One-parameter families of Legendre curves and plane line congruences

Yutaro Kabata
& Masatomo Takahashi

MI 2019-2
(Received August 8, 2019)
One-parameter families of Legendre curves and plane line congruences

Yutaro Kabata and Masatomo Takahashi

July 23, 2019

Abstract
Families of curves in the Euclidean plane naturally contain singular curves, where the frame of classical differential geometry does not work well. We introduce the notions of one-parameter family of Legendre curves in the Euclidean plane, congruent equivalence and curvature. Especially, a one-parameter family of Legendre curves can contain singular curves, and is determined by the curvature up to congruence. We also give properties of one-parameter families of Legendre curves. As applications, we give a relation between one-parameter families of Legendre curves and Legendre surfaces. Moreover, we study plane line congruences (one parameter families of lines in plane) in terms of the curvatures as one-parameter families of Legendre curves.

1 Introduction
We are interested in geometrical differential aspects of families of curves in the Euclidean plane. In the method of classical curve theory, one can never deal with families of curves to which degenerate curves belong: for example, a family of parallel curves in plane where curves naturally become non-immersive as in Figure 1; a one-parameter family of lines in plane as in Figure 2 (the curvature of a line is identically zero). This paper aims to give a new framework to deal with such a general one-parameter family of curves.

As smooth plane curves with singular points, that is, singular plane curves, we may consider frontals and Legendre curves in the unit tangent bundle over the Euclidean plane. In [6], we gave existence and uniqueness theorems of the curvature of Legendre curves. In the present paper, as plane to plane maps, we consider one-parameter families of Legendre curves. We define the notions of a congruent equivalence and a curvature such that the one-parameter family of Legendre curves is determined by the curvature up to congruence, which is a natural expansion of the theory for Legendre curves in [6].

We study a relation between the curvature of the map as a one-parameter families of Legendre curves and differential topological invariants of the map some of which are induced from

2010 Mathematics Subject classification: 58K05, 57R45, 53A55
Key Words and Phrases. One-parameter family of Legendre curves, curvature, plane to plane map, plane line congruence.
singularity theory. Notice that the curvature is a kind of differential geometrical invariant. In the usual sense, differential geometry means the geometry of a map $\mathbb{R}^n \to \mathbb{R}^p$ where the dimension number $p$ of the target space is bigger than the dimension number $n$ of the source space. We emphasize that our approach implies a new direction of differential geometry: differential geometry of a map with general dimensions of the source and target spaces. Note that the different approach to this idea is shown by K. Saji in his unpublished work from the viewpoint of a normal form (cf. [22, 23, 27]).

One of the most typical classes in one-parameter families of Legendre curves is the class of one parameter families of lines in plane (called the plane line congruences in the present paper). Line geometry is a classical subject (cf. [21, 29]), and recently singularity theory provides it with new insights (cf. [16, 17, 18]). For example, singularities generically appearing in line congruences (two-parameter families of lines) in 3-space or ruled surfaces (one parameter families of lines) in 3-space are classified in [17, 18]. We deal with a plane line congruence, and compare the curvature of it with the types of singularities of maps and functions related to it in §5. Especially, in §6, the exact $A$-equivalent types (up to $A_e$-codimension two) of plane line congruences as a map $\mathbb{R}^2 \to \mathbb{R}^2$ are geometrically characterized in terms of the curvature as a one-parameter families of Legendre curves. Note that the singularities of plane line congruences as plane to plane maps are considered as the envelop or evolute in a generalized sense (cf. Figures 8-15).

Moreover, one-parameter families of curves naturally appear when we project surfaces equipped with families of curves into planes. For example, the projections of ruled surfaces to planes give plane line congruences, see Figure 2. There have been a lot of works on the application of singularity theory to the area of vision science (cf. [4, 5, 16, 20]), while they have been mainly concerned with the apparent contour of a surface (the discriminant of a projection mapping restricted to the surface). On the other hand, if the surface is equipped with a family of curves suitably, we have the curvature of the families of the projected curves (as a one-parameter families of Legendre curves) at points even outside the apparent contour. For instance, in §5.1.1, §6.2 and §7.1, we investigate a local nature of a plane (normal) line congruence around a point at which the Jacobian of the map is not equal to zero, but the differential vanishes. Namely, the point is not a singularity of the map but a singularity of the Jacobian. Note also that the Jacobian is characterized by the curvature of the plane line congruence. Thus our method possibly gives new tools to the area of vision science. On the other hand, as smooth surfaces with singular points, that is, singular surfaces, we may consider frontals or framed surfaces in the Euclidean space [1, 2, 10, 11]. Hence we can investigate the
differential geometrical relation between a surface and its projected image in a more general setting than ever before. This application will be discussed in somewhere else by the authors.

The paper is organized as follows: We give the existence and uniqueness theorems of the curvature of one-parameter families of Legendre curves in §2. We also give properties of one-parameter families of Legendre curves in §3. As applications, we give a relation between one-parameter families of Legendre curves and Legendre surfaces in §4. Moreover, in §5-7, we study local geometry of plane line congruences from the viewpoint of the curvatures as one-parameter families of Legendre curves, where normal line congruences are mainly dealt with. In §5, we study what kind of geometrical information is given from $\mathcal{R}$-types of functions in the curvatures of normal line congruences. In §6, we study the relation between higher order information of functions in the curvatures and several geometrical properties of normal line congruences: some new notions (the index of a function, Jacobian constant curve etc.) are defined; and several unstable $\mathcal{A}$-equivalent types of normal line congruences are precisely studied. In §7, we show several examples of plane line congruences with figures.

All maps and manifolds considered here are differential of class $C^\infty$.

Acknowledgement. The first author is partially supported by JSPS KAKENHI Grant Number JP 16J02200. The second author is partially supported by JSPS KAKENHI Grant Number JP 17K05238.

2 Legendre curves and one-parameter families of Legendre curves

Let $\mathbb{R}^2$ be the Euclidean plane equipped with the inner product $a \cdot b = a_1b_1 + a_2b_2$, where $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2$. We denote the norm of $a$ by $|a| = \sqrt{a \cdot a}$.

We review on the theory of Legendre curves in the unit tangent bundle over $\mathbb{R}^2$, in detail see [6]. We say that $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is a Legendre curve if $(\gamma, \nu)^*\theta = 0$ for all $t \in I$, where $\theta$ is a canonical contact form on the unit tangent bundle $T_1\mathbb{R}^2 = \mathbb{R}^2 \times S^1$ over $\mathbb{R}^2$ (cf. [1, 2]). This condition is equivalent to $\dot{\gamma}(t) \cdot \nu(t) = 0$ for all $t \in I$. We say that $\gamma : I \to \mathbb{R}^2$ is a frontal if there exists $\nu : I \to S^1$ such that $(\gamma, \nu)$ is a Legendre curve. Examples of Legendre curves see [14, 15]. We denote by $J(a) = (-a_2, a_1)$ the anticlockwise rotation by $\pi/2$ of a vector $a = (a_1, a_2)$. We have the Frenet formula of a frontal $\gamma$ as follows. We put on $\mu(t) = J(\nu(t))$. Then we call the pair $\{\nu(t), \mu(t)\}$ a moving frame of a frontal $\gamma(t)$ in $\mathbb{R}^2$ and we have the Frenet formula of the frontal (or, Legendre curve),

$$\begin{pmatrix} \nu(t) \\ \mu(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix} \begin{pmatrix} \nu(t) \\ \mu(t) \end{pmatrix}, \quad \gamma(t) = \beta(t) \mu(t),$$

where $\ell(t) = \dot{\nu}(t) \cdot \mu(t)$ and $\beta(t) = \dot{\gamma}(t) \cdot \mu(t)$. We call the pair $(\ell, \beta)$ the curvature of the Legendre curve.

Definition 2.1 Let $(\gamma, \nu)$ and $(\gamma, \tilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves. We say that $(\gamma, \nu)$ and $(\gamma, \tilde{\nu})$ are congruent as Legendre curves if there exist a constant rotation $A \in SO(2)$ and a translation $a$ on $\mathbb{R}^2$ such that $\gamma(t) = A(\gamma(t)) + a$ and $\tilde{\nu}(t) = A(\nu(t))$ for all $t \in I$.

Theorem 2.2 (Existence Theorem for Legendre curves) Let $(\ell, \beta) : I \to \mathbb{R}^2$ be a smooth mapping. There exists a Legendre curve $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ whose associated curvature of the Legendre curve is $(\ell, \beta)$. 3
Theorem 2.3 (Uniqueness Theorem for Legendre curves) Let \((\gamma, \nu)\) and \((\tilde{\gamma}, \tilde{\nu}) : I \to \mathbb{R}^2 \times S^1\) be Legendre curves with the curvatures of Legendre curves \((\ell, \beta)\) and \((\tilde{\ell}, \tilde{\beta})\). Then \((\gamma, \nu)\) and \((\tilde{\gamma}, \tilde{\nu})\) are congruent as Legendre curves if and only if \((\ell, \beta)\) and \((\tilde{\ell}, \tilde{\beta})\) coincide.

We now consider one-parameter families of Legendre curves in the unit tangent bundle \(T_1\mathbb{R}^2\) over \(\mathbb{R}^2\). Let \(U\) be a simply connected domain in \(\mathbb{R}^2\).

Definition 2.4 Let \((f, \nu) : U \to \mathbb{R}^2 \times S^1\) be a smooth mapping. We say that \((f, \nu)\) is a one-parameter family of Legendre curves with respect to \(u\) (respectively, with respect to \(v\)) if \(f_u(u, v) \cdot \nu(u, v) = 0\) (respectively, \(f_v(u, v) \cdot \nu(u, v) = 0\)) for all \((u, v) \in U\).

If \((f, \nu)\) is a one-parameter family of Legendre curves with respect to \(u\), then \((f(\cdot, v), \nu(\cdot, v))\) is a Legendre curve for each fixed parameter \(v\), that is, \((f(\cdot, v), \nu(\cdot, v))\) is an integrable curve with respect to the canonical contact 1-form on \(\mathbb{R}^2 \times S^1\). Therefore, \(f : U \to \mathbb{R}^2\) is a one-parameter family of frontals.

In this paper, we deal with one-parameter families of Legendre curves with respect to \(u\). We define \(\mu(u, v) = J(\nu(u, v))\). Since \(\{\nu(u, v), \mu(u, v)\}\) is a moving frame along \(f(u, v)\) on \(\mathbb{R}^2\), we have the Frenet type formula.

\[
\begin{align*}
\begin{pmatrix}
\nu_u(u, v) \\
\mu_u(u, v)
\end{pmatrix} &= \begin{pmatrix} 0 & \ell(u, v) \\
-\ell(u, v) & 0 \end{pmatrix} \begin{pmatrix}
\nu(u, v) \\
\mu(u, v)
\end{pmatrix}, \\
\begin{pmatrix}
\nu_v(u, v) \\
\mu_v(u, v)
\end{pmatrix} &= \begin{pmatrix} 0 & L(u, v) \\
-L(u, v) & 0 \end{pmatrix} \begin{pmatrix}
\nu(u, v) \\
\mu(u, v)
\end{pmatrix}, \\
f_u(u, v) &= \beta(u, v)\mu(u, v), \\
f_v(u, v) &= A(u, v)\nu(u, v) + B(u, v)\mu(u, v),
\end{align*}
\]

where

\[
\left\lbrace \begin{array}{l}
\ell(u, v) = \nu_u(u, v) \cdot \mu(u, v), \\
L(u, v) = \nu_u(u, v) \cdot \mu(u, v), \\
\beta(u, v) = f_u(u, v) \cdot \mu(u, v), \\
A(u, v) = f_v(u, v) \cdot \nu(u, v), \\
B(u, v) = f_v(u, v) \cdot \mu(u, v).
\end{array} \right\}
\]

(1)

By the integrability conditions \(\nu_{uv}(u, v) = \nu_{vu}(u, v)\) and \(f_{uv}(u, v) = f_{vu}(u, v)\), \((\ell, L, \beta, A, B)\) satisfies the conditions

\[
\left\lbrace \begin{array}{l}
L_u(u, v) = \ell_v(u, v), \\
A_u(u, v) = B(u, v)\ell(u, v) - L(u, v)\beta(u, v), \\
B_u(u, v) = \beta_v(u, v) - A(u, v)\ell(u, v)
\end{array} \right\}
\]

(2)

for all \((u, v) \in U\). We call the mapping \((\ell, L, \beta, A, B)\) with the integrability condition (2) the curvature of the one-parameter family of Legendre curves \((f, \nu)\).

Each component in the above curvature is geometrically defined as in equations in (1). We state some additional properties of them in the following remarks:

Remark 2.5 For a one-parameter family of Legendre curves \((f, \nu)\) with respect to \(u\), \((f, \nu)(u, v_0)\) is a Legendre curve for a fixed value \(v_0\), and \((\ell, \beta)(u, v_0)\) is the curvature of it in the sense of Theorem 2.2.
Remark 2.6 For a smooth mapping \( f: \mathbb{R}^2 \to \mathbb{R}^2, (u, v) \mapsto (f_1(u, v), f_2(u, v)) \), the Jacobian \( \lambda \) of \( f \) is defined as
\[
\lambda(u, v) = J_f(u, v) = \begin{vmatrix} \frac{\partial f_1}{\partial u}(u, v) & \frac{\partial f_1}{\partial v}(u, v) \\ \frac{\partial f_2}{\partial u}(u, v) & \frac{\partial f_2}{\partial v}(u, v) \end{vmatrix},
\]
where \( \cdots \) means the determinant of a matrix. The Jacobian \( \lambda \) is related to information of the local density of the image of the mapping \( f \). Especially, \( \lambda(p) = 0 \) for \( p \in \mathbb{R}^2 \) means \( p \) is a singularity of \( f \). The Jacobian plays an important role in characterizations of singularities (see [12, 19, 26]). For a one-parameter family of Legendre curves \( (f, \nu) \), the Jacobian of \( f \) is written as \( \lambda = -\beta A \), where \( \beta \) and \( A \) are components of the curvature of \( (f, \nu) \). Note also that the differential of the Jacobian with respect to \( u \) or \( v \) measures a ratio of the change of the density of families of curves along special curves.

Remark 2.7 Let \( (f, \nu): U \to \mathbb{R}^2 \times S^1 \) be a one-parameter family of Legendre curves with the curvature \( (\ell, L, \beta, A, B) \). Then \( (f, -\nu) \) is also a one-parameter family of Legendre curves with the curvature \( (\ell, L, -\beta, -A, -B) \). Moreover, \( (-f, \nu) \) is also a one-parameter family of Legendre curves with the curvature \( (\ell, L, -\beta, -A, -B) \).

Definition 2.8 Let \( (f, \nu) \) and \( (\tilde{f}, \tilde{\nu}): U \to \mathbb{R}^2 \times S^1 \) be one-parameter families of Legendre curves. We say that \( (f, \nu) \) and \( (\tilde{f}, \tilde{\nu}) \) are congruent as one-parameter family of Legendre curves if there exist a constant rotation \( A \in SO(2) \) and a constant vector \( a \in \mathbb{R}^2 \) such that \( \tilde{f}(u, v) = A(f(u, v)) + a \) and \( \tilde{\nu}(u, v) = A(\nu(u, v)) \) for all \( (u, v) \in U \).

We gave the existence and uniqueness theorems for one-parameter families of Legendre curves in [24, 28]. However, we give here an explicit construction of one-parameter families of Legendre curves by using the curvatures.

Theorem 2.9 (Existence Theorem for one-parameter families of Legendre curves)
Let \( (\ell, L, \beta, A, B): U \to \mathbb{R}^5 \) be a smooth mapping with the integrability condition. There exists a one-parameter family of Legendre curves \( (f, \nu): U \to \mathbb{R}^2 \times S^1 \) whose associated curvature is \( (\ell, L, \beta, A, B) \).

Proof. Let \( (u_0, v_0) \in U \) be fixed. We define a smooth mapping \( \theta: I \times \Lambda \to \mathbb{R} \) by
\[
\theta(u, v) = \int_{u_0}^u \ell(u, v)du + \int_{v_0}^v L(u_0, v)dv.
\]
Then \( \theta \) satisfies the conditions \( \theta_u(u, v) = \ell(u, v) \) and \( \theta_v(u, v) = L(u, v) \) for all \( (u, v) \in U \). We define \( \nu(u, v) = (\cos \theta(u, v), \sin \theta(u, v)) \) and hence \( \mu(u, v) = (-\sin \theta(u, v), \cos \theta(u, v)) \). We also define a smooth mapping \( f: U \to \mathbb{R}^2 \) by
\[
f(u, v) = \int_{u_0}^u \beta(u, v)\mu(u, v)du + \int_{v_0}^v (A(u_0, v)\nu(u_0, v) + B(u_0, v)\mu(u_0, v))dv.
\]
By a direct calculation, \( f_u(u, v) = \beta(u, v)\mu(u, v) \) and \( f_v(u, v) = A(u, v)\nu(u, v) + B(u, v)\mu(u, v) \). It follows that \( (f, \nu): U \to \mathbb{R}^2 \times S^1 \) is a one-parameter family of Legendre curves with the curvature \( (\ell, L, \beta, A, B) \).

Theorem 2.10 (Uniqueness Theorem for one-parameter families of Legendre curves) 

Let \((f, \nu)\) and \((\tilde{f}, \tilde{\nu}) : U \to \mathbb{R}^2 \times S^1\) be one-parameter families of Legendre curves with the curvatures \((\ell, L, \beta, A, B)\) and \((\ell, \tilde{L}, \tilde{\beta}, \tilde{A}, \tilde{B})\) respectively. Then \((f, \nu)\) and \((\tilde{f}, \tilde{\nu})\) are congruent as one-parameter family of Legendre curves if and only if \((\ell, L, \beta, A, B)\) and \((\ell, \tilde{L}, \tilde{\beta}, \tilde{A}, \tilde{B})\) coincide.

