on discrete differential geometry and its links to visualization

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Abstract

Mathematical visualization as always been a key tool in what we now call discrete differential geometry. The talk will focus on applications of discrete differential geometry in visualization as well as on visualization in discrete differential geometry. An interesting example for such an mutual interplay are for example discrete conjugate nets - meshes with planar quadrilateral faces. They play an important role in architectural geometry where constructions of free form surfaces from pre-manufactured planar panels is desired. Additional constraints occur when one needs to have parallel meshes (like for certain glass steel constructions). The theory of conjugate nets that allow for parallel nets gives rise to interesting definitions of curvature for such surfaces.

1 introduction

Discrete differential geometry is in part motivated by the needs of computer graphics for discrete analogues of classical differential geometry notions.

In some sense computer graphics and differential geometry model the same thing: smooth geometric objects. But while differential geometry models them analytically, describing them through smooth maps, computer graphics needs to model them through approximation with some discrete sets of values, simply because representations in the computer (better in the computers graphics card) are finite. As a consequence many of the techniques that geometers have developed through the last centuries are not directly applicable in computer graphics and visualization. This is where discrete differential geometry comes into play. By developing discrete analogues of the differential geometric notions, methods from the classical theory become usable in a discrete setup. A main paradigm of discrete differential geometry is, that the discretizations should “behave” in the same way the smooth objects do. If for example a class of smooth surfaces is invariant under a certain transformations group, so should be the discretization; or if the class of surfaces posesses a special transformation, the the discretization should heve one as well.

2 the Tractrix of a curve

One of my favorite examples is a very simple one: Given two points linked by a rigid joint. If the first moves on a given curve how moves the second one?[1]

This problem can easily be phrased as a differential equation: If the first point p moves on a curve $p(t)$ the second point q in distance l moves on a curve $q(t)$ given by the conditions:

- $\frac{d}{dt}q(t) \parallel q(t) - p(t) = v$
- $|(q(t) - p(t)| = l$

Fig. 2 shows the tractrix for $p(t)$ being a straight line. In a discretization of this

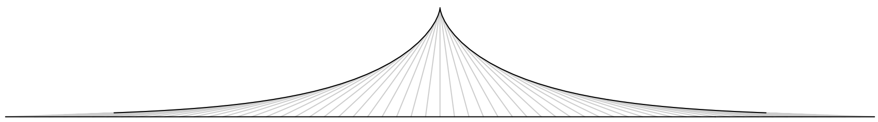

Figure 1: The Tractrix of a straight line

the curve on which the first point p moves would be a sequence of points p_k. Now a straight forward discretization of the Tractrix would be the following: If p_k moves (jumps) to p_{k+1} then q_{k+1} is chosen to be the point on the straight line through p_{k+1} and q_k that is in distance l from p_{k+1} and on the same side as q_k. While this discretization might work well and even converge to the smooth solution eventually, it has a crucial drawback: In contrast to the smooth model the discretization is not time reversible. If $p_{k+1} = p_{k-1}$ then q_{k+1} need not be equal to q_{k-1}.

There is, however, a smooth result that leads to a simple discretization that overcomes this. In the smooth case it turns out that there is a third curve $r(t)$ at distance $2l$ from $p(t)$ so that $q(t) = \frac{1}{2}(p(t) + r(t))$ and $p(t)$ and $r(t)$ are arc-length related: $|\frac{d}{dt}p(t)| = |\frac{d}{dt}r(t)|$. This curve r is usually called a Darboux transform of p and this is one of the special transformations I referd to in the introduction.

The relation between p and r translates easily into the discrete realm: $|p_{k+1} - p_k| = |r_{k+1} - r_k|$. And since $|p_k - r_k| = |p_{k+1} - r_{k+1}| = 2l$ one sees that each four points p_k, p_{k+1}, r_{k+1}, and r_k form a parallelogram of a parallelogram folded along one of its diagonals. The second solution is the one we are interested in. The discrete Tractrix of p can now be defined as simply $q_k = \frac{1}{2}(p_k + r_k)$ and since the construction of r_{k+1} from p_k, p_{k+1}, and r_k is completely symmetric in k and $k+1$ time reversibility is build in the definition this time.

