Representation Theory and Number Theory, and their Applications to Statistics and Engineering

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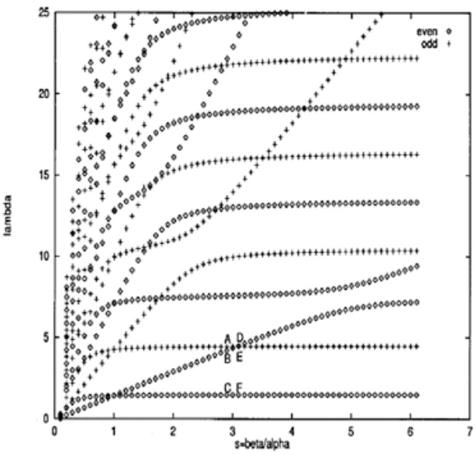
Research Interests: Representation Theory and Modular Forms

My research to date has included various problems in number theory and zeta functions based on representation theory, and their applications to mathematical physics. For a long time, mathematics has been considered a language to describe natural sciences. Similarly, representation theory is a language to describe symmetry and, as with the whole body of mathematics, has a wide range of applications. It has been extremely useful in diverse areas of mathematics and theoretical physics. For instance, if symmetry is considered, certain cases that are seemingly complex can be reduced to easy-to-handle problems. In these cases, representation theory is a powerful tool. There is also research on representation theory for its own sake, as with other fields of mathematics. Below are descriptions of my recent research and projects now in progress.

[Spectrum of Non-Commutative Harmonic Oscillators]



From:Numer. Funct. Anal. Optim. 23, (2002), K. Nagatou, M. T. Nakao, M. Wakayama

In recent years, the spectra of self-adjoint operators with non-commutative coefficients have been investigated intensively. Such spectra are studied not only in the context of mathematics but also experimental physics. In such research, the Dirac operator is of great historical importance, while the Rabi model and the Jaynes-Cumming model have been used extensively in the study of cavity QED. The non-commutative harmonic oscillator (NcHO), which Alberto Parmeggiani (University of Bologna) and I introduced approximately fifteen years ago, constitutes a system of ordinary differential equations with two kinds of non-commutativity, inherited from both the matrix structure and the canonical commutation relations. In special cases, such a system defines a pair of quantum harmonic oscillators. Although a physical system described by the NcHO has yet to be discovered, it has been an important problem to determine the structure of the ground state. Moreover, since the eigenvalues of NcHOs build a continuous curve (when the ratio of the two defining parameters is varied), it comes as an important problem to analyze the behavior of eigenvalue curves, in particular, the characterization of crossing / avoided crossing of these curves.

In general, we do not know if there exist creation and annihilation operators of NcHOs, and this is difficult to predict in any given case. Also, the actual determination of the spectrum is quite difficult. Since its introduction, however, it has become apparent that the problem of determining the spectra of NcHO is a deep problem, with broad implications, as it is related to, for instance, the monodromy problem of Heun's differential equations, and the moduli of elliptic curves and automorphic forms (through the spectral zeta function) quite explicitly. In particular, we have found recently that a nicer description of special values of the spectral zeta function can be given in terms of "residual modular forms," which represent a generalization of Eichler integrals (and Abelian integrals), and certain (newly-defined) cohomology groups.

[Mathematical modeling for digital image expressions]

This is a part of the research project "Mathematics for Expressive Image Synthesis," led by Ken Anjyo (OLM Digital, Inc.), which is one of the CREST research projects in the JST (Japan Science and Technology Agency) mathematics program. We are presently investigating this topic with Hiroyuki Ochiai and Yoshihiro Mizoguchi of this institute. My particular interest here is to obtain a detailed description that could provide a suitable mathematical framework for expressive image synthesis from group theoretical and functional analysis points of view (for instance, the theory of spherical harmonics) and to study discrete models using the spectral analysis of graph Laplacians.

[Representation theory and invariant theory for α-determinants]

Probability theory and statistics have required the notion of the α-determinant to be introduced (D. Vere-Jones, 1988). An α-determinant interpolates a determinant (α=-1) and permanent (α=1) of a matrix. I started studying its representation theory several years ago and have been working on modules of a general linear group GL_n generated by its (complex) powers in collaboration with Kazufumi Kimoto (University of the Ryukyus) and Sho Matsumoto (Nagoya University). When the power is 1, we developed a theory that interpolates the skew-symmetric tensor and symmetric tensor representations at the representation level. This study also has the potential for a new perspective towards a theory of special functions to be proposed. With the discovery of a new relative invariant called the wreath-determinant, we are continuing our research from the invariant and combinatorial theoretic viewpoints. The relationship with zonal polynomials, which are also important in statistics, has been discovered. To date, the α-determinant has multiple connections to statistics as well as a strong connection with probability theory discovered through research on positivity.

[Harmonic analysis on symmetric spaces and its application to statistics]

In recent years, the information geometric approach to optimization theory has become more widely used in various types of research. Typical examples of differential manifolds, which provide the foundation for the formulation of various problems, include symmetric cones formed by positive-definite symmetric matrices. These manifolds can be treated group-theoretically using Lie algebra and Jordan algebra. There is indeed a rich history of research on symmetric spaces, and through this research, many important mathematical techniques have been developed. Recently, some statisticians have also started to conduct research from such a group-theoretical point of view. My current objectives are to develop special functions, including Meixner-Pollaczek polynomials associated with Hermitian symmetric spaces (work with Jacques Faraut of Univ. Paris VI) and their application to stochastic processes, and to investigate the number-theoretical and representation-theoretical structures of statistical / information manifolds, such as the group theoretical treatment of the geodesics for multivariate normal models and, more generally, elliptic models.