**Proof.** Suppose that \((f, \nu)\) and \((\tilde{f}, \tilde{\nu})\) are congruent as one-parameter family of Legendre curves. Then there exists a constant rotation \(A \in SO(2)\) and a constant vector \(a \in \mathbb{R}^2\) such that \(\tilde{f}(u, v) = A(f(u, v)) + a\) and \(\tilde{\nu}(u, v) = A(\nu(u, v))\) for all \((u, v) \in U\). It follows that \(\tilde{\mu}(u, v) = A(\mu(u, v))\). By a direct calculation, \((\ell, L, \beta, A, B)\) and \((\ell, \tilde{L}, \tilde{\beta}, \tilde{A}, \tilde{B})\) coincide.

Conversely, let \((u_0, v_0) \in U\) be fixed. By using congruence as one-parameter family of Legendre curves, we may \((f, \nu)(u_0, v_0) = (\tilde{f}, \tilde{\nu})(u_0, v_0)\). By the construction in the proof of Theorem 2.9, \((f, \nu)(u, v) = (\tilde{f}, \tilde{\nu})(u, v)\) for all \((u, v) \in U\). \(\square\)

3 Properties of one-parameter families of Legendre curves

Let \((f, \nu) : U \to \mathbb{R}^2 \times S^1\) be a one-parameter family of Legendre curves with respect to \(u\) and \((\ell, L, \beta, A, B)\) be the curvature.

We say that \(\phi : \tilde{U} \to U\) is a one-parameter parameter change if \(\phi\) is a diffeomorphism of the form \(\phi(p, q) = (u(p, q), v(q))\).

**Proposition 3.1** Under the above notations, \((\tilde{f}, \tilde{\nu}) = (f \circ \phi, \nu \circ \phi) : \tilde{U} \to \mathbb{R}^2 \times S^1\) is a one-parameter family of Legendre curves with respect to \(p\) and the curvature \((\ell, \tilde{L}, \tilde{\beta}, \tilde{A}, \tilde{B})\) is given by

\[
\begin{align*}
\tilde{\ell}(p, q) &= \ell(\phi(p, q))u_p(p, q), \\
\tilde{L}(p, q) &= \ell(\phi(p, q))u_q(p, q) + L(\phi(p, q))v_q(q), \\
\tilde{\beta}(p, q) &= \beta(\phi(p, q))u_p(p, q), \\
\tilde{A}(p, q) &= A(\phi(p, q))v_q(q), \\
\tilde{B}(p, q) &= \beta(\phi(p, q))u_q(p, q) + B(\phi(p, q))v_q(q).
\end{align*}
\]

**Proof.** Since \(\tilde{f}_p(p, q) \cdot \tilde{\nu}(p, q) = f_u(\phi(p, q))u_p(p, q) \cdot \nu(\phi(p, q)) = 0\) for all \((p, q) \in \tilde{U}\), \((\tilde{f}, \tilde{\nu})\) is a one-parameter family of Legendre curves with respect to \(p\). By a direct calculation, we have the curvature \((\ell, \tilde{L}, \tilde{\beta}, \tilde{A}, \tilde{B})\). \(\square\)

Next we consider a diffeomorphism on the target \(\mathbb{R}^2\).

**Proposition 3.2** Let \((f, \nu) : U \to \mathbb{R}^2 \times S^1\) be a one-parameter family of Legendre curves with respect to \(u\) and \((\ell, L, \beta, A, B)\) be the curvature. Suppose that \(\Phi : \mathbb{R}^2 \to \mathbb{R}^2\) is a diffeomorphism. Then there exists a smooth mapping \(\tilde{\nu} : U \to S^1\) such that \((\Phi \circ f, \tilde{\nu}) : U \to \mathbb{R}^2 \times S^1\) is a one-parameter family of Legendre curves with respect to \(u\).

**Proof.** We denote \(\Phi(x, y) = (\phi_1(x, y), \phi_2(x, y))\), \(f(u, v) = (x(u, v), y(u, v))\) and \(\nu(u, v) = (a(u, v), b(u, v))\). By the Frenet type formula, \(f_u(u, v) = \beta(u, v)\mu(u, v)\), that is, \(x_u(u, v) = \beta(u, v) \mu(u, v)\) and \(y_u(u, v) = \beta(u, v) \mu(u, v)\).

Next, we consider a diffeomorphism on the target \(\mathbb{R}^2\).
\[ -\beta(u, v)b(u, v) \text{ and } y_a(u, v) = \beta(u, v)a(u, v). \] We define
\[
\mathcal{P}(u, v) = (\phi_{2y}(x(u, v), y(u, v))a(u, v) - \phi_{2x}(x(u, v), y(u, v))b(u, v),
-\phi_{1y}(x(u, v), y(u, v))a(u, v) + \phi_{1x}(x(u, v), y(u, v))b(u, v))
\]
and \( \widetilde{\nu}(u, v) = \mathcal{P}(u, v)/|\mathcal{P}(u, v)|. \) Note that \(|\mathcal{P}(u, v)| \neq 0\) for all \((u, v) \in U\) and \( \mathcal{P}(u, v) = J_{\Phi}(u, v)^tD_{\Phi}^{-1}(f(u, v))\nu(u, v), \) where
\[
D_{\Phi}(u, v) = \begin{pmatrix} \phi_{1x} & \phi_{1y} \\ \phi_{2x} & \phi_{2y} \end{pmatrix}(u, v), \quad J_{\Phi}(u, v) = \det D_{\Phi}(u, v)
\]
and \( ^tA \) is the transpose matrix of \( A. \) Then \( \widetilde{\nu} : U \to S^1 \) is a smooth mapping. Since \( \Phi \circ f(u, v) = (\phi_1(x(u, v), y(u, v)), \phi_2(x(u, v), y(u, v))) \), we have
\[
(\Phi \circ f)_a(u, v) = \begin{pmatrix} \phi_{1x}(x(u, v), y(u, v))x_a(u, v) + \phi_{1y}(x(u, v), y(u, v))y_a(u, v), \\ \phi_{2x}(x(u, v), y(u, v))x_a(u, v) + \phi_{2y}(x(u, v), y(u, v))y_a(u, v) \end{pmatrix}
\]
\[ = \beta(u, v)\begin{pmatrix} -\phi_{1x}(x(u, v), y(u, v))b(u, v) + \phi_{1y}(x(u, v), y(u, v))a(u, v), \\ -\phi_{2x}(x(u, v), y(u, v))b(u, v) + \phi_{2y}(x(u, v), y(u, v))a(u, v) \end{pmatrix}. \]

By a direct calculation, we have \((\Phi \circ f)_a(u, v) \cdot \tilde{\nu}(u, v) = 0\) for all \((u, v) \in U.\) Hence \((\Phi \circ f, \tilde{\nu})\) is a one-parameter family of Legendre curves with respect to \( u. \)

**Remark 3.3** By a direct calculation, we have the curvature \((\tilde{\ell}, \tilde{L}, \tilde{\beta}, \tilde{A}, \tilde{B})\) of \((\Phi \circ f, \tilde{\nu})\) as follows:
\[
\tilde{\ell} = (1/|\mathcal{P}|^2)\left( -\beta((\phi_{2xx}b^2 - 2\phi_{2xy}ab + \phi_{2yy}a^2)(\phi_{1y}a - \phi_{1x}b)
-\phi_{1xx}b^2 - 2\phi_{1xy}ab + \phi_{1yy}a^2)(\phi_{2y}a - \phi_{2x}b) + \ell(\phi_{1x}\phi_{2y} - \phi_{2x}\phi_{1y}) \right),
\]
\[
\tilde{L} = (1/|\mathcal{P}|^2)\left( ((\phi_{2xy}x + \phi_{2yy}y)\alpha - (\phi_{2xx}x + \phi_{2xy}y)\alpha)(\phi_{1y}a - \phi_{1x}b)
+(-\phi_{1xy}x + \phi_{1yy}y)\alpha + (\phi_{1xx}x + \phi_{1xy}y)\alpha)(\phi_{2y}a - \phi_{2x}b) + L(\phi_{1x}\phi_{2y} - \phi_{2x}\phi_{1y}) \right),
\]
\[
\tilde{\beta} = |\mathcal{P}|\beta,
\]
\[
\tilde{A} = (1/|\mathcal{P}|)A(\phi_{1x}\phi_{2y} - \phi_{2x}\phi_{1y}),
\]
\[
\tilde{B} = |\mathcal{P}|B + (1/|\mathcal{P}|)A\left( (\phi_{1x}a + \phi_{1y}b)(\phi_{1y}a - \phi_{1x}b) + (\phi_{2x}a + \phi_{2y}b)(\phi_{2y}a - \phi_{2x}b) \right).
\]

We now consider existence conditions for a one-parameter family of Legendre curves of a given map \( f : U \to \mathbb{R}^2. \)

**Proposition 3.4** Let \( f : U \to \mathbb{R}^2, f(u, v) = (x(u, v), y(u, v)) \) be a smooth mapping and \( p = (u_0, v_0) \in U. \)

(1) If \( \text{rank } df = 2 \) at \( p \in U, \) then there exists \( \nu \) around \( p \) such that \((f, \nu)\) is a one-parameter family of Legendre curves with respect to \( u \) around \( p. \)
(2) If rank $df = 1$ at $p \in U$, then there exists $\nu$ around $p$ such that $(f, \nu)$ is a one-parameter family of Legendre curves with respect to $u$ or with respect to $v$ around $p$.

(3) Let rank $df = 0$ at $p \in U$. Suppose that there exist smooth map germs $\lambda_1, \lambda_2 : (U, p) \rightarrow \mathbb{R}$ with $\lambda_1(p) \neq 0, \lambda_2(p) \neq 0$, $k_1, k_2$ are natural numbers and $\ell_1, \ell_2$ are non-negative integers such that $x(u, v) = \lambda_1(u, v)(u - u_0)^{k_1}(v - v_0)^{\ell_1}$ and $y(u, v) = \lambda_2(u, v)(u - u_0)^{k_2}(v - v_0)^{\ell_2}$. If $1 \leq \ell_1 < \ell_2$ and $\ell_1 \leq \ell_2$ (respectively, $k_1 \geq k_2 \geq 1$ and $\ell_1 \geq \ell_2$), then there exists $\nu$ around $p$ such that $(f, \nu)$ is a one-parameter family of Legendre curves with respect to $u$ around $p$.

(4) Suppose that $f(u, v)$ is given by the form $f(u, v) = (x(u, v), y(v))$. Then there exists $\nu$ such that $(f, \nu)$ is a one-parameter family of Legendre curves with respect to $u$.

Proof. (1) By Proposition 3.2, we may assume that $f(u, v) = (u, v)$ around $p$. If we take $\nu : U \rightarrow S^1, \nu(u, v) = (0, 1)$, then $(f, \nu)$ is a one-parameter family of Legendre curves with respect to $u$ around $p$.

(2) By Propositions 3.1 and 3.2, we may assume that $f(u, v) = (u, f(u, v))$ or $f(u, v) = (v, f(u, v))$ around $p$. Suppose that $f(u, v) = (u, f(u, v))$. If we take

$$\nu : U \rightarrow S^1, \nu(u, v) = (1/\sqrt{f_u^2(u, v) + 1})(-f_u(u, v), 1),$$

then $(f, \nu)$ is a one-parameter family of Legendre curves with respect to $u$ around $p$. On the other hand, suppose that $f(u, v) = (v, f(u, v))$. If we take

$$\nu : U \rightarrow S^1, \nu(u, v) = (1/\sqrt{f_u^2(u, v) + 1})(-f_v(u, v), 1),$$

then $(f, \nu)$ is a one-parameter family of Legendre curves with respect to $v$ around $p$.

(3) We may assume that $p = (u_0, v_0) = (0, 0)$. By a direct calculation, we have

$$f_u(u, v) = ((\lambda_{1u}(u, v)u + \lambda_1(u, v)k_1)u^{k_1-1}v^\ell_1, (\lambda_{2u}(u, v)u + \lambda_2(u, v)k_2)u^{k_2-1}v^\ell_2).$$

If $1 \leq k_1 \leq k_2$ and $\ell_1 \leq \ell_2$, then there exist non-negative integers $n_1, n_2$ such that $k_2 = k_1 + n_1$ and $\ell_2 = \ell_1 + n_2$. If we take $\nu : U \rightarrow S^1$ by

$$\nu(u, v) = \frac{(-\lambda_{2u}(u, v)u + \lambda_2(u, v)k_2)u^{n_1}v^n + \lambda_1(u, v)u + \lambda_1(u, v)k_1}{\sqrt{(\lambda_{2u}(u, v)u + \lambda_2(u, v)k_2)^2u^{n_1}v^{n_2} + (\lambda_{1u}(u, v)u + \lambda_1(u, v)k_1)^2}},$$

then $(f, \nu)$ is a one-parameter family of Legendre curves with respect to $u$ around $p$. In the case $k_1 \geq k_2 \geq 1$ and $\ell_1 \geq \ell_2$, we can prove similarly.

(4) Suppose that $f(u, v) = (x(u, v), y(v))$. Then $f_u(u, v) = (x_u(u, v), 0)$. If we take $\nu : U \rightarrow S^1, \nu(u, v) = (0, 1), (f, \nu)$ is a one-parameter family of Legendre curves with respect to $u$. \qed

**Example 3.5** Let $(f, \nu) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $f(u, v) = (uv, v^2), \nu(u, v) = (0, 1)$. Since $f_u(u, v) = (v, 0)$, $(f, \nu)$ is a one-parameter family of Legendre curves with respect to $u$ (cf. Proposition 3.4 (4)).

On the other hand, let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $f(u, v) = (uv, v^2)$. Then $f_u(u, v) = (v, 2u)$. There does not exists $\nu : \mathbb{R}^2 \rightarrow S^1$ such that $(f, \nu)$ is a one-parameter family of Legendre curves with respect to $u$, see Example 4.3. The figures are in Figure 3.

**Example 3.6** (Parallel curves of Legendre curves) Let $(\gamma, \nu) : I \rightarrow \mathbb{R}^2 \times S^1$ be a Legendre curve with the curvature $(\ell, \beta)$. The parallel curve $\gamma^k : I \rightarrow \mathbb{R}^2$ is given by $\gamma^k(t) = \gamma(t) + k\nu(t)$.
for fixed $k \in \mathbb{R}$ (cf. [7, 8]). Then $(\gamma^k, \nu) : I \to \mathbb{R}^2 \times S^1$ is also a Legendre curve with the curvature $(\beta + k\ell, \ell)$. We define $(f, \tilde{v}) : I \times \mathbb{R} \to \mathbb{R}^2 \times S^1$ by

$$f(t, k) = \gamma^k(t) = \gamma(t) + k\nu(t), \quad \tilde{v}(t, k) = \nu(t).$$

Then $(f, \tilde{v})$ is a one-parameter family of Legendre curves with respect to $t$ with the curvature $(\ell, 0, \beta + k\ell, 1, 0)$. Figure 1 shows an example of a family of parallel curves.

Moreover, we define $(f, \tilde{v}) : \mathbb{R} \times I \to \mathbb{R}^2 \times S^1$ by

$$f(k, t) = \gamma^k(t) = \gamma(t) + k\nu(t), \quad \tilde{v}(k, t) = \mu(t).$$

Then $(f, \tilde{v})$ is a one-parameter family of Legendre curves with respect to $k$ with the curvature $(0, \ell, -1, \beta + k\ell, 0)$. The mapping $f$ is an example of plane line congruences, see section 5.

4 Relations between one-parameter families of Legendre curves and Legendre surfaces

We give a relation between one-parameter families of Legendre curves and Legendre surfaces.

We say that $(x, n) : U \to \mathbb{R}^3 \times S^2$ is a Legendre surface if $(x, n)^*\theta = 0$ for all $(u, v) \in U$, where $\theta$ is a canonical contact form on the unit tangent bundle $T_1\mathbb{R}^3 = \mathbb{R}^3 \times S^2$ over $\mathbb{R}^3$ (cf. [1, 2]). This condition is equivalent to $x_u(u, v) \cdot n(u, v) = 0$ and $x_v(u, v) \cdot n(u, v) = 0$ for all $(u, v) \in U$. We say that $x : U \to \mathbb{R}^3$ is a frontal if there exists $n : U \to S^2$ such that $(x, n)$ is a Legendre surface.

**Proposition 4.1**

(1) Let $(f, \nu) : U \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves with respect to $u$, where $f(u, v) = (x(u, v), y(u, v))$ and $\nu(u, v) = (a(u, v), b(u, v))$. Then $x : U \to \mathbb{R}^3, x(u, v) = (x(u, v), y(u, v), v)$ is a frontal. More precisely, $(x, n) : U \to \mathbb{R}^3 \times S^2$ is a Legendre surface, where

$$n(u, v) = \frac{(a(u, v), b(u, v), -b(u, v)y_v(u, v) - a(u, v)x_v(u, v))}{\sqrt{1 + (b(u, v)y_v(u, v) + a(u, v)x_v(u, v))^2}}.$$  

(2) Let $(x, n) : U \to \mathbb{R}^3 \times S^2$ be a Legendre surface of the form $x(u, v) = (x(u, v), y(u, v), v)$ and $n(u, v) = (a(u, v), b(u, v), c(u, v))$. Then $(f, \nu) : U \to \mathbb{R}^2 \times S^1$ is a one-parameter family of Legendre curves with respect to $u$, where

$$f(u, v) = (x(u, v), y(u, v)), \quad \nu(u, v) = \frac{(a(u, v), b(u, v))}{\sqrt{a^2(u, v) + b^2(u, v)}}.$$
Proof. (1) Since \( \mathbf{x}_u(u,v) = (x_u(u,v), y_u(u,v), 0) \) and \( \mathbf{x}_v(u,v) = (x_v(u,v), y_v(u,v), 1) \), we have \( \mathbf{x}_u(u,v) \cdot \mathbf{n}(u,v) = 0 \) and \( \mathbf{x}_v(u,v) \cdot \mathbf{n}(u,v) = 0 \). It follows that \( (\mathbf{x}, \mathbf{n}) \) is a Legendre surface.