This is a very basic example, but it shows how insights from the smooth theory can help make better (in the sense of better behaving) discretizations. One can find more on the discrete Tractrix in [H08].

[1] This problem is usually attributed to Leibniz (1646-1716). He states it in his 1693 Leipziger Acta eruditorum Problem: "In the xy-plane drag a point P with a tightly strained string PZ of length a. The "drag point" Z shall propagate along the positive y-axis, and at the beginning P shall be in $(a, 0)$. Which curve is described by P?" For comprehension Leibniz imagined a pocket watch on a chain. But as source of the problem he mentions the Paris Architect Claude Perault.

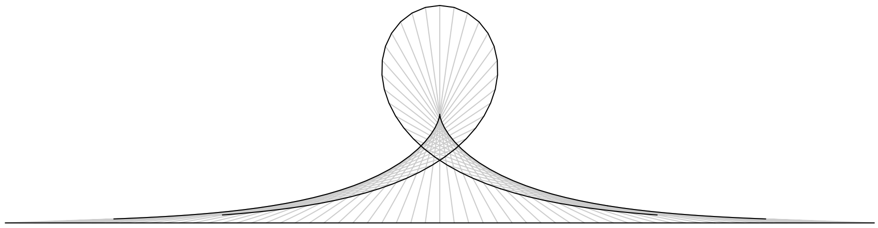

Figure 2: The Tractrix and the Darboux transform of a straight line (sometimes called Euler loop).

3 Conjugate meshes, offset meshes, and the Steiner formula

Another example are discrete conjugate nets: quadrilateral meshes (all faces are quadrilateral and usually a map from $\mathbb{Z}^2$ to $\mathbb{R}^3$ (or $\mathbb{R}^n$)) with planar faces. They were first introduced by Sauer [S70] and for higher dimensions by Doliwa and Santini [DoS97]); see for example [P08] and references therein. In discrete differential geometry quadrilateral meshes are desired when discretizing parameterized surfaces, since the lattice direction correspond to the parameter lines[2] and meshes with planar faces are in fact polyhedral surfaces.

For a similar reason quadrilateral meshes are useful in computer graphics when ever one needs to texture map a surface – the texture coordinates for a mesh are nothing else than a discretized parameterization. However, general non planar quadrilaterals have the drawback that they usually need to be triangulated for rendering (most state of the art graphics hardware expects triangles for rendering) and the choice of triangulation affects the visual result. Only if the quadrilaterals are planar the choice of triangulation has practically no influence.

Besides that planar quadrilateral meshes are of importance for architectural geometry [P08]. When realizing free form surfaces in glass-steel constructions (see Fig. 3) it is desirable to have flat facets, since curved glass panels are very expensive.

There is a smooth counterpart to quadrilateral meshes with planar faces:

Definition 3.1 *An immersion*[3] $f : \mathbb{R}^2 \to \mathbb{R}^3$ *is called a conjugate net, if*

$$\frac{\partial^2}{\partial x \partial y} f \in \operatorname{span}\{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\}.$$

This notion actually belongs to projective geometry (as does the notion of a quadrilateral mesh with planar faces, so here we have again fullfiled our paradigm regarding the transformation goups). The condition, that the mixed second derivative lies in the span of the first ones, says roughly that the points $f(x, y), f(x + \epsilon, y), f(x, y + \epsilon)$, and $f(x + \epsilon, y + \epsilon)$ all do lie in a plane for very small ϵ.

[2] But there are discrete differential geometry for other combinatorics like triangle meshes, as well.

