbert Spohn Akito Suzuki	Enhanced Binding in Quantum Field Theory　204pages	March 12, 2012
COE Lecture Note Vol.39	Institute of Mathematics for Industry, Kyushu University	Multiscale Mathematics; Hierarchy of collective phenomena and interrelations between hierarchical structures　180pages	March 13, 2012
COE Lecture Note Vol.40	井ノ口順一 太田　泰広 筧　　三郎 梶原　健司 松浦　　望	離散可積分系・離散微分幾何チュートリアル 2012　152pages	March 15, 2012
COE Lecture Note Vol.41	Institute of Mathematics for Industry, Kyushu University	Forum “Math-for-Industry” 2012 “Information Recovery and Discovery”　91pages	October 22, 2012
COE Lecture Note Vol.42	佐伯　　修 若山　正人 山本　昌宏	Study Group Workshop 2012 Abstract, Lecture & Report　178pages	November 19, 2012
COE Lecture Note Vol.43	Institute of Mathematics for Industry, Kyushu University	Combinatorics and Numerical Analysis Joint Workshop　103pages	December 27, 2012
COE Lecture Note Vol.44	萩原　　学	モダン符号理論からポストモダン符号理論への展望　107pages	January 30, 2013
COE Lecture Note Vol.45	金山　　寛	Joint Research Workshop of Institute of Mathematics for Industry (IMI), Kyushu University “Propagation of Ultra-large-scale Computation by the Domain-decomposition-method for Industrial Problems (PUCDIP 2012)” 121pages	February 19, 2013
COE Lecture Note Vol.46	西井　龍映 栄　伸一郎 岡田　勘三 落合　啓之 小磯　深幸 斎藤　新悟 白井　朋之	科学・技術の研究課題への数学アプローチ ―数学モデリングの基礎と展開―　325pages	February 28, 2013
COE Lecture Note Vol.47	SOO TECK LEE	BRANCHING RULES AND BRANCHING ALGEBRAS FOR THE COMPLEX CLASSICAL GROUPS　40pages	March 8, 2013