(2) Since \( \mathbf{x}_u(u,v) = (x_u(u,v), y_u(u,v), 0) \) and \( \mathbf{x}_v(u,v) = (x_v(u,v), y_v(u,v), 1) \), we have \( x_u(u,v)a(u,v) + y_u(u,v)b(u,v) = 0 \) and \( x_v(u,v)a(u,v) + y_v(u,v)b(u,v) + c(u,v) = 0 \). If \( a(u,v) = b(u,v) = 0 \), then \( c(u,v) = 0 \). It is a contradiction the fact that \( \mathbf{n}(u,v) \in S^2 \). Therefore, we have \( (a(u,v), b(u,v)) \neq (0,0) \) for all \( (u,v) \in U \). It follows that \( (f, \nu) \) is a one-parameter family of Legendre curves.

Remark 4.2 Let \( (f, \nu) : U \rightarrow \mathbb{R}^2 \times S^1 \) be a one-parameter family of Legendre curves with respect to \( u \). Then \( (\mathbf{x}, \mathbf{\nu}_1, \mathbf{\nu}_2) : U \rightarrow \mathbb{R}^3 \times \Delta \) is also a one-parameter family of framed curves with respect to \( u \), where \( \mathbf{x}(u,v) = (f(u,v), \nu_1(u,v), \nu_2(u,v)) = (\nu(u,v), 0) \) and \( \mathbf{\nu}_2(u,v) = (0,0,1) \) (cf. [11, 24]). Moreover, \( (\mathbf{x}, \mathbf{n}, \mathbf{s}) : U \rightarrow \mathbb{R}^3 \times \Delta \) is a framed surface, where

\[
\mathbf{n}(u,v) = \frac{(a(u,v), b(u,v), \neg b(u,v) y_v(u,v) - a(u,v) x_v(u,v))}{\sqrt{1 + (b(u,v) y_v(u,v) + a(u,v) x_v(u,v))^2}}, \quad \mathbf{s}(u,v) = (-b(u,v), a(u,v), 0)
\]

(cf. [10, 11]).

Example 4.3 (Example 3.5) Let \( (f, \nu) : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be given by \( f(u,v) = (uv, v^2) \), \( \nu(u,v) = (0, 1) \). Then \( \mathbf{x} : \mathbb{R}^2 \rightarrow \mathbb{R}^3, \mathbf{x}(u,v) = (uv, v^2, v) \) is a frontal by Proposition 4.1.

On the other hand, let \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be given by \( f(u,v) = (uv, u^2) \). Then \( \mathbf{x} : \mathbb{R}^2 \rightarrow \mathbb{R}^3, \mathbf{x}(u,v) = (uv, u^2, v) \) is a cross cap. It is an example which is not a frontal.

5 Plane line congruences

We deal with local geometry of plane line congruences (one-parameter families of lines in plane). Let \( I \) be an open interval, and \( \gamma : I \rightarrow \mathbb{R}^2 \) and \( \mathbf{e} : I \rightarrow S^1(\subseteq \mathbb{R}^2) \) be smooth mappings. We define a plane line congruence as a map of the following form:

\[
f : \mathbb{R} \times I \rightarrow \mathbb{R}^2, \quad (u,v) \mapsto \gamma(v) + u \mathbf{e}(v).
\]

\( \gamma \) and \( \mathbf{e} \) are respectively called the base and direction curves of the plane line congruence \( f \), and the pair \( (\gamma, \mathbf{e}) \) is often regarded as the plane line congruence \( f \) itself.

Proposition 5.1 The mapping \( (f, \nu) : \mathbb{R} \times I \rightarrow \mathbb{R}^2 \times S^1, f(u,v) = \gamma(v) + u \mathbf{e}(v), \nu(u,v) = J(\mathbf{e}(v)) \) is a one-parameter family of Legendre curves with respect to \( u \), and the curvature is given as follows:

\[
\ell(u,v) = 0, \quad L(u,v) = |\mathbf{e}(v) \mathbf{e}'(v)|, \quad \beta(u,v) = -1, \quad A(u,v) = \overline{A}(v) + uL(v), \quad B(u,v) = |J(\mathbf{e}(v)) \gamma'(v)|,
\]

where \( ' \) means \( d/dv \), \( \cdot, \cdot \) means the determinant of the vectors in \( \mathbb{R}^2 \) and \( \overline{A}(v) = |\mathbf{e}(v) \gamma'(v)| \). Especially \( L, A, B \) are functions depending only on the parameter \( v \).
**Remark 5.3** The above characterizations say that the congruent invariants $L$ and $A$ of the mapping $f$ of a plane line congruence is determined just by a function $\lambda$ satisfying $\lambda_{uu} = 0$. 

**Proof.** Since $f_u(u,v) \cdot \nu(u,v) = e(v) \cdot J(e(v)) = 0$ for all $(u,v) \in \mathbb{R} \times I$, $(f,\nu)$ is a one-parameter family of Legendre curves with respect to $u$. We calculate the curvature of $(f,\nu)$. Put $\mu(u,v) = J(\nu(v)) = -e(v)$. Since $\nu_u(u,v) = 0$ and $f_u(u,v) = e(v)$, we have $\ell(u,v) = 0$ and $\beta(u,v) = -1$. By $\mu_u(u,v) = -e'(v)$,

$$L(u,v) = |\mu(u,v) \mu_v(u,v)| = |e(v) e'(v)|.$$ 

Finally, we have $f_\nu(u,v) = \gamma'(v) + u e'(v)$,

$$A(u,v) = |f_\nu(u,v) \mu(u,v)| = |e(v) \gamma'(v)| + u |e(v) e'(v)|$$

and

$$B(u,v) = |\nu(u,v) f_\nu(u,v)| = |J(e(v)) \gamma'(v)|.$$ 

\[\Box\]

Conversely, we can construct the mapping of a plane line congruence for given smooth functions $L, \AA, B : I \to \mathbb{R}$ and a fixed point $v_0 \in I$, following the construction in the proof of Theorem 2.9: First put $\theta(v) = \int_{v_0}^{v} L(v) dv$, $\nu(v) = (\cos \theta(v), \sin \theta(v))$ and hence $\mu(v) = (-\sin \theta(v), \cos \theta(v))$. Then we get mappings $\gamma, e : I \to \mathbb{R}^2$ as follows

$$\gamma(v) = \int_{v_0}^{v} (\AA(v) \nu(v) + B(v) \mu(v)) dv, \quad e(v) = -\mu(v).$$

Then $(f,\nu) : \mathbb{R} \times I \to \mathbb{R}^2 \times S^1, f(u,v) = \gamma(v) + u e(v), \nu(u,v) = J(e(v))$ is a one-parameter family of Legendre curves with respect to $u$ with the curvature $(0, L, -1, \AA + uL, B)$.

**Remark 5.2** The congruent type of a plane line congruence is distinguished even by the choice of the coordinate of the parameter $v$, though the family of lines in the plane are the same.

We are interested in local geometry of plane line congruences, thus deal with germs of direction and base curves $\gamma, e : (I,0) \to \mathbb{R}^2$ constructed from function germs $L, \AA, B : (I,0) \to \mathbb{R}$. Recall that $L, \AA, B$ are clearly characterized by geometry of $\gamma, e$. Note also that $A$ and $L$ coincide with the following functions which are meaningful with respect to a geometry of the mapping $f(u,v) = \gamma(v) + u e(v)$.

**The Jacobian:** For a plane line congruence $f(u,v) = \gamma(v) + u e(v)$, the Jacobian $\lambda(u,v)$ coincides with the function $A(u,v)$:

$$\lambda(u,v) = |e(v) \gamma'(v)| + u |e(v) e'(v)| = \AA(v) + u L(v) = A(u,v).$$

The differential of the Jacobian with respect to $u$: $u$ is a special parameter of the map of a plane line congruence parametrizing each line for fixed $v$, and the derivative of the Jacobian with respect to $u$ coincides with the function $L$:

$$\frac{\partial \lambda}{\partial u}(u,v) = L(v).$$

**Remark 5.3** The above characterizations say that the congruent invariants $L$ and $A$ of the mapping $f$ of a plane line congruence is determined just by a function $\lambda$ satisfying $\lambda_{uu} = 0$. 

11
Two map germs \(f, g: (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)\) are said to be \(A\)-equivalent (written as \(f \sim_A g\)) if there exist diffeomorphism germs \(\phi: (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)\) and \(\tau: (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)\) such that \(f = \tau \circ g \circ \phi\). When \(\tau\) is the identity in the above, \(f\) and \(g\) are said to be \(R\)-equivalent (written as \(f \sim_R g\)). A lot of works on the classification of germs with the above equivalences have been done in the context of singularity theory. See [1, 2, 3, 12, 16] for basic idea of singularity theory, and [19, 25, 26] for classification of map germs \((\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)\) with respect to \(A\)-equivalence. The \(A\) or \(R\)-equivalent class of mappings or functions are worth studying from the viewpoint of differential topology. Especially, we investigate the relation between the congruent invariants of a plane line congruence and the \(A\) or \(R\)-equivalent classes of mappings or functions related to the plane line congruence.

A one-variable function \(f: (I, t_0) \to (\mathbb{R}, 0)\) has type \(A_k\) at \(t_0 \in I\) if \(f^{(i)}(t_0) = 0\) for \(i = 1, \ldots, k\) and \(f^{(k+1)}(t_0) \neq 0\). Then \(f\) is \(R\)-equivalent to \(\pm v^{k+1}\) (cf. [3]). The following relation between \(e\) and \(L\) immediately holds:

**Proposition 5.4** For \(k \geq 1\), \(e: (I, 0) \to S^1\) is \(R\)-equivalent to the germ of type \(A_k\) (whose normal form is \(\pm v^{k+1}\)) if and only if \(L: (I, 0) \to \mathbb{R}\) is \(R\)-equivalent to the germ of type \(A_{k-1}\) and \(L(0) = 0\). In addition, when \(L(0) \neq 0\), \(e\) is always of type \(A_0\).

**Proof.** Taking a suitable coordinate, \(e\) is locally expressed as \(\theta(v) = \int_0^v L(v)dv\) around \(0 \in I\).

In addition, the next Proposition says that we can locally parametrize any plane line congruence by a normal line congruence that is studied in the next subsection:

**Proposition 5.5** For a given germ \((\gamma, e): (I, 0) \to \mathbb{R}^2\), put \(\epsilon(v) = \int_0^v B(v)dv\) and \(\tilde{\gamma} = \gamma + \epsilon e:\ (\mathbb{R}, 0) \to \mathbb{R}^2\). Then \((\tilde{\gamma}, e)\) satisfies the followings:

1. \((\gamma, e)(v)\) and \((\tilde{\gamma}, e)(v)\) define the same line for any \(v\).

2. \(|J(e) \tilde{\gamma}'|(v) = 0\) for any \(v\).

**Proof.** Since \(\hat{\gamma}(v) + u \epsilon(v) = \gamma(v) + (u + \epsilon(v))e(v)\), the statement (1) is clear. The statement (2) follows from a direct calculation:

\[
|J(e) \hat{\gamma}'| = |J(e) \gamma'| + \epsilon'|J(e) e| + \epsilon |J(e) e'| = B + B(-1) + \epsilon \cdot 0 = 0.
\]

**Remark 5.6** Let \((\tilde{\gamma}, e)\) in Proposition 5.5 have the curvature \((\ell, L, \beta, \hat{A}, \hat{B})\). Then the curvature is expressed in terms of \((\ell, L, \beta, A, B)\) (the curvature of \((\gamma, e)\)) as follows:

\[
\ell = \ell = 0, \quad L = \hat{L}, \quad \beta = \beta = -1, \quad \hat{A} = A + \epsilon L, \quad \hat{B} = 0.
\]

Especially, in the same use of the notation “hat”, \(\hat{A} = \overline{A} + \epsilon L\).

### 5.1 Normal line congruences

We deal with a normal line congruence: a plane line congruence satisfying \(B = |J(e) \gamma'| \equiv 0\). Therefore the congruent type of a normal line congruence and thus the corresponding base and
direction curves $\gamma,e$ are uniquely determined by the invariants $\overline{A}$ and $L$. It is also remarkable that the pair $(L,\overline{A})$ coincides with the curvature $(\ell_{\gamma},\beta_{\gamma})$ of the Legendre curve $(\gamma,e): I \to \mathbb{R}^2 \times S^1$. Especially, the non-injective case with $L = \ell_{\gamma} \neq 0$ is well studied in [7].

**Remark 5.7** Conversely, let a function $\lambda(u,v)$ satisfying $\lambda_{uu} = 0$ be given, then we can uniquely determine functions $A(u,v)$ and $L(v)$ thus a congruent type of normal line congruences. Especially $\lambda$ coincides with the Jacobian of the congruent type (precisely, the representatives).

The followings are the Taylor expansion (or normal form) of such $\gamma(v) = (\gamma_1(v), \gamma_2(v))$ and $e(v) = (e_1(v), e_2(v))$ with $L_i = (d^iL/dv^i)(0)$ and $\overline{A}_i = (d^i\overline{A}/dv^i)(0)$:

$$
\begin{align*}
\gamma_1(v) &= \overline{A}_0 v + \frac{1}{2!} \overline{A}_1 v^2 + \frac{1}{3!} (-\overline{A}_0 L^2_0 + \overline{A}_2) v^3 + \frac{1}{4!} (-3\overline{A}_0 L_0 L_1 - 3\overline{A}_1 L_0^2 + \overline{A}_3) v^4 + \cdots, \\
\gamma_2(v) &= \frac{1}{2!} \overline{A}_0 L_0 v^2 + \frac{1}{3!} (\overline{A}_0 L_1 + 2\overline{A}_1 L_0) v^3 + \frac{1}{4!} (\overline{A}_0 (L_2 - L_0^3) + 3\overline{A}_1 L_1 + 3\overline{A}_2 L_0) v^4 + \cdots, \\
e_1(v) &= L_0 v + \frac{1}{2!} L_1 v^2 + \frac{1}{3!} (-L_0^3 + L_2) v^3 + \frac{1}{4!} (-6L_0^2 L_1 + L_3) v^4 + \cdots, \\
e_2(v) &= -1 + \frac{1}{2!} L_0^2 v^2 + \frac{3}{3!} L_0 L_1 v^3 + \frac{1}{4!} (-L_0^4 + 3L_1^2 + 4L_0^2 L_2) v^4 + \cdots.
\end{align*}
$$

Recall that the Jacobian $\lambda$ of the map of the plane line congruence $f(u,v) = \gamma(v) + u e(v)$ is expressed as

$$
\lambda(u,v) = \overline{A}(v) + u L(v).
$$

From the above expressions, the following is easily seen.

**Proposition 5.8** The followings are equivalent:

1. $\lambda(0) = 0$, that is, $f$ is not immersive at 0.
2. $\overline{A}(0) = |e(0) \gamma'(0)| = 0$.
3. $\gamma$ is not immersive at 0.

Since $\overline{A}$ and $L$ are function germs of one variable, they are $\mathcal{R}$-equivalent to one of the $A_k$-types. From now on, we fix the $\mathcal{R}$-types of $\overline{A}$ and $L$ respectively, and study the $A$ or $\mathcal{R}$-types of singularities in some functions or mappings which give us geometric information of normal line congruences. We must remark that the types of $\overline{A}$ and $L$ are not always independent of the coordinate change of the source space to $(\gamma,e)$ (see §6.1). Note also that normal line congruences are divided into four types depending on $L_0, \overline{A}_0$:

$$
(1) \overline{A}_0, L_0 \neq 0, \ (2) \overline{A}_0 \neq 0, L_0 = 0, \ (3) \overline{A}_0 = 0, L_0 \neq 0, \ (4) \overline{A}_0 = L_0 = 0.
$$

Since neither $e, \gamma, f, \lambda$ nor $\lambda_u$ are singular, we do not deal with the case (1).

### 5.1.1 Inflection regular base curves

Consider the case (2) $\overline{A}_0 \neq 0$ and $L_0 = 0$. Then the map of a normal line congruence $f$ is immersive at 0; but $e$ and $\lambda_u$ are not regular, and $\gamma$ is immersive but inflectional or more degenerated. Note that $e$ and $\lambda_u$ depend only on $L$, while $\lambda$ and $\gamma$ depend on both $\overline{A}$ and $L$. 

13
Let \( \overline{A} \) be of type \( A_{k_1-1} \) and \( L \) be of type \( A_{k_2-1} \), and write
\[
\overline{A}(v) = \overline{A}_0 + \frac{\overline{A}_{k_1}}{k_1!} v^{k_1} + \cdots, \quad L(v) = \frac{L_{k_2}}{k_2!} v^{k_2} + \cdots
\]
where \( \overline{A}_{k_1}, L_{k_2} \neq 0 \) \((k_1, k_2 \geq 1)\). In the following we show types of leading terms of certain functions or mappings.

