New viewpoints of mathematics via “rigorous numerics”

Kaname MATSUE

Degree: Ph.D.(Science)(Kyoto University)
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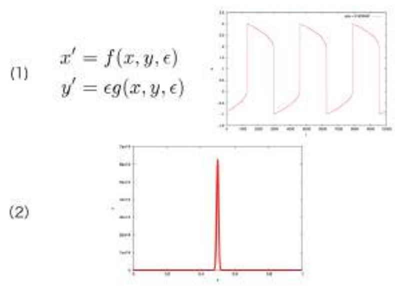
Mathematics is applied to various fields for explaining events such as natural phenomena, there are still many problems which abstract mathematical arguments cannot be applied directly.

For example, “singular perturbation method” for differential equation (1) provides the existence of solutions for “sufficiently small parameter ε”. However, this mathematics does not tell us how small or large such ε can be treated. On the other hand, we can treat this problems by numerical simulations with concrete ε. However, we cannot get any information if such ε is included in a category of “singular perturbations”. In particular, there is an unavoidable gap between mathematical results and numerical ones.

There is also a case that the special structure of numerical solutions cannot be validated, like blow-up solutions of differential equations. We cannot treat “infinity” rigorously on computers, and hence we often regard large solutions in a suitable situation as blow-up solutions (Figure 1). However, are they truly blow-up solutions ? Ordinary numerics can never answer the question. Since such special structures are assumed in advance in mathematical analysis, then arguments with unclear assumptions may cause wrong results.

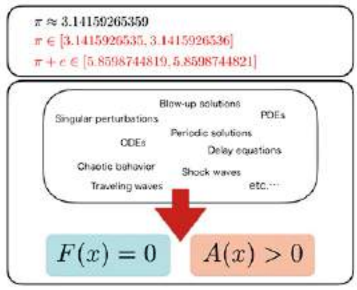
Figure 1.



(1) A differential equation with singular perturbation structure, and a numerical example for some system with ε=0.005. Is the valueε=0.005 “sufficiently small” so that singular perturbation theory can be applied ?
(2) A numerical solution of a differential equation. We can see that this solution “blows up” at a finite time, but is this solution truly a “blow-up solution” ?

“Validated numerics”, which bridge the above unavoidable gaps between mathematical analysis and numerics, are recently well-studied. Validated numerics are one of applications of “interval arithmetic”, which regards intervals as fundamental units instead of numbers, and do all operations among intervals. All numerical errors (such as truncation and rounding errors) can be included in intervals, and hence “rigorous numerics” can be realized on computers. Studies of validated numerics have begun with linear algebra or solvers of nonlinear equations, and nowadays there are a lot of applications to differential equations, dynamical systems and so on. The key idea of applications is reduction of problems so that rigorous numerics can be actually applied within suitable processes. Many of problems are reduced to zero-finding problems or inequality arguments (Figure 2).

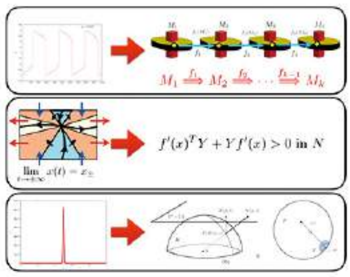
Figure 2. Ideas of validated numerics.



Top : represent all “numbers” by intervals, and do all operations among the category of intervals. “Rigorous numerics” can be applied on computers in the sense that the true value is always contained in intervals. Bottom : Reduce all problems to several simple ones so that rigorous numerics can be applied. Many of them are reduced to zero-finding problem (left), or inequalities (right).

Recently, I study computer assisted proofs of systems with singularities, such as singular perturbation problems, blow-up solutions and shock waves containing discontinuities. These objects possess completely different features and difficult issues from both mathematical and numerical viewpoints. I have succeeded the reduction of existence problems for these objects to topological equations called covering relations and several inequalities by mathematical theories such as geometric singular perturbation theory, compactification and desingularizations, and reduction of systems. Then “rigorous numerics” of solutions of (1) with explicit range of ε, blow-up solutions and discontinuous solutions can be achieved.

Figure 3.
Rigorous numerics to dynamical systems.



Top : Represent trajectories by “covering relations”: graphs involving zero-finding problems. Middle : Reduce “asymptotic behavior” to positive definiteness of a matrix. Bottom : Compactification embedding the phase space into a compact manifold. “The infinity” corresponds to boundary of the manifold. Using these tools, we can validate singular perturbed solutions or blow-up solutions on computers.

Rigorous numerics has a potential which extends applicability of mathematics to numerics in various concrete problems extensively. On the other hand, the methodology relies on “reductions of problems” for practical computations, and we have to study the “root” of targets deeply so that we can make the reduction in a natural way.

I thus believe that rigorous numerics is the field which requires not only numerical computations but also a lot of mathematical considerations, and which will develop new mathematical issues through numerics.