Global Analysis of Variational Problems for Surfaces

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Research Interests: Differential Geometry

Geometric variational problems are a fundamental subject in differential geometry. I am working on mainly variational problems for hypersurfaces in Riemannian manifolds with constant curvature. In general, a solution of a variational problem is said to be stable if the second variation of the energy is non-negative; in particular, a solution that attains a minimum energy is stable. Therefore, it is important to study the stability of solutions from both mathematical and physical points of view. My main interests are the existence, uniqueness, stability, and global properties of these solutions.

My recent subjects of study are surfaces with constant mean curvature (CMC surfaces) and surfaces with constant anisotropic mean curvature (CAMC surfaces) (Fig. 1). The former are critical points of area (that is, isotropic surface energy) for volume-preserving variations, while the latter are critical points of anisotropic surface energy for such variations. Thus, CMC surfaces serve as mathematical models of thin liquid bubbles, and CAMC surfaces serve as mathematical models of, for example, certain kinds of small liquid crystals. Usually, CMC surfaces are regarded as a special case of CAMC surfaces.

CMC surfaces are a classical subject and are still very actively studied. In addition, CAMC surfaces are now studied in many research areas of not only mathematics but also other fields, e.g., physics and technology, as both basic research and applied science. There had not been much geometric research of CAMC surfaces until recently, but a series of joint studies by Professor Bennett Palmer (Idaho State University, U.S.A.) and myself has produced a new development in this field. We have obtained many important results about geometric properties, representation formulas, the Gauss map (the unit normal) and its removable set of CAMC surfaces, existence and uniqueness for stable solutions of free boundary problems for anisotropic surface energies, etc.

Recently we proved that any CAMC surface that is a topological sphere is a rescaling of the Wulff shape. We hope that this result can contribute toward determining the shape of materials that have anisotropic surface energies.

At present, my greatest interest is constructing a bifurcation theory of solutions to geometric variational problems with constraint. It is especially interesting to study bifurcations from stable solutions with symmetry to unstable solutions with the same symmetry and stable solutions with lower symmetry (Fig. 2), which may be important from both mathematical and physical points of view. I would like to emphasize that the problem of stability for variational problems with constraint is much more complicated than that for variational problems without constraint, and variational problems with constraint appear naturally in various situations. In fact, CMC and CAMC surfaces are such examples. I hope that this research will not only contribute to the development of mathematics, but also give mathematical assurance of various physical phenomena.

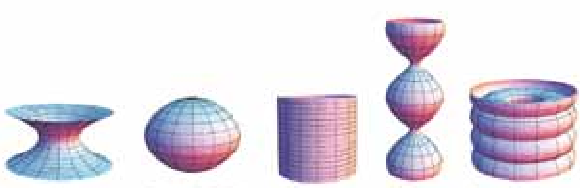

Fig. 1: CAMC surfaces of revolution, which serve as mathematical models of certain liquid crystals.

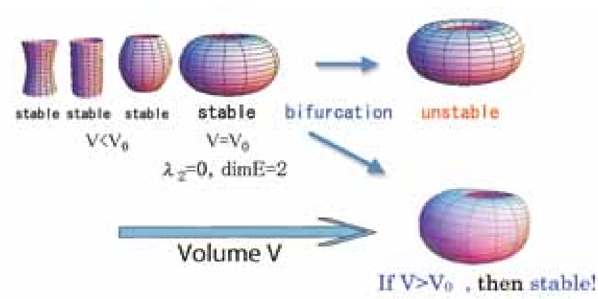

Fig. 2: Equilibrium surfaces of area with volume constraint spanned by two coaxial circles with the same radius. Stable solutions can have lower symmetry than unstable solutions.