**Proposition 5.9 (\( \lambda \) with \( \overline{A}_0 \neq 0 \) and \( L_0 = 0 \))** In the above setting, there are 3 cases with respect to the Jacobian \( \lambda \):

1. for \( k_1 = 1 \), \( \lambda \) is regular;
2. for \( k_1 \geq k_2 + 1 \),
   \[
   \lambda(u,v) = \overline{A}_0 + v^{k_2} \left( \frac{L_{k_2}}{k_2!} u + \cdots \right) \sim_R \overline{A}_0 + uv^{k_2};
   \]
3. for \( k_2 + 1 > k_1 \geq 2 \),
   \[
   \lambda(u,v) = \overline{A}_0 + \frac{\overline{A}_{k_1}}{k_1!} v^{k_1} (1 + \cdots) \sim_R \overline{A}_0 + v^{k_1}.
   \]

**Proof.** Since the Jacobian of the map of a normal line congruence is expressed as
\[
\lambda(u,v) = \left( \frac{\overline{A}_0}{k_1!} + \frac{\overline{A}_{k_1}}{k_1!} v^{k_1} + \cdots \right) + u \left( \frac{L_{k_2}}{k_2!} v^{k_2} + \cdots \right),
\]
we have the result. \( \square \)

**Proposition 5.10 (\( \gamma \) with \( \overline{A}_0 \neq 0 \) and \( L_0 = 0 \))** In the above setting, \( \gamma \) has an inflection point at 0. Especially, it has the contact type of \((k_2 + 1)\)-th order with the tangent line at 0.

**Proof.** Recall that \( \gamma \) is constructed as
\[
\gamma(v) = (\gamma_1(v), \gamma_2(v)) = \int_0^v \left( \overline{A}(v) \cos \theta(v), \overline{A}(v) \sin \theta(v) \right) dv
\]
with \( \theta(v) = \int_0^v L(v)dv \). Thus we have
\[
\gamma_1(v) = \int_0^v \left( \frac{\overline{A}_0}{k_1!} + \frac{\overline{A}_{k_1}}{k_1!} v^{k_1} + \cdots \right) \cos \left( \frac{L_{k_2}}{(k_2 + 1)!} v^{k_2+1} + \cdots \right) dv = \overline{A}_0 v + \cdots,
\]
\[
\gamma_2(v) = \int_0^v \left( \frac{\overline{A}_0}{k_1!} + \frac{\overline{A}_{k_1}}{k_1!} v^{k_1} + \cdots \right) \sin \left( \frac{L_{k_2}}{(k_2 + 1)!} v^{k_2+1} + \cdots \right) dv = \overline{A}_0 L_{k_2} \frac{1}{(k_2 + 2)!} v^{k_2+2} + \cdots.
\]

Summing up the above results for relatively small numbers of \( k_1 \) and \( k_2 \), we get the Table 1.
<table>
<thead>
<tr>
<th>$\mathcal{A}$</th>
<th>$L = \ell_\gamma = \lambda_u$</th>
<th>$\lambda$</th>
<th>$e$</th>
<th>$\gamma$</th>
<th>$L/\mathcal{A}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_0$</td>
<td>$A_0$</td>
<td>regular</td>
<td>$A_1$</td>
<td>(1, 3)</td>
<td>(1, 2)</td>
</tr>
<tr>
<td></td>
<td>$A_1$</td>
<td>regular</td>
<td>$A_2$</td>
<td>(1, 4)</td>
<td>(1, 2)</td>
</tr>
<tr>
<td></td>
<td>$A_2$</td>
<td>regular</td>
<td>$A_3$</td>
<td>(1, 5)</td>
<td>(1, 2)</td>
</tr>
<tr>
<td></td>
<td>$A_{\geq 3}$</td>
<td>regular</td>
<td>$A_{&gt; 4}$</td>
<td>(1, 6)</td>
<td>(1, 2)</td>
</tr>
<tr>
<td>$A_1$</td>
<td>$A_0$</td>
<td>$uv$</td>
<td>$A_1$</td>
<td>(1, 3)</td>
<td>(2, 3)</td>
</tr>
<tr>
<td></td>
<td>$A_1$</td>
<td>$v^2$</td>
<td>$A_2$</td>
<td>(1, 4)</td>
<td>(2, 3)</td>
</tr>
<tr>
<td></td>
<td>$A_2$</td>
<td>$v^2$</td>
<td>$A_3$</td>
<td>(1, 5)</td>
<td>(2, 3)</td>
</tr>
<tr>
<td></td>
<td>$A_{\geq 3}$</td>
<td>$v^2$</td>
<td>$A_{&gt; 4}$</td>
<td>(1, 6)</td>
<td>(2, 3)</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$A_0$</td>
<td>$uv$</td>
<td>$A_1$</td>
<td>(1, 3)</td>
<td>(3, 4)</td>
</tr>
<tr>
<td></td>
<td>$A_1$</td>
<td>$uv^2$</td>
<td>$A_2$</td>
<td>(1, 4)</td>
<td>(3, 4)</td>
</tr>
<tr>
<td></td>
<td>$A_2$</td>
<td>$v^3$</td>
<td>$A_3$</td>
<td>(1, 5)</td>
<td>(3, 4)</td>
</tr>
<tr>
<td></td>
<td>$A_{\geq 3}$</td>
<td>$v^3$</td>
<td>$A_{&gt; 4}$</td>
<td>(1, 6)</td>
<td>(3, 4)</td>
</tr>
</tbody>
</table>

Table 1: The types of $\lambda, e, \gamma$ are shown for given types of $\mathcal{A}, L$. $f$ and $\gamma$ are regular, but $\gamma$ has an inflection point at the origin, and the order is determined by the type of $L = \ell_\gamma$; Especially, $(m_1, m_2)$ in the column means the degrees of leading terms of $\gamma = (\gamma_1, \gamma_2)$. The last column shows 2-multi indices (see §6.1) of $L/\mathcal{A}$.

### 5.1.2 Singular mappings and base curves

Consider the case (3) $\mathcal{A}_0 = 0$ and $L_0 \neq 0$. Then the germs $f$ and $\gamma$ are not immersive; on the other hand $e$ and $\lambda$ are regular. Note that this is non-injective case in the sense of [7], and is well studied with respect to the base curve and the evolute. Let $\mathcal{A}$ be of type $A_{k_1 - 1}$ and $L$ be of type $A_{k_2 - 1}$, and write

$$\mathcal{A}(v) = \frac{A_{k_1}}{k_1!} v^{k_1} + \cdots, \quad L(v) = L_0 + \frac{L_{k_2}}{k_2!} v^{k_2} + \cdots$$

where $A_{k_1}, L_{k_2} \neq 0$ ($k_1, k_2 \geq 1$).

**Proposition 5.11 (\gamma with $A_0 = 0$ and $L_0 \neq 0$)** In the above setting, $\gamma$ is $\mathcal{A}$-equivalent to a map germ of the form $(t^{k_1 + 1}, t^{k_1 + 2} + h.o.t)$.

**Proof.** The base curve $\gamma(v) = (\gamma_1(v), \gamma_2(v))$ is as follows:

$$\gamma_1(v) = \int_0^v \frac{A_{k_1}}{k_1!} v^{k_1} \cos \left( L_0 v + \frac{L_{k_2}}{(k_2 + 1)!} v^{k_2 + 1} + \cdots \right) \, dv = \frac{A_{k_1}}{(k_1 + 1)!} v^{k_1 + 1} + \cdots,$$
$$\gamma_2(v) = \int_0^v \frac{A_{k_1}}{k_1!} v^{k_1} \sin \left( L_0 v + \frac{L_{k_2}}{(k_2 + 1)!} v^{k_2 + 1} + \cdots \right) \, dv = L_0 \frac{A_{k_1}(k_1 + 1)}{(k_1 + 2)!} v^{k_1 + 2} + \cdots.$$  

**Proposition 5.12 (f with $A_0 = 0$ and $L_0 \neq 0$)** In the above setting, $f$ is $\mathcal{A}$-equivalent to a map germ of the form

$$(u, uv + v^{k_1 + 1} + h.o.t)$$

for the map of the normal line congruence $f(u, v) = \gamma(v) + \text{ue}(v) = (ue_1(v) + \gamma_1(v), ue_2(v) + \gamma_2(v))$. Especially, $f$ is $\mathcal{A}$-equivalent to of type fold $(u, v^2)$ for $k_1 = 1$; cusp $(u, uv + v^3)$ for $k_1 = 2$; swallowtail $(u, uv + v^4)$ for $k_1 = 4$.  

15
\[ \bar{A} = L = \ell \gamma = \lambda \]

<table>
<thead>
<tr>
<th>( \bar{A} )</th>
<th>( L = \ell ) = \lambda )</th>
<th>( f )</th>
<th>( \gamma )</th>
<th>( \bar{A}/L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_0 )</td>
<td>( A_0 )</td>
<td>fold</td>
<td>( (2, 3) )</td>
<td>( (1, \geq 2) )</td>
</tr>
<tr>
<td>( A_0 )</td>
<td>( A_1 )</td>
<td>fold</td>
<td>( (2, 3) )</td>
<td>( (1, \geq 2) )</td>
</tr>
<tr>
<td>( A_0 )</td>
<td>( A_2 )</td>
<td>fold</td>
<td>( (2, 3) )</td>
<td>( (1, \geq 2) )</td>
</tr>
<tr>
<td>( A_0 )</td>
<td>( A_{\geq 3} )</td>
<td>fold</td>
<td>( (2, 3) )</td>
<td>( (1, \geq 2) )</td>
</tr>
<tr>
<td>( A_1 )</td>
<td>( A_0 )</td>
<td>cusp</td>
<td>( (3, 4) )</td>
<td>( (2, \geq 3) )</td>
</tr>
<tr>
<td>( A_1 )</td>
<td>( A_1 )</td>
<td>cusp</td>
<td>( (3, 4) )</td>
<td>( (2, \geq 3) )</td>
</tr>
<tr>
<td>( A_1 )</td>
<td>( A_2 )</td>
<td>cusp</td>
<td>( (3, 4) )</td>
<td>( (2, \geq 3) )</td>
</tr>
<tr>
<td>( A_1 )</td>
<td>( A_{\geq 3} )</td>
<td>cusp</td>
<td>( (3, 4) )</td>
<td>( (2, \geq 3) )</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>( A_0 )</td>
<td>swallowtail</td>
<td>( (4, 5) )</td>
<td>( (3, \geq 4) )</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>( A_1 )</td>
<td>swallowtail</td>
<td>( (4, 5) )</td>
<td>( (3, \geq 4) )</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>( A_2 )</td>
<td>swallowtail</td>
<td>( (4, 5) )</td>
<td>( (3, \geq 4) )</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>( A_{\geq 3} )</td>
<td>swallowtail</td>
<td>( (4, 5) )</td>
<td>( (3, \geq 4) )</td>
</tr>
</tbody>
</table>

Table 2: The types of \( f, \gamma \) are shown for given types of \( \bar{A}, L \). \( \lambda \) and \( e \) are regular. \( \gamma \) is singular and \( (m_1, m_2) \) in the column means the degrees of leading terms of \( \gamma = (\gamma_1, \gamma_2) \). The last column shows 2-multi indices (see §6.1) of \( \bar{A}/L \).

**Proof.** Remark that the leading terms of \( e_1 \) and \( e_2 \) are expressed as

\[
\begin{align*}
    e_1(v) &= L_0v + h.o.t, \\
    e_2(v) &= -1 + \frac{1}{2!}L_0^2v^2 + h.o.t.
\end{align*}
\]

Since \( e_2(0) \neq 0 \), we can replace \( \bar{u}(u, v) = \bar{u}e_2(v) + \gamma_2(v) \). Then

\[
f(\bar{u}, v) = \left( \frac{(\bar{u} - \bar{u}_2(v))}{e_2(v)}e_1(v) + \gamma_1(v), \bar{u} \right)
\]

and exchanging the components it is \( \mathcal{A} \)-equivalent to

\[
\left( \bar{u}, \bar{u}, \frac{e_1(v)}{e_2(v)} - \frac{\gamma_2(v)e_1(v)}{e_2(v)} + \gamma_1(v) \right).
\]

Here

\[
\frac{e_1(v)}{e_2(v)} = -L_0v + h.o.t, \quad \frac{\gamma_2(v)e_1(v)}{e_2(v)} + \gamma_1(v) = \frac{\bar{A}k_1}{k_1 + 1}v^{k_1+1} + h.o.t,
\]

thus \( f \) is \( \mathcal{A} \)-equivalent to

\[
(u, uv + v^{k_1+1} + h.o.t).
\]

In addition, whether the germs are \( \mathcal{A} \)-equivalent to of type fold, cusp or swallowtail is determined by the leading terms of the second component in the above form (cf. [25]).

Summing up the above results for relatively small numbers of \( k_1 \) and \( k_2 \), we get the Table 2. When \( k_1 \geq 5 \), we need analyze higher order terms in the germ in order to determine the exact \( \mathcal{A} \)-type. Refer to §6.4 for the detail with some examples.
5.1.3 Singular mappings, base curves and direction curves

Consider the case (4) $A_0 = L_0 = 0$. Then $\gamma, f, \lambda$ and $\lambda_u$ are singular. Let $A$ be of type $A_{k_1 - 1}$ and $L$ be of type $A_{k_2 - 1}$, and write

$$\overline{A}(v) = \frac{A_{k_1}}{k_1!} v^{k_1} + \cdots, \quad L(v) = \frac{L_{k_2}}{k_2!} v^{k_2} + \cdots$$

where $\overline{A}_{k_1}, L_{k_2} \neq 0 (k_1, k_2 \geq 1)$.

**Proposition 5.13** ($\lambda$ with $A_0 = L_0 = 0$) In the above setting, there are 3 cases with respect to the Jacobian $\lambda$:

1. for $k_1 = 1$, $\lambda$ is regular;
2. for $k_1 \geq k_2 + 1$, $\lambda(u, v) \sim_R uv^{k_2}$;
3. for $k_2 + 1 > k_1 \geq 2$, $\lambda(u, v) \sim_R \pm v^{k_1}$;

**Proof.** Since the Jacobian of the map of a normal line congruence is expressed as

$$\lambda(u, v) = \left( \frac{\overline{A}_{k_1}}{k_1!} v^{k_1} + \cdots \right) + u \left( \frac{L_{k_2}}{k_2!} v^{k_2} + \cdots \right),$$

we have the result. $\square$

**Proposition 5.14** ($\gamma$ with $A_0 = L_0 = 0$) In the above setting, the leading terms of the Taylor expansions of $\gamma_1$ and $\gamma_2$ are

$$\gamma_1(v) = \frac{\overline{A}_{k_1}}{(k_1 + 1)!} v^{k_1+1} + \cdots, \quad \gamma_2(v) = \frac{\overline{A}_{k_1} L_{k_2}}{k_1!(k_2 + 1)!(k_1 + k_2 + 2)} v^{(k_1+k_2+2)} + \cdots.$$

**Proof.** The base curve $\gamma(v) = (\gamma_1(v), \gamma_2(v))$ is as follows:

$$\gamma_1(v) = \int_0^v \left( \frac{\overline{A}_{k_1}}{k_1!} v^{k_1} + \cdots \right) \cos \left( \frac{L_{k_2}}{(k_2 + 1)!} v^{k_2+1} + \cdots \right) \, dv = \frac{\overline{A}_{k_1}}{(k_1 + 1)!} v^{k_1+1} + \cdots,$$

$$\gamma_2(v) = \int_0^v \left( \frac{\overline{A}_{k_1}}{k_1!} v^{k_1} + \cdots \right) \sin \left( \frac{L_{k_2}}{(k_2 + 1)!} v^{k_2+1} + \cdots \right) \, dv$$

$$= \frac{\overline{A}_{k_1} L_{k_2}}{k_1!(k_2 + 1)!(k_1 + k_2 + 2)} v^{(k_1+k_2+2)} + \cdots.$$ $\square$

**Proposition 5.15** ($f$ with $A_0 = L_0 = 0$) In the above setting, there are 3 cases for the map of the normal line congruence $f(u, v) = \gamma(v) + u e(v) = (u\gamma_1(v) + \gamma_1(v), u\gamma_2(v) + \gamma_2(v))$:

1. for $k_1 \leq k_2$, $f$ is $A$-equivalent to $(u, v^{k_1+1})$;
<table>
<thead>
<tr>
<th>$\bar{A}$</th>
<th>$L = \ell_\gamma = \lambda_u$</th>
<th>$\lambda$</th>
<th>$f_2$</th>
<th>$e$</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_0$</td>
<td>$A_0$</td>
<td>regular</td>
<td>$v^2$</td>
<td>$A_1$</td>
<td>(2, 4)</td>
</tr>
<tr>
<td>$A_1$</td>
<td>regular</td>
<td>$v^2$</td>
<td>$A_2$</td>
<td>(2, 5)</td>
<td></td>
</tr>
<tr>
<td>$A_2$</td>
<td>regular</td>
<td>$v^2$</td>
<td>$A_3$</td>
<td>(2, 6)</td>
<td></td>
</tr>
<tr>
<td>$A_{\geq 3}$</td>
<td>regular</td>
<td>$v^2$</td>
<td>$A_{\geq 4}$</td>
<td>(2, $\geq 7$)</td>
<td></td>
</tr>
<tr>
<td>$A_1$</td>
<td>$A_0$</td>
<td>$uv$</td>
<td>$u^2v - v^3$</td>
<td>$A_1$</td>
<td>(3, 5)</td>
</tr>
<tr>
<td>$A_1$</td>
<td>$v^2$</td>
<td>$v^3$</td>
<td>$A_2$</td>
<td>(3, 6)</td>
<td></td>
</tr>
<tr>
<td>$A_2$</td>
<td>$v^3$</td>
<td>$v^3$</td>
<td>$A_3$</td>
<td>(3, 7)</td>
<td></td>
</tr>
<tr>
<td>$A_{\geq 3}$</td>
<td>$v^3$</td>
<td>$v^3$</td>
<td>$A_{\geq 4}$</td>
<td>(3, $\geq 8$)</td>
<td></td>
</tr>
<tr>
<td>$A_2$</td>
<td>$A_0$</td>
<td>$uv^2$</td>
<td>$wu^2v^2 + v^3$</td>
<td>$A_1$</td>
<td>(4, 6)</td>
</tr>
<tr>
<td>$A_1$</td>
<td>$v^4$</td>
<td>$v^3 + v^4$</td>
<td>$A_2$</td>
<td>(4, 7)</td>
<td></td>
</tr>
<tr>
<td>$A_2$</td>
<td>$v^3$</td>
<td>$v^4$</td>
<td>$A_3$</td>
<td>(4, 8)</td>
<td></td>
</tr>
<tr>
<td>$A_{\geq 3}$</td>
<td>$v^3$</td>
<td>$v^4$</td>
<td>$A_{\geq 4}$</td>
<td>(4, $\geq 9$)</td>
<td></td>
</tr>
</tbody>
</table>

Table 3: The types of $\lambda, f, e, \gamma$ are shown for given types of $\bar{A}, L$. The 4-th column shows the second components of a mapping $(u, f_2(u, v))$ which is a representative of the $\bar{A}$-type of the mapping of the normal line congruence $f$. Note that an item with $*$ means the type of a jet, thus several $\bar{A}$-types exist over the type of the jet.