[3] a smooth map for which the differential has maximal rank everywhere

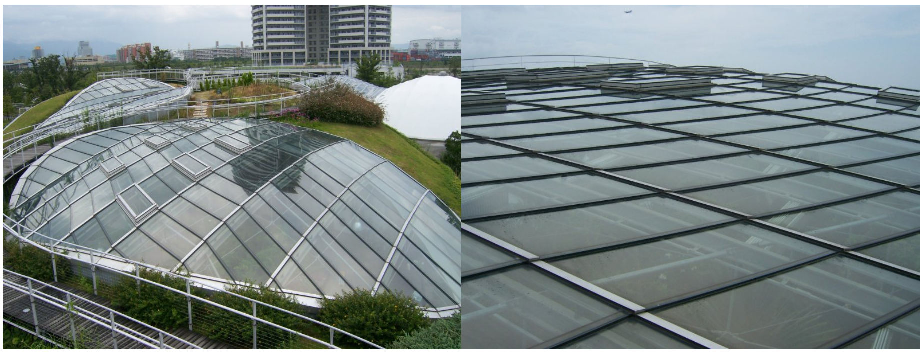

Figure 3: a discrete conjugate mesh (City Island Park, Fukuoka)

For smooth surfaces f with normal field N there is an interesting formula, the Steiner formula, that lets one compute the mean and Gaußian curvature from the area $A(f)$ and the area of an offset surface $f_t = f + tN$:

$$A(f_t) = A(f) + 2tH(f) + t^2K(f)$$

where $H(f)$ is the integral over the mean curvature of the surface f and likewise $K(f)$ is the integral over the Gauß curvature of f.

One can use it to define discrete curvatures for meshes with planar faces once one has defined what parallel meshes are. The definition of parallel surfaces involves the normals and in the discrete case there is some choice. It is natural to call a mesh an offset mesh of another, if it is parallel (in the sense that all corresponding edges are parallel) and in constant distance. Here the distance can be taken at either vertices, edges, or faces giving offset meshes with different properties. For architectural applications for example these offset meshes are important, since the glass-steel constructions are not infinitesimally thin objects but the steel frame for example might have a considerable (fixed) width which would make it necessary that the mesh has a parallel mesh with constant edge offset. If on the other hand double glass panels are needed these usual should have a constant distance leading to face offset meshes.

The existence of parallel surfaces allows the definition of curvatures. If Q is a quadrilateral in a quadrilateral mesh with planar faces and Q_t is its offset quadrilateral at distance t, one finds that the area $A(Q_t)$ of Q_t changes quadratic in the distance t. In particular

$$A(Q_t) = (1 + 2tH + t^2K)A(Q)$$

with $A(Q)$ being the area of Q and H and K some numbers that we can interpret as discrete mean and Gauß curvatures.

In particular surfaces with constant curvatures can now be defined. Fig. 4 is a discrete surface of vanishing mean curvature - a discrete minimal surface.

For more on this see [PLWBW07] or [PLWBW07] and references therein , which is also a valuable source on the architectural applications.

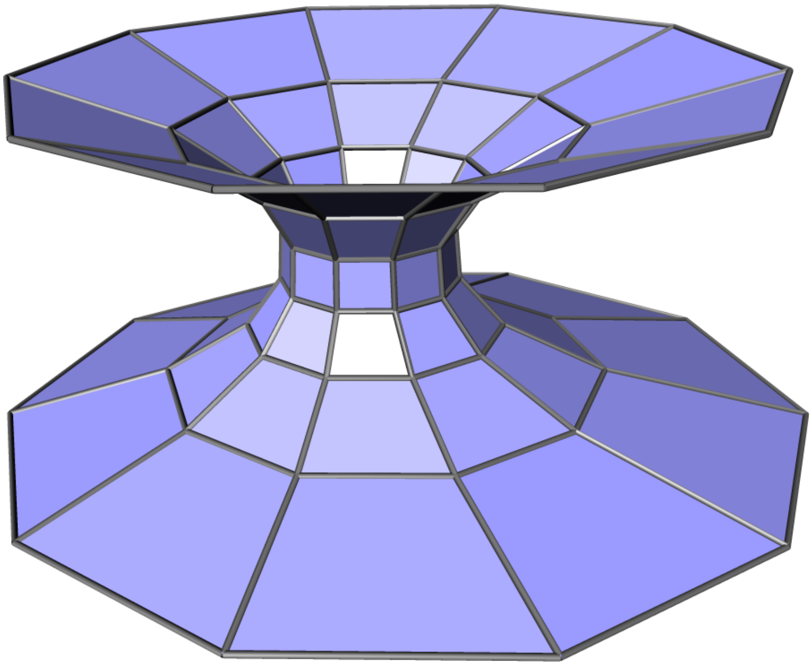

Figure 4: A discrete minimal surface: the Catenoid.