2. for $k_1 - 1 = k_2$, $f$ is $\bar{A}$-equivalent to $(u, (u + v)v^{k_1} + h.o.t)$. Especially, when $k_1 = 2$, $f$ is $\bar{A}$-equivalent to of type beaks $(u, u^2v - v^3)$;

3. for $k_1 > k_2 + 1$, $f$ is $\bar{A}$-equivalent to $(u, uv^{k_2+1} + v^{k_1+1} + h.o.t)$.

Proof. First, remark that the leading terms of $e_1$ and $e_2$ are $\frac{L_{k_2}}{(k_2 + 1)!}v^{k_2+1}$ and $-1$, respectively. Thus $f(u, v)$ is $\bar{A}$-equivalent to the following form:

\[
\left( u, \frac{e_1(v)}{e_2(v)} - \gamma_2(v)e_1(v) + \gamma_1(v) \right) = \left( \bar{u}, \frac{L_{k_2}}{(k_2 + 1)!}vu^{k_2+1}(1 + \ldots) + \frac{\bar{A}_{k_1}}{(k_1 + 1)!}v^{k_1+1}(1 + \ldots) \right).
\]

Note that when $k_1 = 2, k_2 = 1$,

\[ f \sim_{\bar{A}} (u, (u + v)v^2 + h.o.t) \sim_{\bar{A}} (u, u^2v - v^3 + h.o.t). \]

From 3-$\bar{A}$-determinacy of the beaks-type map germ (refer to [25]), we can determine the $\bar{A}$-type of the above germ as beaks $(u, v^3 - u^2v)$.

Summing up the above results for relatively small numbers of $k_1$ and $k_2$, we get the Table 3.

6 Several geometrical aspects of plane line congruences

In §5.1, we focused on properties of normal line congruences which depend on the $R$-types of functions in the curvature. In this section, we show results on geometry of normal line congruences depending on higher order information of the functions.

For later use, we define a detailed type of a function germ:
**Definition 6.1** Take a one-variable smooth function germ \( f : (I, x_0) \to \mathbb{R} , \ x \mapsto f(x) \).

1. We say that the 1st index of \( f \) with respect to \( x \) is a non-negative integer \( \ell_1 \) if \( f^{(i)}(x_0) = 0 \) for any integer \( i \) with \( 0 \leq i \leq \ell_1 - 1 \) and \( f^{(\ell_1)}(x_0) \neq 0 \) (here we define \( f^{(0)}(x_0) := f(x_0) \)); and the 1st index is \( \infty \) if \( f^{(i)}(x_0) = 0 \) for any integer \( i \geq 0 \).

2. For an integer \( n \geq 2 \), we say that the \( n \)-th index of \( f \) with respect to \( x \) is \( \ell_n \in \mathbb{N} \) if \( f^{(i)}(x_0) = 0 \) for any integer \( i \) with \( \ell_{n-1} < i < \ell_n \) and \( f^{(\ell_n)}(x_0) \neq 0 \), where \( \ell_{n-1} \) is an \( n-1 \)-th index of \( f \).

3. Suppose \( f \) has the finite \( n \)-th index with respect to \( x \), and the \( i \)-th index is \( \ell_i \) for an integer \( i \) with \( 0 \leq i \leq n \). The tuple \( (\ell_1, \ldots, \ell_n) \) is called the \( n \)-multi index of \( f \) with respect to \( x \).

**Example 6.2** Let \( f : (I, 0) \to \mathbb{R} \) be a one-variable smooth function germ which is written as

\[
f(x) = \frac{a_{\ell_1}}{\ell_1!} x^{\ell_1} + \frac{a_{\ell_2}}{\ell_2!} x^{\ell_2} + \frac{a_{\ell_3}}{\ell_3!} x^{\ell_3} + \cdots + \frac{a_{\ell_m}}{\ell_m!} x^{\ell_m} + \mathrm{h.o.t.}
\]

with integers \( 0 \leq \ell_1 < \ell_2 < \ell_3 < \cdots < \ell_m \) and non-zero real values \( a_{\ell_1}, a_{\ell_2}, \ldots, a_{\ell_m} \). The \( i \)-th index of \( f \) is \( \ell_i \) for an integer \( i \) with \( 1 \leq i \leq m \), and the \( m \)-multi index is \( (\ell_1, \ldots, \ell_m) \).

**Remark 6.3** The multi-index for \( f : (I, x_0) \to \mathbb{R} \) is not coordinate free (in other words, not invariant under \( \mathcal{R} \)-equivalence), however it plays an interesting role in a context of local differential geometry. This notion is a natural expansion of the type of the curvature function of a plane curve at a degenerate point such as a vertex or inflection point (cf. [16]). Clearly, a vertex point in a plane curve can be characterized by that the curvature function at the point has the 2-multi index \( (0, m) \) for \( m \geq 2 \) (\( m = 2 \) for an ordinary vertex). In addition, an inflection point can be characterized by that the curvature function at the point has the 2-multi index \( (\ell_1, \ell_2) \) for \( \ell_1 \geq 1 \) (\( \ell_1 = 1 \) for an ordinary inflection). In the above settings, \( m \) measures the degree of the contact of the curve with circles; \( \ell_1 \) measures the degree of the contact of the curve with lines. One aim of introducing multi-index is focusing on the second index \( \ell_2 \) (especially for the case \( \ell_1 \geq 1 \)) to the components of the curvature of a Legendre curve appearing in our setting.

The notion of multi index is characterized by the following equivalence of functions.

**Definition 6.4** Let \( f, g : (I, x_0) \to \mathbb{R} \) be smooth functions.

1. When the 1st indices of \( f, g \) are 0, \( f, g \) are \( \mathcal{R}_{(0, \ell_2, \ldots, \ell_n)} \)-equivalent if

\[
f - j^{\ell_1-1} f(x_0) \sim_{\mathcal{R}} g - j^{\ell_1-1} g(x_0)
\]

for any integer \( i \) with \( 1 < i \leq n \).

2. When the 1st indices of \( f, g \) are more than 0, \( f, g \) are \( \mathcal{R}_{(\ell_1, \ldots, \ell_n)} \)-equivalent if

\[
f - j^{\ell_1-1} f(x_0) \sim_{\mathcal{R}} g - j^{\ell_1-1} g(x_0)
\]

for any integer \( i \) with \( 1 \leq i \leq n \).

The next proposition follows from the definition (see also Example 6.2).
Proposition 6.5 Suppose \( f : (I, x_0) \to \mathbb{R} \) is a smooth function germ.

1. The followings are equivalent:
   \( (a) \) \( f \) has a \( n \)-multi index \((0, \ell_2, \ldots, \ell_n)\).
   \( (b) \) \( f \rightarrow j^{\ell_i-1} f(x_0) \) is \( \mathcal{R}(0,\ell_2,\ldots,\ell_n) \)-equivalent to \( \mathcal{A}_{\ell_1-1} \)-type for a non-negative integer \( i \) with \( \ell_2 \leq i \leq n \).

2. The followings are equivalent:
   \( (a) \) \( f \) has a \( \ell_1 \)-multi index \((\ell_1, \ldots, \ell_n)\) for a positive integer \( \ell_1 \).
   \( (b) \) \( f \rightarrow j^{\ell_1-1} f(x_0) \) is \( \mathcal{R}(\ell_1,\ldots,\ell_n) \)-equivalent to \( \mathcal{A}_{\ell_1-1} \)-type for a non-negative integer \( i \) with \( \ell_1 \leq i \leq n \).

As an example, the function \( f \) in Example 6.2 is \( \mathcal{R}(\ell_1,\ldots,\ell_n) \)-equivalent to \( \pm x^{\ell_1} \pm x^{\ell_2} \pm x^{\ell_3} \pm \cdots \pm x^{\ell_m} \).

6.1 The 2-multi indices of \( L/\overline{A} \) and \( \overline{A}/L \)

Let \((\gamma, e) : (I, 0) \to \mathbb{R}^2 \times S^1\) and \((\tilde{\gamma}, \tilde{e}) : (I, 0) \to \mathbb{R}^2 \times S^1\) express plane line congruence germs. Assume that \((\gamma, e) \sim_{\mathcal{R}} (\tilde{\gamma}, \tilde{e})\), then we have the relations of curvatures between \((\ell, L, \beta, A, B)\) of \( f(u, v) = \gamma(v) + u e(v) \) and \((\tilde{\ell}, \tilde{L}, \tilde{\beta}, \tilde{A}, \tilde{B})\) of \( \tilde{f}(u, q) = \tilde{\gamma}(q) + u \tilde{e}(q) \) as follows:

\[
\begin{align*}
\tilde{\ell}(u, q) &= 0, \\
\tilde{L}(u, q) &= L(u, v(q))v_q(q), \\
\tilde{\beta}(u, q) &= -1, \\
\tilde{A}(u, q) &= A(u, v(q))v_q(q) = \overline{A}(v(q))v_q(q) + u L(u, v(q))v_q(q), \\
\tilde{B}(u, q) &= B(u, v(q))v_q(q)
\end{align*}
\]

(3)

where \( \phi(u, q) = (u, v(q)) \) is a one-parameter parameter change of a special form in the source space (see Proposition 3.1). Thus the \( \mathcal{R} \)-types of function germs in the curvature of a plane line congruence germs \((\gamma, e) : (I, 0) \to \mathbb{R}^2 \times S^1\) as a one-parameter families of Legendre curves depends on the coordinate of \((\gamma, e)\). On the other hand, it is easily seen that the ratio of two functions in \( L, A, B, \overline{A} \) is invariant. Especially, we study functions of the form \( L/\overline{A} \) or \( \overline{A}/L \) in the following.

From now on, we consider a normal line congruence \((\gamma, e) : (I, 0) \to \mathbb{R}^2 \times S^1\) which is characterized by function germs \( L, \overline{A} : (I, 0) \to \mathbb{R} \). If \( \overline{A}(0) \neq 0 \), the base curve \( \gamma \) is regular and [7] shows that \(-L/\overline{A}\) is equal to the curvature \( \kappa \) of \( \gamma \) as a regular curve. Thus the ratio of \( L \) and \( \overline{A} \) plays an important role, and we want to expand the notion.

Let \( \overline{A}, L : (I, 0) \to \mathbb{R} \) be function germs with the 2-multi indices \((a_1, a_2)\) and \((\ell_1, \ell_2)\), respectively. If \( \ell_1 \geq a_1 \) (resp. \( a_1 \geq \ell_1 \)), then we can define a function germ \(-L/\overline{A}\) (resp. \(-\overline{A}/L\)). Remark that, from the equation (3), the above \(-L/\overline{A}\) (resp. \(-\overline{A}/L\)) is invariant under the coordinate change of \((\gamma, e)\).

Write

\[
\overline{A}(v) = \frac{\overline{A}_{a_1}}{a_1!} v^{a_1} + \frac{\overline{A}_{a_2}}{a_2!} v^{a_2} + \cdots, \quad L(v) = \frac{L_{\ell_1}}{\ell_1!} v^{\ell_1} + \frac{L_{\ell_2}}{\ell_2!} v^{\ell_2} + \cdots.
\]

(4)

Then we have the following formula for the 2-multi index of the above newly defined functions.
Proposition 6.6  
1. When $\ell_1 \geq a_1$, 
\[
2\text{-multi index of } -\frac{L}{A} = \left\{ \begin{array}{ll}
(\ell_1 - a_1, \ell_2 - a_1) & (\ell_1 - \ell_2 > a_1 - a_2) \\
(\ell_1 - a_1, *) & (\ell_1 - \ell_2 = a_1 - a_2) \\
(\ell_1 - a_1, \ell_1 + a_2 - 2a_1) & (\ell_1 - \ell_2 < a_1 - a_2)
\end{array} \right\}.
\]

Here $*$ is a number more than or equal to $\ell_2 - a_1$, where the equality holds when 
\[a_2!\ell_1!L_{\ell_2}A_{a_1} - a_1!\ell_2!L_{\ell_1}A_{a_2} = 0.\]

2. When $a_1 \geq \ell_1$, 
\[
2\text{-multi index of } -\frac{A}{L} = \left\{ \begin{array}{ll}
(a_1 - \ell_1, a_2 - \ell_1) & (a_1 - a_2 > \ell_1 - \ell_2) \\
(a_1 - \ell_1, *) & (a_1 - a_2 = \ell_1 - \ell_2) \\
(a_1 - \ell_1, a_1 + \ell_2 - 2\ell_1) & (a_1 - a_2 < \ell_1 - \ell_2)
\end{array} \right\}.
\]

Here $*$ is a number more than or equal to $a_2 - \ell_1$, where the equality holds when 
\[\ell_2!a_1!A_{a_2}L_{\ell_1} - \ell_1!a_2!A_{a_1}L_{\ell_2} = 0.\]

Proof. Proving the statement (1) is enough. When $\ell_1 \geq a_1$, the generalized curvature is written as 
\[
\frac{L}{A} = \frac{a_1!}{\ell_1!A_{a_1}} L_{\ell_1} v^ {\ell_1 - a_1} + \left\{ \begin{array}{ll}
\frac{a_1! L_{\ell_2}^2}{\ell_2!^2 A_{a_1}} v^{\ell_2 - a_1} & (\ell_1 - \ell_2 > a_1 - a_2) \\
\left( \frac{a_1! L_{\ell_2}}{\ell_2!^2 A_{a_1}} - \frac{(a_1!)^2 L_{\ell_1}A_{a_2}}{\ell_1!a_2! A_{a_1}} \right) v^{\ell_2 - a_1} & (\ell_1 - \ell_2 = a_1 - a_2) \\
-\frac{(a_1!)^2 L_{\ell_1}A_{a_2}}{\ell_1!a_2! A_{a_1}} v^{\ell_1 + a_2 - 2a_1} & (\ell_1 - \ell_2 < a_1 - a_2)
\end{array} \right\} + \ldots;
\]

especially, when $\ell_1 - \ell_2 = a_1 - a_2$, the coefficient of the second term is written as 
\[
\frac{a_1!}{\ell_1!\ell_2!a_2! A_{a_1}^2} (a_2!\ell_1!L_{\ell_2}A_{a_1} - a_1!\ell_2!L_{\ell_1}A_{a_2}).
\]

Thus we have the statement (1). \qed

Remark 6.7 Generally, the second indices of $L/A$ and $A/L$ depend on the choice of coordinate $v$. However, in the case $a_1 = \ell_1$ the second indices of $L/A$ and $A/L$ become independent of the choice of the coordinate $v$, like the degree of a vertex point of a regular curve.

6.2 Jacobian constant curves

According to §5, for a normal line congruence $f(u, v) = \gamma(v) + u\mathbf{e}(v)$, the Jacobian coincides with the function $A(u, v)$:
\[
\lambda(u, v) = A(v) + u\mathbf{N}(v) = A(u, v).
\]

In general, $\lambda^{-1}(0)$ (or $f(\lambda^{-1}(0))$) is called the set of singularities of the mapping $f$, at which $f$ is not immersive. The singular set of mapping is a main object to study of singularity theory.
On the other hand, even when $\lambda(0, 0) \neq 0$, the level set $\lambda^{-1}(\lambda(0, 0))$ is sometimes an important geometrical feature of a normal line congruence (cf. §7.1).

Write
\[
\overline{A}(v) = \overline{A}(0) + \frac{A_{a_1}}{a_1!}v^{a_1} + \cdots, \quad L(v) = \frac{L_{\ell_1}}{\ell_1!}v^{\ell_1} + \cdots
\]
for integers $a_1 \geq 1, \ell_1 \geq 0$ and real values $\overline{A}_{a_1}, L_{\ell_1} \neq 0$. The diffeomorphic types of the level set germ $\lambda^{-1}(\lambda(0, 0))$ at $(0, 0)$ in the $uv$-plane are divided into the following three cases:

(i) When $\ell_1 = 0$,
\[
\lambda(u, v) = \lambda(0, 0) \iff u = -\frac{\overline{A}(v) - \overline{A}(0)}{L(v)}
\]
and $f(\lambda^{-1}(\lambda(0, 0)))$ is a Legendre curve (see Proposition 6.8);

(ii) When $\ell_1 \geq a_1 \geq 1$,
\[
\lambda(u, v) = \lambda(0, 0) \iff v^{a_1} \left(\frac{\overline{A}_{a_1}}{a_1!} + \text{h.o.t.}\right) = 0 \iff v = 0
\]
and $f(\lambda^{-1}(\lambda(0, 0)))$ is the straight line along the direction $e(0)$;

(iii) When $a_1 > \ell_1 \geq 1$,
\[
\lambda(u, v) = \lambda(0, 0) \iff v^{\ell_1} \left(u + \frac{\overline{A}(v) - \overline{A}(0)}{L(v)}\right) = 0 \iff v = 0 \text{ or } u = -\frac{\overline{A}(v) - \overline{A}(0)}{L(v)},
\]
and $f(\lambda^{-1}(\lambda(0, 0)))$ consists of the straight line along the direction $e(0)$ and another Legendre curve (see Proposition 6.8).

In the cases (i) and (iii), there exists a branch of $f(\lambda^{-1}(\lambda(0, 0)))$ expressed by
\[
\hat{\gamma} := \gamma - \frac{\overline{A} - \overline{A}(0)}{L}e.
\]
If there exists the above germ $\hat{\gamma}$ which goes through $f(0, 0)$, we call it the Jacobian constant curve (for short, JC curve) of the normal line congruence $f$ at $f(0, 0)$. Put
\[
\hat{\gamma}(v) := \gamma(v) - \frac{\overline{A}(v) - \overline{A}(0)}{L(v)}e(v)
\]

\[
\hat{e}(v) := \begin{cases} \frac{\overline{A}(0)e(v) + \frac{d}{dv} \left(\overline{A}(v) - \overline{A}(0)\right)}{\sqrt{\left(\frac{d}{dv} \left(\overline{A}(v) - \overline{A}(0)\right)\right)^2 + \overline{A}'(0)^2}} & \text{when } \overline{A}(0) \neq 0, \\ e(v) & \text{when } \overline{A}(0) = 0. \end{cases}
\]

The next statement immediately follows from direct calculations (see [9] for the case $\overline{A}(0) = 0$).

**Proposition 6.8** $(\hat{\gamma}, \hat{e})$ is a Legendre curve, and the curvature $(\ell, \beta)$ of it is given as follows:

\[
\ell(v) := \begin{cases} L(v) + \frac{-\overline{A}(v) \frac{d^2}{dv^2} \left(\overline{A}(v) - \overline{A}(0)\right)}{\left(\frac{d}{dv} \left(\overline{A}(v) - \overline{A}(0)\right)\right)^2 + \overline{A}'(0)^2} & \text{when } \overline{A}(0) \neq 0, \\ \frac{d}{dv} \left(\overline{A}(v) - \overline{A}(0)\right) & \text{when } \overline{A}(0) = 0. \end{cases}
\]
\[
\beta(v) = \begin{cases} 
\sqrt{\left( \frac{d}{dv} \left( \frac{\overline{A}(v)}{L(v)} \right) \right)^2 + \overline{A}(0)} & \text{when } \overline{A}(0) \neq 0, \\
\frac{d}{dv} \left( \frac{\overline{A}(v)}{L(v)} \right) & \text{when } \overline{A}(0) = 0.
\end{cases}
\]

**Remark 6.9** The JC curve of a normal line congruence \( f \) at a point is exactly the evolute (or envelope) when it is a branch of the singular value set of \( f \) (i.e. \( f(\lambda^{-1}(0)) \)). It is well known that the evolute of a curve locally never has intersections with normal lines at inflection points of the curve; while general JC curves at the points can exist. Especially, the existence of JC curves is local invariant under the congruent equivalence, see Table 1 and §7.1.

### 6.3 Evolutes and normal line congruences

We consider the cotangent bundle \( \pi : T^*\mathbb{R}^2 \to \mathbb{R}^2 \) over \( \mathbb{R}^2 \). Let \( (x, y, p, q) \) be the canonical coordinate and \( \omega = dp \wedge dx + dq \wedge dy \) be a canonical symplectic form on \( T^*\mathbb{R}^2 \).

For a plane line congruence \( f : \mathbb{R} \times I \to \mathbb{R}^2, f(u, v) = \gamma(v) + ue(v), \) we define \( \tilde{f} : \mathbb{R} \times I \to T^*\mathbb{R}^2 \) by \( \tilde{f}(u, v) = (f(u, v), e(v)) \). Then \( \tilde{f}^*\omega = e'(v) \cdot e(v) dv \wedge du = 0 \). Hence \( \tilde{f} \) is a Lagrangian mapping (cf. [1, 2, 16]). Moreover, if \( L(v) \neq 0 \) for all \( v \in I \), then \( \tilde{f} \) is a Lagrangian immersion. The caustic \( C_f \) of \( \tilde{f} \) is defined by the set of the critical value of \( \pi \circ \tilde{f} = f \). In this case, the caustic \( C_f \) is given by \( \{ \gamma(v) - (\overline{A}(v)/L(v)) e(v) | v \in I \} \). Moreover, if we consider \( \hat{f}(u, v) = \hat{\gamma}(v) + u e(v), \) where \( \hat{\gamma} \) is in Proposition 5.5, then \( C_{\hat{f}} = C_f \). Since \( \hat{f} \) is a normal line congruence, \( (\hat{\gamma}, e) : I \to \mathbb{R}^2 \times S^1 \) is a Legendre curve (a Legendre immersion when \( L \neq 0 \)) with the curvature \( (\ell, \beta) = (L, \overline{A} + \epsilon L) \). It follows that the caustic \( C_{\hat{f}} \) is given by the evolute of \( \hat{\gamma} \) (cf. [7]). In fact, the evolute of \( \hat{\gamma}, E\hat{v}(\hat{\gamma}) : I \to \mathbb{R}^2 \) is given by

\[
E\hat{v}(\hat{\gamma})(v) = \hat{\gamma}(v) - \frac{\beta_3(v)}{\ell_3(v)} e(v)
= \gamma(v) + \epsilon(v) e(v) - \frac{\overline{A}(v) + \epsilon(v)L(v)}{L(v)} e(v)
= \gamma(v) - \frac{\overline{A}(v)}{L(v)} e(v).
\]

### 6.4 \( \mathcal{A} \)-types of normal line congruences

In §5.1, we studied the \( \mathcal{A} \)-types of germs of normal line congruences, where conditions to determine just first or second terms of the Taylor expansions of the map germs are given for some cases. In this section, we show the conditions for singularities of \( \mathcal{A} \)-codimension \( \leq 4 \) to appear in map germs of normal line congruences. Here the \( \mathcal{A} \)-codimension means a codimension of the \( \mathcal{A} \)-orbit in the space of map germs (see [16, 25] for the details).

#### 6.4.1 Butterfly

Set \( \overline{A}_0 = 0, L_0 \neq 0, \overline{A} \sim_R A_3 \), then \( f \sim_\mathcal{A} (u, uw + v^5 + h.o.t) \) (see section 5.1.2). The normal forms of the \( \mathcal{A} \)-orbits over this type are either \( (u, uw + v^5 \pm v^7) \) or \( (u, uw + v^5) \), which are \( 7, \mathcal{A} \)-determined [25]. The first type is called butterfly. The difference is determined by higher order terms of the given germ, that is, higher order terms of \( \overline{A}, L \).
Proposition 6.10 Assume that $\overline{A}, L$ are written as
\[
\overline{A}(v) = \frac{A_{4}}{4!}v^{4} + \frac{A_{5}}{5!}v^{5} + \frac{A_{6}}{6!}v^{6} + \cdots, \quad \text{and} \quad L(v) = \frac{L_{0}}{1!}v + \frac{L_{2}}{2!}v^{2} + \cdots
\]
where $A_{4}, L_{0} \neq 0$. Then $f$ is $\mathcal{A}$-equivalent to

- $(u, uv + v^{5} \pm v^{7})$ if and only if $P \geq 0$;
- $(u, uv + v^{5})$ if and only if $P = 0$

where
\[
P := -15(160L_{0}^{4} - 147L_{2}^{4} + 112L_{0}L_{2})\overline{A}_{4}^{2} + 6L_{0}(7L_{1}\overline{A}_{5} + 8L_{0}\overline{A}_{6})\overline{A}_{4} - 35L_{0}^{2}\overline{A}_{5}^{2}.
\]

Especially, $f$ is always equivalent to $(u, uv + v^{5} - v^{7})$ if the 2nd index of $L$ is more than 2 (that is, $L_{1} = L_{2} = 0$) and that of $\overline{A}$ is more than 6 (that is, $\overline{A}_{5} = \overline{A}_{6} = 0$).

Proof. The claim follows from direct calculations using criteria of $\mathcal{A}$-types in [19]. First, by routine coordinate changes, $f$ is $\mathcal{A}$-equivalent to the form
\[
(x, xy + y^{5} + \sum_{7 \geq i+j \geq 6} c_{ij}x^{i}y^{j} + \text{h.o.t})
\]
where $c_{ij}$ are polynomials consisting of $A_{i}, L_{i}$ as variables. Especially, the data of the following coefficients are important:
\[
c_{06} = \frac{-15L_{1}A_{4} + L_{0}\overline{A}_{5}}{6\overline{A}_{4}L_{0}}, \quad c_{07} = \frac{-5(10L_{0}^{4} - 42L_{2}^{4} + 7L_{0}L_{2})\overline{A}_{4} - 21L_{0}L_{4}\overline{A}_{5} + L_{0}^{2}\overline{A}_{6}}{42\overline{A}_{4}L_{0}^{2}}.
\]
From criteria (2) of Proposition 3.3 in [19], we see that the value
\[
c_{07} - \frac{5}{8}c_{06} = \frac{-15(160L_{0}^{4} - 147L_{2}^{4} + 112L_{0}L_{2})\overline{A}_{4}^{2} + 6L_{0}(7L_{1}\overline{A}_{5} + 8L_{0}\overline{A}_{6})\overline{A}_{4} - 35L_{0}^{2}\overline{A}_{5}^{2}}{2016\overline{A}_{4}L_{0}^{2}}
\]
determines the $\mathcal{A}$-type of $f$.  \qed

Remark 6.11 The remark on index types of $\overline{A}, L$ in Proposition 6.10 can be stated also as follows: $f$ is always equivalent to $(u, uv + v^{5} - v^{7})$ if $L$ is $\mathcal{R}_{(0, \ell_{2})}$-equivalent to $L_{0} \pm x^{\ell_{2}}$ for $\ell_{2} \geq 3$ and $\overline{A}$ is $\mathcal{R}_{(4, \ell_{2})}$-equivalent to $\pm x^{4} \pm x^{\ell_{2}}$ for $\ell_{2} \geq 7$.

6.4.2 Gulls

Set $\overline{A}_{0} = 0, L_{0} = 0, \overline{A} \sim_{\mathcal{A}} A_{2}$ and $L \sim_{\mathcal{R}} A_{0}$, then $f \sim_{\mathcal{A}} (u, uv^{2} + v^{4} + \text{h.o.t})(\text{see Section 5.1.3}).$ The $\mathcal{A}$-orbits over the above form have the representatives $(u, uv^{2} + v^{4} + v^{2p+1})$ with $p \geq 2$ which are $2p + 1$-determined (refer to [25]). The type with $p = 2$ is called gulls.

Proposition 6.12 Assume that $\overline{A}, L$ are written as
\[
\overline{A}(v) = \frac{A_{3}}{3!}v^{3} + \frac{A_{4}}{4!}v^{4} + \cdots, \quad \text{and} \quad L(v) = \frac{L_{1}}{1!}v + \frac{L_{2}}{2!}v^{2} + \cdots
\]
where $A_{3}, L_{1} \neq 0$. Then $f$ is $\mathcal{A}$-equivalent to
• \((u, uv^2 + v^4 + v^5)\) if and only if \(-10L_2\overline{A}_3 + 3L_1\overline{A}_4 \neq 0;\)

• \((u, uv^2 + v^4 + v^7)\) if and only if \(-10L_2\overline{A}_3 + 3L_1\overline{A}_4 = 0, Q \neq 0\)

where

\[Q := -7(10L_2^3 + 10L_1L_4L_3 - 3L_2^2L_4)\overline{A}_3 + 21L_1^2L_2\overline{A}_5 - 3L_2^3\overline{A}_6.\]

Especially, \(f\) is not equivalent to \((u, uv^2 + v^4 + v^5)\) if the 2nd index of \(L\) is more than 2 (that is, \(L_2 = 0\)) and that of \(\overline{A}\) is more than 4 (that is, \(\overline{A}_4 = 0\)).

Proof. As in the proof of Proposition 6.10, the claim also follows from direct calculations using criteria of \(\mathcal{A}\)-types in [19]. By routine coordinate changes, \(f\) is \(\mathcal{A}\)-equivalent to the form

\[(x, xy^2 + y^4 + \sum_{i+j\geq 5} c_{ij}x^iy^j + h.o.t)\]

where \(c_{ij}\) are polynomials consisting of \(L_i\) as variables. Especially,

\[c_{23} = 0, \quad c_{05} = \frac{-10L_2\overline{A}_3 + 3L_1\overline{A}_4}{15A_3L_1}, \quad c_{15} = \frac{5L_2^3 - 10L_1L_2L_3 + 3L_2^2L_4}{180L_1^3},\]

\[c_{07} = \frac{-35L_2^3A_3 + 105L_1L_2^2\overline{A}_4 - 63L_2L_2\overline{A}_5 + 9L_2^2\overline{A}_6}{1890A_3^2L_1^3}.\]

From criteria (2) of Proposition 3.5 in [19], we see that if \(c_{05} \neq 0,\) then \(f\) is of type gulls; and if \(c_{05} = 0\) and \(c_{07} - 2c_{15} + 4c_{23} \neq 0,\) then \(f\) is \(\mathcal{A}\)-equivalent to \((x, xy^2 + y^4 + y^7).\)

\[\Box\]

Remark 6.13 The remark on index types of \(\overline{A}, L\) in Proposition 6.12 can be stated also as follows: \(f\) is not equivalent to \((u, uv^2 + v^4 + v^5)\) if \(L\) is \(\mathcal{R}_{(1,\ell_2)}\)-equivalent to \(\pm x \pm x^{\ell_2}\) for \(\ell_2 \geq 3\) and \(\overline{A}\) is \(\mathcal{R}_{(3,\ell_2)}\)-equivalent to \(\pm x^3 \pm x^{\ell_2}\) for \(\ell_2 \geq 5.\)

6.4.3 Summary

Summing up the above results including parts of Propositions in §5.1, we have the following characterizations of singular germs of \(\mathcal{A}\)-codimension up to 4 appearing in the maps of normal line congruences. Examples of the figures for those \(\mathcal{A}\)-types are seen in §7.2 and §7.3.

Theorem 6.14 The following table shows all \(\mathcal{A}\)-types and characterizations of germs \((\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)\) with \(\mathcal{A}\)-codimension \(\leq 4\) appearing in the maps of normal line congruences \((\overline{A}_0 = 0\) is always assumed). Especially, the \(\mathcal{A}\)-types of lips: \((u, u^2v + v^3)\) \((\mathcal{A}\)-codimension = 3), goose: \((u, v^3 + u^3v)\) \((\mathcal{A}\)-codimension = 4) or of corank 2 never appear.

Proof. The absences of lips and goose types follow from Proposition 5.15. The absences of germs of corank 2 immediately follow from the form of the map of a plane line congruence. \(\Box\)

Remark 6.15 The absences of lips and goose types can be explained also in terms of the Jacobian \(\lambda\). In [19, 26], it is shown that a map germ is \(\mathcal{A}\)-equivalent to lips, then the Jacobian is \(\mathcal{R}\)-equivalent to \(A_1^2 : x^2 + y^2\); or to an \(\mathcal{A}\)-type of the form \((x, y^3 + x^ky)\) for \(k \geq 3\) (the germ is goose when \(k = 3\)), then the Jacobian is \(\mathcal{R}\)-equivalent to \(A_k : x^2 + y^k\). With the above facts, since the Jacobian \(\lambda\) for the map of a plane line congruence satisfies \(\lambda_{uj} = 0\) as seen in Remark 5.3, the map is never \(\mathcal{A}\)-equivalent to lips or an \(\mathcal{A}\)-type of the form \((x, y^3 + x^ky)\) for \(k \geq 3\). Note also that, according to Proposition 3.4, the absent germs of corank 1 in normal line congruences can be realized by one parameter families of Legendre curves.

25
A-cod | A-cod | type | characterization |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>fold : $(u, v^2)$</td>
<td>$\overline{A} \sim \mathcal{R} A_0$</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>cusp : $(u, uv + v^3)$</td>
<td>$\overline{A} \sim \mathcal{R} A_1$ and $L_0 \neq 0$</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>swallowtail : $(u, uv + v^4)$</td>
<td>$\overline{A} \sim \mathcal{R} A_2$ and $L_0 \neq 0$</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>beaks : $(u, u^2v - v^3)$</td>
<td>$\overline{A} \sim \mathcal{R} A_1, L_0 = 0$ and $L \sim \mathcal{R} A_0$</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>butterfly : $(u, uv + v^5 \pm v^7)$ and a condition in Proposition 6.10</td>
<td>$\overline{A} \sim \mathcal{R} A_3, L_0 \neq 0$</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>gulls : $(u, uv^2 + v^4 + v^5)$ and a condition in Proposition 6.12</td>
<td>$\overline{A} \sim \mathcal{R} A_2, L \sim \mathcal{R} A_0, L_0 = 0$</td>
</tr>
</tbody>
</table>

Table 4: $\mathcal{A}$-types of $\mathcal{A}_e$ (resp. $\mathcal{A}$)-codimension $\leq 2$ (resp. 4) appearing in maps of plane line congruences. See [25] for the definition of $\mathcal{A}_e$-codimension.