4 discrete conformal maps

The central problem of texturing a surface is the choice of the mapping. In general there is no isometric map from a surface in the plane. Thus any texture map will inevitable have distortions and the question is not how to prevent this (it is impossible) but how to distribute it. Often a reasonable choice is a conformal map, that does not preserve distances but does preserve angles. Again ideas from discrete differential geometry can help here: Discretizations of conformal maps in the plane have been studied and there are good discretizations using circle packings or circle patterns. The use of circle seems unexpected from a computer graphics point of view, but the group of conformal transformations in space are generated by inversions on spheres and these inversions preserve circles. Thus phrasing a discretization in terms of circles (and their intersection angles) has the the desired invariance build in from the beginning. For more on this see e.g. [KSS06].

5 teaching and edutainment

Visualization is of course a great teaching tool as well and here mathematical visualization learns a lot from computer graphics and game development. Texturing objects with slight bumpiness and decal maps for example helps getting a better feel for realism and provides marks on the surface, the eye can use for orientation.

In a recent public lecture I showed a “walk through” of a Klein bottle

Figure 5: A bear textured with help of a discrete conformal map.

with jReality (`www.jreality.de`). The immersive environment and the "ego-shooter" like navigation helps here to get a better feel for depth, size, and spacial orientation. In fact the visualization group at TU Berlin experimented for a while with using the "Far Cry" engine to visualize surfaces (`http://www.math.tu-berlin.de/geometrie/gallery/vr/vr.shtml`). At that point however, game engines were not prepared for rendering complicated dynamically changing geometries. At the moment the physics engine jBullet gets attached to jReality to add even more realism to the virtual environments.

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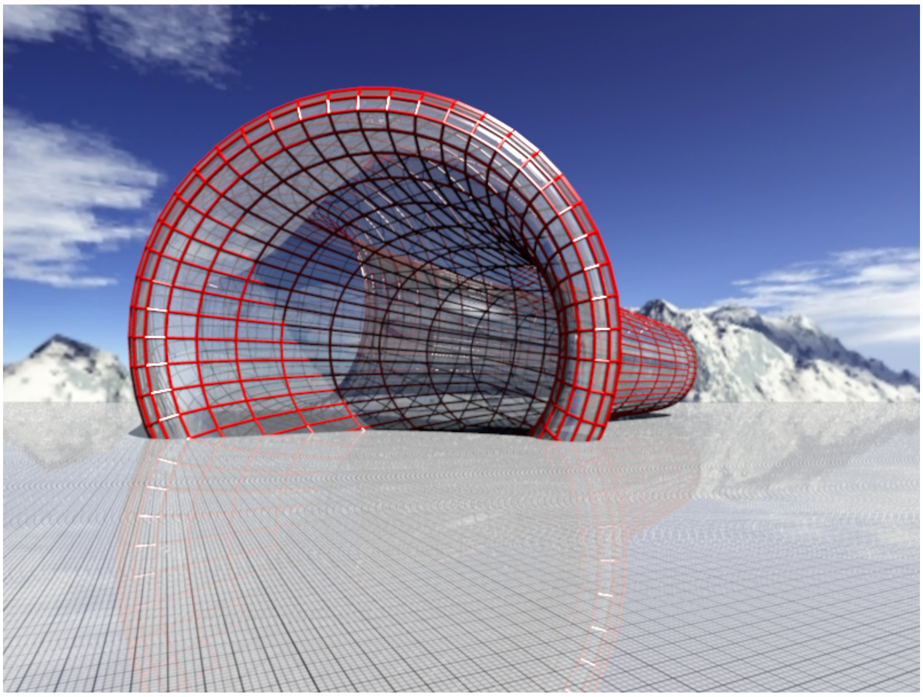

Figure 6: A Klein bottle in jReality

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