7 Examples of normal line congruences with figures

In this section, we show several figures of normal line congruences as examples.

7.1 Inffective regular base curves

We deal with normal line congruences of types in §5.1.1. The Figures 4 - 7 show the figures of normal line congruences to base curves of of the form $\gamma_{(a,b)}(v) = (v + v^{a+2}, 2v^{b+3})$ for pairs of non-negative integers $(a, b)$. $\overline{A}(0) \neq 0$ and $L(0) = 0$ hold for the curvatures $(\overline{A}, L)$ to the normal line congruences $f_{(a,b)}(u, v) = \gamma_{(a,b)}(v) + u\mathbf{e}(v)$, where $\mathbf{e}$ is the normal vector of $\gamma_{(a,b)}$ constructed as in §5. The blue curve expresses $\gamma_{(a,b)}$, which has an inflection point at $v = 0$. The red curve expresses the set

$$f_{(a,b)}(\lambda^{-1}(\lambda(0, 0))) - \{\text{the normal line of } \gamma_{(a,b)} \text{ at } \gamma_{(a,b)}(0)\}.$$

In Figures 4 - 5, the red curves never go through the origin, that is, the JC curves at $f_{(a,b)}(0, 0)$ (defined in §6.2) never exist; while the red curves in Figures 6 - 7 are the JC curves at $f_{(a,b)}(0, 0)$. Note that all figures are drawn as the image of the domain with $-0.9 \leq u \leq 0.9$ and $-0.5 \leq v \leq 0.5$ in the $(u, v)$-plane, and the straight lines parametrized by $u$ are plotted par 1/30 intervals to the domain of $v$.

7.2 Singular mappings and base curves

We deal with normal line congruences of types in §5.1.2. The Figures 8 - 11 show the figures of normal line congruences to base curves of of the form $\gamma_a(v) = (v^{a+2}, v^{a+3} + v^{a+4})$ for non-negative integers $a$. $\overline{A}(0) = 0$ and $L(0) \neq 0$ hold for the curvatures $(\overline{A}, L)$ to the normal line congruences $f_a(u, v) = \gamma_a(v) + u\mathbf{e}(v)$, where $\mathbf{e}$ is the normal vector of $\gamma_a$ constructed as in §5. The blue curve expresses $\gamma_a$. Note that all figures are drawn as the image of the domain with $-0.05 \leq u \leq 0.05$ and $-0.5 \leq v \leq 0.5$ in the $(u, v)$-plane, and the straight lines parametrized by $u$ are plotted par 1/30 intervals to the domain of $v$. 
7.3 Singular mappings, base curves and direction curves

We deal with normal line congruences of types in 5.1.3. The Figures 12 - 14 show the figures of normal line congruences to base curves of of the form $\gamma_a(v) = (v^2 + 2v^3, 2v^3)$ for non-negative integers $a$, and the Figure 15 shows the figure of normal line congruences to base curves of of the form $\gamma(v) = (v^4 + v^5, -2v^6 + v^9)$. $\overline{A}(0) = L(0) = 0$ hold for the curvatures $(\overline{A}, L)$ to the normal line congruences. The blue curve expresses $\gamma_a$ or $\gamma$. The red curve expresses the JC curve of the normal line congruence at the origin (see (2) of Proposition 5.13 or case (ii) of §6.2). Remark also that the map of the normal line congruence to $\gamma_2$ is $\mathcal{A}$-equivalent to the germ of type $(u, uv^2 + v^4 + v^5)$, which is called gulls, at $(u, v) = (0, 0)$; while that to $\gamma$ is $\mathcal{A}$-equivalent to the germ of type $(u, uv^2 + v^4 + v^5)$ at $(u, v) = (0, 0)$ (see §6.4.2). Note that all figures are drawn as the image of the domain with $-0.5 \leq u \leq 0.5$ and $-0.6 \leq v \leq 0.6$ in the $(u, v)$-plane, and the straight lines parametrized by $u$ are plotted par 1/30 intervals to the domain of $v$. 

Figure 4: The normal line congruence to $\gamma(0.0)(v) = (v + v^2, 2v^3)$. The $\mathcal{R}$-type of $\overline{A}$ (resp. $L$) at $v = 0$ is $A_0$ (resp. $A_0$).

Figure 5: The normal line congruence to $\gamma(0.1)(v) = (v + v^2, 2v^4)$. The $\mathcal{R}$-type of $\overline{A}$ (resp. $L$) at $v = 0$ is $A_0$ (resp. $A_1$).

Figure 6: The normal line congruence to $\gamma(1.0)(v) = (v + v^3, 2v^3)$. The $\mathcal{R}$-type of $\overline{A}$ (resp. $L$) at $v = 0$ is $A_1$ (resp. $A_0$).

Figure 7: The normal line congruence to $\gamma(2.0)(v) = (v + v^4, 2v^3)$. The $\mathcal{R}$-type of $\overline{A}$ (resp. $L$) at $v = 0$ is $A_2$ (resp. $A_0$).
Figure 8: The normal line congruence to $\gamma_0(v) = (v^2, v^3 + v^4)$. The $R$-type of $A$ (resp. $L$) at $v = 0$ is $A_0$ (resp. $A_0$). The $A$-equivalent type of the map of the normal line congruence is fold.

Figure 9: The normal line congruence to $\gamma_1(v) = (v^3, v^4 + v^5)$ . The $R$-type of $A$ (resp. $L$) at $v = 0$ is $A_1$ (resp. $A_0$). The $A$-equivalent type of the map of the normal line congruence is cusp.

References


Figure 10: The normal line congruence to $\gamma_2(v) = (v^4, v^5 + v^6)$. The $\mathcal{R}$-type of $\mathcal{A}$ (resp. $\mathcal{L}$) at $v = 0$ is $A_2$ (resp. $A_0$). The $\mathcal{A}$-equivalent type of the map of the normal line congruence is swallowtail.

Figure 11: The normal line congruence to $\gamma_3(v) = (v^5, v^6 + v^7)$. The $\mathcal{R}$-type of $\mathcal{A}$ (resp. $\mathcal{L}$) at $v = 0$ is $A_3$ (resp. $A_0$). The $\mathcal{A}$-equivalent type of the map of the normal line congruence is butterfly.


Figure 12: The normal line congruence to $\gamma_0(v) = (v^2, -2v^4 + v^5)$. The $R$-type of $\mathcal{A}$ (resp. $L$) at $v = 0$ is $A_0$ (resp. $A_0$). The $A$-equivalent type of the map of the normal line congruence is fold. Here the JC curve at the origin does not exist.

Figure 13: The normal line congruence to $\gamma_1(v) = (v^3, -2v^5 + v^6)$. The $R$-type of $\mathcal{A}$ (resp. $L$) at $v = 0$ is $A_1$ (resp. $A_0$). The $A$-equivalent type of the map of the normal line congruence is baeks.


Yutaro Kabata,
Kyushu University, Fukuoka 819-0395, Japan
E-mail address: kabata@imi.kyushu-u.ac.jp

Masatomo Takahashi,
Muroran Institute of Technology, Muroran 050-8585, Japan,
E-mail address: masatomo@mmm.muroran-it.ac.jp
Figure 14: The normal line congruence to $\gamma_2(v) = (v^4, -2v^6 + v^7)$ . The $\mathcal{R}$-type of $A$ (resp. $L$) at $v = 0$ is $A_2$ (resp. $A_0$). The $A$-equivalent type of the map of the normal line congruence is gulls.

Figure 15: The normal line congruence to $\gamma_d(v) = (v^4 + v^6, -2v^6 + v^9)$ . The $\mathcal{R}$-types of $A$ and $L$ are the same with those to $\gamma_2(v) = (v^4, -2v^6 + v^7)$, however the $A$-type of the map of the normal line congruence to $\gamma_d$ is different from that to $\gamma_2$. 
MI

MI2008-1 Takahiro ITO, Shuichi INOKUCHI & Yoshihiro MIZOGUCHI
Abstract collision systems simulated by cellular automata

MI2008-2 Eiji ONODERA
The initial value problem for a third-order dispersive flow into compact almost Hermitian manifolds

MI2008-3 Hiroaki KIDO
On isosceles sets in the 4-dimensional Euclidean space

MI2008-4 Hirofumi NOTSU
Numerical computations of cavity flow problems by a pressure stabilized characteristic-curve finite element scheme

MI2008-5 Yoshiyasu OZEKI
Torsion points of abelian varieties with values in infinite extensions over a p-adic field

MI2008-6 Yoshiyuki TOMIYAMA
Lifting Galois representations over arbitrary number fields

MI2008-7 Takehiro HIROTSU & Setsuo TANIGUCHI
The random walk model revisited

MI2008-8 Silvia GANDY, Masaaki KANNO, Hirokazu ANAI & Kazuhiro YOKOYAMA
Optimizing a particular real root of a polynomial by a special cylindrical algebraic decomposition

MI2008-9 Kazufumi KIMOTO, Sho MATSUMOTO & Masato WAKAYAMA
Alpha-determinant cyclic modules and Jacobi polynomials

MI2008-10 Sangyeol LEE & Hiroki MASUDA
Jarque-Bera Normality Test for the Driving Lévy Process of a Discretely Observed Univariate SDE

MI2008-11 Hiroyuki CHIHARA & Eiji ONODERA
A third order dispersive flow for closed curves into almost Hermitian manifolds

MI2008-12 Takehiko KINOSHITA, Kouji HASHIMOTO and Mitsuo T. NAKAO
On the $L^2$ a priori error estimates to the finite element solution of elliptic problems with singular adjoint operator

MI2008-13 Jacques FARAUT and Masato WAKAYAMA
Hermitian symmetric spaces of tube type and multivariate Meixner-Pollaczek polynomials
MI2008-14  Takashi NAKAMURA
Riemann zeta-values, Euler polynomials and the best constant of Sobolev inequality

MI2008-15  Takashi NAKAMURA
Some topics related to Hurwitz-Lerch zeta functions

MI2009-1  Yasuhide FUKUMOTO
Global time evolution of viscous vortex rings

MI2009-2  Hidetoshi MATSUI & Sadanori KONISHI
Regularized functional regression modeling for functional response and predictors

MI2009-3  Hidetoshi MATSUI & Sadanori KONISHI
Variable selection for functional regression model via the $L_1$ regularization

MI2009-4  Shuichi KAWANO & Sadanori KONISHI
Nonlinear logistic discrimination via regularized Gaussian basis expansions

MI2009-5  Toshiro HIRANOUCHI & Yuichiro TAGUCHII
Flat modules and Groebner bases over truncated discrete valuation rings

MI2009-6  Kenji KAJIWARA & Yasuhiro OHTA
Bilinearization and Casorati determinant solutions to non-autonomous 1+1 dimensional discrete soliton equations

MI2009-7  Yoshiyuki KAGEI
Asymptotic behavior of solutions of the compressible Navier-Stokes equation around the plane Couette flow

MI2009-8  Shohei TATEISHI, Hidetoshi MATSUI & Sadanori KONISHI
Nonlinear regression modeling via the lasso-type regularization

MI2009-9  Takeshi TAKAISHI & Masato KIMURA
Phase field model for mode III crack growth in two dimensional elasticity

MI2009-10  Shingo SAITO
Generalisation of Mack’s formula for claims reserving with arbitrary exponents for the variance assumption

MI2009-11  Kenji KAJIWARA, Masanobu KANEKO, Atsushi NOBE & Teruhisa TSUDA
Ultradiscretization of a solvable two-dimensional chaotic map associated with the Hesse cubic curve

MI2009-12  Tetsu MASUDA
Hypergeometric $\tau$-functions of the q-Painlevé system of type $E_8^{(1)}$

MI2009-13  Hidenao IWANE, Hitoshi YANAMI, Hirokazu ANAI & Kazuhiro YOKOYAMA
A Practical Implementation of a Symbolic-Numeric Cylindrical Algebraic Decomposition for Quantifier Elimination

MI2009-14  Yasunori MAEKAWA
On Gaussian decay estimates of solutions to some linear elliptic equations and its applications
MI2009-15 Yuya ISHIHARA & Yoshiyuki KAGEI
Large time behavior of the semigroup on \( L^p \) spaces associated with the linearized compressible Navier-Stokes equation in a cylindrical domain

MI2009-16 Chikashi ARITA, Atsuo KUNIBA, Kazumitsu SAKAI & Tsuyoshi SAWABE
Spectrum in multi-species asymmetric simple exclusion process on a ring

MI2009-17 Masato WAKAYAMA & Keitaro YAMAMOTO
Non-linear algebraic differential equations satisfied by certain family of elliptic functions

MI2009-18 Me Me NAING & Yasuhide FUKUMOTO
Local Instability of an Elliptical Flow Subjected to a Coriolis Force

MI2009-19 Mitsunori KAYANO & Sadanori KONISHI
Sparse functional principal component analysis via regularized basis expansions and its application

MI2009-20 Shuichi KAWANO & Sadanori KONISHI
Semi-supervised logistic discrimination via regularized Gaussian basis expansions

MI2009-21 Hiroshi YOSHIDA, Yoshihiro MIWA & Masanobu KANEKO
Elliptic curves and Fibonacci numbers arising from Lindenmayer system with symbolic computations

MI2009-22 Eiji ONODERA
A remark on the global existence of a third order dispersive flow into locally Hermitian symmetric spaces

MI2009-23 Stjepan LUGOMER & Yasuhide FUKUMOTO
Generation of ribbons, helicoids and complex scherk surface in laser-matter Interactions

MI2009-24 Yu KAWAKAMI
Recent progress in value distribution of the hyperbolic Gauss map

MI2009-25 Takehiko KINOSHITA & Mitsuhiro T. NAKAO
On very accurate enclosure of the optimal constant in the a priori error estimates for \( H_0^2 \)-projection

MI2009-26 Manabu YOSHIDA
Ramification of local fields and Fontaine’s property (Pm)

MI2009-27 Yu KAWAKAMI
Value distribution of the hyperbolic Gauss maps for flat fronts in hyperbolic three-space

MI2009-28 Masahisa TABATA
Numerical simulation of fluid movement in an hourglass by an energy-stable finite element scheme

MI2009-29 Yoshiyuki KAGEI & Yasunori MAEKAWA
Asymptotic behaviors of solutions to evolution equations in the presence of translation and scaling invariance
MI2009-30 Yoshiyuki KAGEI & Yasunori MAEKAWA
On asymptotic behaviors of solutions to parabolic systems modelling chemotaxis

MI2009-31 Masato WAKAYAMA & Yoshinori YAMASAKI
Hecke’s zeros and higher depth determinants

MI2009-32 Olivier PIRONNEAU & Masahisa TABATA
Stability and convergence of a Galerkin-characteristics finite element scheme of lumped mass type

MI2009-33 Chikashi ARITA
Queueing process with excluded-volume effect

MI2009-34 Kenji KAJIWARA, Nobutaka NAKAZONO & Teruhisa TSUDA
Projective reduction of the discrete Painlevé system of type$(A_2 + A_1)^{(1)}$

MI2009-35 Yosuke MIZUYAMA, Takamasa SHINDE, Masahisa TABATA & Daisuke TAGAMI
Finite element computation for scattering problems of micro-hologram using DtN map

MI2009-36 Reiichiro KAWAI & Hiroki MASUDA
Exact simulation of finite variation tempered stable Ornstein-Uhlenbeck processes

MI2009-37 Hiroki MASUDA
On statistical aspects in calibrating a geometric skewed stable asset price model

MI2010-1 Hiroki MASUDA
Approximate self-weighted LAD estimation of discretely observed ergodic Ornstein-Uhlenbeck processes

MI2010-2 Reiichiro KAWAI & Hiroki MASUDA
Infinite variation tempered stable Ornstein-Uhlenbeck processes with discrete observations

MI2010-3 Kei HIROSE, Shuichi KAWANO, Daisuke MIIKE & Sadanori KONISHI
Hyper-parameter selection in Bayesian structural equation models

MI2010-4 Nobuyuki IKEDA & Setsuo TANIGUCHI
The Itô-Nisio theorem, quadratic Wiener functionals, and 1-solitons

MI2010-5 Shohei TATEISHI & Sadanori KONISHI
Nonlinear regression modeling and detecting change point via the relevance vector machine

MI2010-6 Shuichi KAWANO, Toshihiro MISUMI & Sadanori KONISHI
Semi-supervised logistic discrimination via graph-based regularization

MI2010-7 Teruhisa TSUDA
UC hierarchy and monodromy preserving deformation

MI2010-8 Takahiro ITO
Abstract collision systems on groups
MI2010-9 Hiroshi YOSHIDA, Kinji KIMURA, Naoki YOSHIDA, Junko TANAKA & Yoshihiro MIWA  
An algebraic approach to underdetermined experiments

MI2010-10 Kei HIROSE & Sadanori KONISHI  
Variable selection via the grouped weighted lasso for factor analysis models

MI2010-11 Katsusuke NABESHIMA & Hiroshi YOSHIDA  
Derivation of specific conditions with Comprehensive Groebner Systems

MI2010-12 Yoshiyuki KAGEI, Yu NAGAFUCHI & Takeshi SUDOU  
Decay estimates on solutions of the linearized compressible Navier-Stokes equation around a Poiseuille type flow

MI2010-13 Reiichiro KAWAI & Hiroki MASUDA  
On simulation of tempered stable random variates

MI2010-14 Yoshiyasu OZEKI  
Non-existence of certain Galois representations with a uniform tame inertia weight

MI2010-15 Me Me NAING & Yasuhide FUKUMOTO  
Local Instability of a Rotating Flow Driven by Precession of Arbitrary Frequency

MI2010-16 Yu KAWAKAMI & Daisuke NAKAJO  
The value distribution of the Gauss map of improper affine spheres

MI2010-17 Kazunori YASUTAKE  
On the classification of rank 2 almost Fano bundles on projective space

MI2010-18 Toshimitsu TAKAESU  
Scaling limits for the system of semi-relativistic particles coupled to a scalar bose field

MI2010-19 Reiichiro KAWAI & Hiroki MASUDA  
Local asymptotic normality for normal inverse Gaussian Lévy processes with high-frequency sampling

MI2010-20 Yasuhide FUKUMOTO, Makoto HIROTA & Youichi MIE  
Lagrangian approach to weakly nonlinear stability of an elliptical flow

MI2010-21 Hiroki MASUDA  
Approximate quadratic estimating function for discretely observed Lévy driven SDEs with application to a noise normality test

MI2010-22 Toshimitsu TAKAESU  
A Generalized Scaling Limit and its Application to the Semi-Relativistic Particles System Coupled to a Bose Field with Removing Ultraviolet Cutoffs

MI2010-23 Takahiro ITO, Mitsuhiko FUJIO, Shuichi INOKUCHI & Yoshihiro MIZOGUCHI  
Composition, union and division of cellular automata on groups

MI2010-24 Toshimitsu TAKAESU  
A Hardy’s Uncertainty Principle Lemma in Weak Commutation Relations of Heisenberg-Lie Algebra
MI2010-25  Toshimitsu TAKAESU  
On the Essential Self-Adjointness of Anti-Commutative Operators

MI2010-26  Reiichiro KAWAI & Hiroki MASUDA  
On the local asymptotic behavior of the likelihood function for Meixner Lévy processes under high-frequency sampling

MI2010-27  Chikashi ARITA & Daichi YANAGISAWA  
Exclusive Queueing Process with Discrete Time

MI2010-28  Jun-ichi INOGUCHI, Kenji KAJIWARA, Nozomu MATSUURA & Yasuhiro OHTA  
Motion and Bäcklund transformations of discrete plane curves

MI2010-29  Takanori YASUDA, Masaya YASUDA, Takeshi SHIMOYAMA & Jun KOGURE  
On the Number of the Pairing-friendly Curves

MI2010-30  Chikashi ARITA & Kohei MOTEGI  
Spin-spin correlation functions of the $q$-VBS state of an integer spin model

MI2010-31  Shohei TATEISHI & Sadanori KONISHI  
Nonlinear regression modeling and spike detection via Gaussian basis expansions

MI2010-32  Nobutaka NAKAZONO  
Hypergeometric $\tau$ functions of the $q$-Painlevé systems of type $(A_2 + A_1)^{(1)}$

MI2010-33  Yoshiyuki KAGEI  
Global existence of solutions to the compressible Navier-Stokes equation around parallel flows

MI2010-34  Nobushige KUROKAWA, Masato WAKAYAMA & Yoshinori YAMASAKI  
Milnor-Selberg zeta functions and zeta regularizations

MI2010-35  Kissani PERERA & Yoshihiro MIZOGUCHI  
Laplacian energy of directed graphs and minimizing maximum outdegree algorithms

MI2010-36  Takanori YASUDA  
CAP representations of inner forms of $Sp(4)$ with respect to Klingen parabolic subgroup

MI2010-37  Chikashi ARITA & Andreas SCHADSCHNEIDER  
Dynamical analysis of the exclusive queueing process

MI2011-1  Yasuhide FUKUMOTO & Alexander B. SAMOKHIN  
Singular electromagnetic modes in an anisotropic medium

MI2011-2  Hiroki KONDO, Shingo SAITO & Setsuo TANIGUCHI  
Asymptotic tail dependence of the normal copula

MI2011-3  Takehiro HIROTsu, Hiroki KONDO, Shingo SAITO, Takuya SATO, Tatsushi TANAKA & Setsuo TANiguchi  
Anderson-Darling test and the Malliavin calculus

MI2011-4  Hiroshi INOUE, Shohei TATEISHI & Sadanori KONISHI  
Nonlinear regression modeling via Compressed Sensing
MI2011-5 Hiroshi INOUE  
Implications in Compressed Sensing and the Restricted Isometry Property

MI2011-6 Daeju KIM & Sadanori KONISHI  
Predictive information criterion for nonlinear regression model based on basis expansion methods

MI2011-7 Shohei TATEISHI, Chiaki KINJYO & Sadanori KONISHI  
Group variable selection via relevance vector machine

MI2011-8 Jan BREZINA & Yoshiyuki KAGEI  
Decay properties of solutions to the linearized compressible Navier-Stokes equation around time-periodic parallel flow  
Group variable selection via relevance vector machine

MI2011-9 Chikashi ARITA, Arvind AYYER, Kirone MALLICK & Sylvain PROLHAC  
Recursive structures in the multispecies TASEP

MI2011-10 Kazunori YASUTAKE  
On projective space bundle with nef normalized tautological line bundle

MI2011-11 Hisashi ANDO, Mike HAY, Kenji KAJIWARA & Tetsu MASUDA  
An explicit formula for the discrete power function associated with circle patterns of Schramm type

MI2011-12 Yoshiyuki KAGEI  
Asymptotic behavior of solutions to the compressible Navier-Stokes equation around a parallel flow

MI2011-13 Vladimír CHALUPECKÝ & Adrian MUNTEAN  
Semi-discrete finite difference multiscale scheme for a concrete corrosion model: approximation estimates and convergence

MI2011-14 Jun-ichi INOGUCHI, Kenji KAJIWARA, Nozomu MATSUURA & Yasuhiro OHTA  
Explicit solutions to the semi-discrete modified KdV equation and motion of discrete plane curves

MI2011-15 Hiroshi INOUE  
A generalization of restricted isometry property and applications to compressed sensing

MI2011-16 Yu KAWAKAMI  
A ramification theorem for the ratio of canonical forms of flat surfaces in hyperbolic three-space

MI2011-17 Naoyuki KAMIYAMA  
Matroid intersection with priority constraints

MI2012-1 Kazufumi KIMOTO & Masato WAKAYAMA  
Spectrum of non-commutative harmonic oscillators and residual modular forms

MI2012-2 Hiroki MASUDA  
Mighty convergence of the Gaussian quasi-likelihood random fields for ergodic Levy driven SDE observed at high frequency
MI2012-3 Hiroshi INOUE
A Weak RIP of theory of compressed sensing and LASSO

MI2012-4 Yasuhide FUKUMOTO & Youich MIE
Hamiltonian bifurcation theory for a rotating flow subject to elliptic straining field

MI2012-5 Yu KAWAKAMI
On the maximal number of exceptional values of Gauss maps for various classes of surfaces

MI2012-6 Marcio GAMEIRO, Yasuaki HIRAOKA, Shunsuke IZUMI, Miroslav KRAMAR, Konstantin MISCHAIKOW & Vidit NANDA
Topological Measurement of Protein Compressibility via Persistence Diagrams

MI2012-7 Nobutaka NAKAZONO & Seiji NISHIOKA
Solutions to a $q$-analog of Painlevé III equation of type $D_7^{(1)}$

MI2012-8 Naoyuki KAMIYAMA
A new approach to the Pareto stable matching problem

MI2012-9 Jan BREZINA & Yoshiyuki KAGEI
Spectral properties of the linearized compressible Navier-Stokes equation around time-periodic parallel flow

MI2012-10 Jan BREZINA
Asymptotic behavior of solutions to the compressible Navier-Stokes equation around a time-periodic parallel flow

MI2012-11 Daeju KIM, Shuichi KAWANO & Yoshiyuki NINOMIYA
Adaptive basis expansion via the extended fused lasso

MI2012-12 Masato WAKAYAMA
On simplicity of the lowest eigenvalue of non-commutative harmonic oscillators

MI2012-13 Masatoshi OKITA
On the convergence rates for the compressible Navier-Stokes equations with potential force

MI2013-1 Abduwuiali PAERHATI & Yasuhide FUKUMOTO
A Counter-example to Thomson-Tait-Chetayev’s Theorem

MI2013-2 Yasuhide FUKUMOTO & Hirofumi SAKUMA
A unified view of topological invariants of barotropic and baroclinic fluids and their application to formal stability analysis of three-dimensional ideal gas flows

MI2013-3 Hiroki MASUDA
Asymptotics for functionals of self-normalized residuals of discretely observed stochastic processes

MI2013-4 Naoyuki KAMIYAMA
On Counting Output Patterns of Logic Circuits

MI2013-5 Hiroshi INOUE
RIPless Theory for Compressed Sensing
MI2013-6 Hiroshi INOUE
Improved bounds on Restricted isometry for compressed sensing

MI2013-7 Hidetoshi MATSUI
Variable and boundary selection for functional data via multiclass logistic regression modeling

MI2013-8 Hidetoshi MATSUI
Variable selection for varying coefficient models with the sparse regularization

MI2013-9 Naoyuki KAMIYAMA
Packing Arborescences in Acyclic Temporal Networks

MI2013-10 Masato WAKAYAMA
Equivalence between the eigenvalue problem of non-commutative harmonic oscillators and existence of holomorphic solutions of Heun’s differential equations, eigenstates degeneration, and Rabi’s model

MI2013-11 Masatoshi OKITA
Optimal decay rate for strong solutions in critical spaces to the compressible Navier-Stokes equations

MI2013-12 Shuichi KAWANO, Ibuki HOSHINA, Kazuki MATSUDA & Sadanori KONISHI
Predictive model selection criteria for Bayesian lasso

MI2013-13 Hayato CHIBA
The First Painleve Equation on the Weighted Projective Space

MI2013-14 Hidetoshi MATSUI
Variable selection for functional linear models with functional predictors and a functional response

MI2013-15 Naoyuki KAMIYAMA
The Fault-Tolerant Facility Location Problem with Submodular Penalties

MI2013-16 Hidetoshi MATSUI
Selection of classification boundaries using the logistic regression

MI2014-1 Naoyuki KAMIYAMA
Popular Matchings under Matroid Constraints

MI2014-2 Yasuhide FUKUMOTO & Youichi MIE
Lagrangian approach to weakly nonlinear interaction of Kelvin waves and a symmetry-breaking bifurcation of a rotating flow

MI2014-3 Reika AOYAMA
Decay estimates on solutions of the linearized compressible Navier-Stokes equation around a Parallel flow in a cylindrical domain

MI2014-4 Naoyuki KAMIYAMA
The Popular Condensation Problem under Matroid Constraints
MI2014-5 Yoshiyuki KAGEI & Kazuyuki TSUDA  
Existence and stability of time periodic solution to the compressible Navier-Stokes equation for time periodic external force with symmetry

MI2014-6 This paper was withdrawn by the authors.

MI2014-7 Masatoshi OKITA  
On decay estimate of strong solutions in critical spaces for the compressible Navier-Stokes equations

MI2014-8 Rong ZOU & Yasuhide FUKUMOTO  
Local stability analysis of azimuthal magnetorotational instability of ideal MHD flows

MI2014-9 Yoshiyuki KAGEI & Naoki MAKIO  
Spectral properties of the linearized semigroup of the compressible Navier-Stokes equation on a periodic layer

MI2014-10 Kazuyuki TSUDA  
On the existence and stability of time periodic solution to the compressible Navier-Stokes equation on the whole space

MI2014-11 Yoshiyuki KAGEI & Takaaki NISHIDA  
Instability of plane Poiseuille flow in viscous compressible gas

MI2014-12 Chien-Chung HUANG, Naonori KAKIMURA & Naoyuki KAMIYAMA  
Exact and approximation algorithms for weighted matroid intersection

MI2014-13 Yusuke SHIMIZU  
Moment convergence of regularized least-squares estimator for linear regression model

MI2015-1 Hidetoshi MATSUI & Yuta UMEZU  
Sparse regularization for multivariate linear models for functional data

MI2015-2 Reika AOYAMA & Yoshiyuki KAGEI  
Spectral properties of the semigroup for the linearized compressible Navier-Stokes equation around a parallel flow in a cylindrical domain

MI2015-3 Naoyuki KAMIYAMA  
Stable Matchings with Ties, Master Preference Lists, and Matroid Constraints

MI2015-4 Reika AOYAMA & Yoshiyuki KAGEI  
Large time behavior of solutions to the compressible Navier-Stokes equations around a parallel flow in a cylindrical domain

MI2015-5 Kazuyuki TSUDA  
Existence and stability of time periodic solution to the compressible Navier-Stokes-Korteweg system on \( \mathbb{R}^3 \)

MI2015-6 Naoyuki KAMIYAMA  
Popular Matchings with Ties and Matroid Constraints
MI2015-7 Shoichi EGUCHI & Hiroki MASUDA
Quasi-Bayesian model comparison for LAQ models

MI2015-8 Yoshiyuki KAGEI & Ryouta OOMACHI
Stability of time periodic solution of the Navier-Stokes equation on the half-space under oscillatory moving boundary condition

MI2016-1 Momonari KUDO
Analysis of an algorithm to compute the cohomology groups of coherent sheaves and its applications

MI2016-2 Yoshiyuki KAGEI & Masatoshi OKITA
Asymptotic profiles for the compressible Navier-Stokes equations on the whole space

MI2016-3 Shota ENOMOTO & Yoshiyuki KAGEI
Asymptotic behavior of the linearized semigroup at space-periodic stationary solution of the compressible Navier-Stokes equation

MI2016-4 Hiroki MASUDA
Non-Gaussian quasi-likelihood estimation of locally stable SDE

MI2016-5 Yoshiyuki KAGEI & Takaaki NISHIDA
On Chorin’s method for stationary solutions of the Oberbeck-Boussinesq equation

MI2016-6 Hayato WAKI & Florin NAE
Boundary modeling in model-based calibration for automotive engines via the vertex representation of the convex hulls

MI2016-7 Kazuyuki TSUDA
Time periodic problem for the compressible Navier-Stokes equation on $\mathbb{R}^2$ with antisymmetry

MI2016-8 Abulizi AIHAITI, Shota ENOMOTO & Yoshiyuki KAGEI
Large time behavior of solutions to the compressible Navier-Stokes equations in an infinite layer under slip boundary condition

MI2016-9 Fermín Franco MEDRANO, Yasuhide FUKUMOTO, Clara M. VELTE & Azur HODŽIĆ
Gas entrainment rate coefficient of an ideal momentum atomizing liquid jet

MI2016-10 Naoyuki KAMIYAMA, Akifumi KIRA, Hirokazu ANAI, Hidenao IWANE & Kotaro OHORI
Coalition Structure Generation with Subadditivity Constraints

MI2016-11 Akifumi KIRA, Hidenao IWANE, Hirokazu ANAI, Yutaka KIMURA & Katsuki FU-JISAWA
An indirect search algorithm for disaster restoration with precedence and synchronization constraints

MI2016-12 Shota ENOMOTO
Large time behavior of the solutions around spatially periodic solution to the compressible Navier-Stokes equation
MI2016-13 Naoyuki KAMIYAMA
Popular Matchings with Two-Sided Preference Lists and Matroid Constraints

MI2016-14 Naoyuki KAMIYAMA
An Algorithm for the Evacuation Problem based on Parametric Submodular Function Minimization

MI2016-15 Naoyuki KAMIYAMA
A Note on Submodular Function Minimization with Covering Type Linear Constraints

MI2017-1 Yuki MIYACHI & Yasuhide FUKUMOTO
Gyroscopic Analogy of Coriolis Effect of Rotating Stratified Flows Confined in a Spheroid

MI2017-2 Yoshiyuki KAGEI, Takaaki NISHIDA & Yuka TERAMOTO
On the spectrum for the artificial compressible system

MI2017-3 Naoyuki KAMIYAMA
Pareto Stable Matchings under One-Sided Matroid Constraints

MI2017-4 Naoyuki KAMIYAMA
Submodular Function Minimization with Submodular Set Covering Constraints and Precedence Constraints

MI2017-5 Yuka TERAMOTO
Stability of bifurcating stationary solutions of the artificial compressible system

MI2017-6 Ummu HABIBAH, Hironori NAKAGAWA & Yasuhide FUKUMOTO
Finite-thickness effect on speed of a counter-rotating vortex pair at high Reynolds numbers

MI2017-7 Keigo WADA & Yasuhide FUKUMOTO
$M^2$ expansion for effect of compressibility on Darrieus-Landau instability of a premixed flame

MI2017-8 Liangbing JIN, Thi Thai LE & Yasuhide FUKUMOTO
Frictional effect on stability of discontinuity surface of tangential velocity in shallow water

MI2018-1 Ryo TAKADA
Strongly stratified limit for the 3D inviscid Boussinesq equations

MI2018-2 Keigo WADA & Yasuhide FUKUMOTO
Effect of Compressibility on Heat-loss and Darrieus-Landau Instability of a Premixed Flame

MI2018-3 Yusuke ISHIGAKI
Global existence of solutions of the compressible viscoelastic fluid around a parallel flow

MI2018-4 Abulizi AIHAITI & Yoshiyuki KAGEI
Asymptotic behavior of solutions of the compressible Navier-Stokes equations in a cylinder under the slip boundary condition
MI2018-5  Ryo TAKADA
Long time solutions for the 2D inviscid Boussinesq equations with strong stratification

MI2019-1  Mikihiro FUJII
Long time existence and the asymptotic behavior of solutions for the 2D quasi-geostrophic equation with large dispersive forcing

MI2019-2  Yutaro KABATA & Masatomo TAKAHASHI
One-parameter families of Legendre curves and plane line congruences