Mathematical quantum field theory and related topics
Editors: Asao ARAI, Izumi OJIMA, Fumio HIROSHIMA
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About MI Lecture Note Series

The Math-for-Industry (MI) Lecture Note Series is the successor to the COE Lecture Notes, which were published for the 21st COE Program “Development of Dynamic Mathematics with High Functionality,” sponsored by Japan’s Ministry of Education, Culture, Sports, Science and Technology (MEXT) from 2003 to 2007. The MI Lecture Note Series has published the notes of lectures organized under the following two programs: “Training Program for Ph.D. and New Master’s Degree in Mathematics as Required by Industry,” adopted as a Support Program for Improving Graduate School Education by MEXT from 2007 to 2009; and “Education-and-Research Hub for Mathematics-for-Industry,” adopted as a Global COE Program by MEXT from 2008 to 2012.

In accordance with the establishment of the Institute of Mathematics for Industry (IMI) in April 2011 and the authorization of IMI’s Joint Research Center for Advanced and Fundamental Mathematics-for-Industry as a MEXT Joint Usage / Research Center in April 2013, hereafter the MI Lecture Notes Series will publish lecture notes and proceedings by worldwide researchers of MI to contribute to the development of MI.

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Director
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Mathematical quantum field theory and related topics

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Preface

The international conference “Mathematical Quantum Field Theory and Related Topics” was held in the Institute of Mathematics for Industry (IMI) of Kyushu University from June 6, 2016 through June 8, 2016. The conference was organized to discuss recent progresses and problems in mathematical physics, experiment physics, theoretical physics, operator algebra, operator theory and related topics. We also introduced in this conference a lecture recording system but unfortunately speakers who accepted to record his lecture were few.

The conference itself was very hot like as the temperature of the summer season of Fukuoka and we had 14 talks and more than 40 participants who were very keen to join the discussions. Participants came from France, Italy, Germany, Denmark and Japan, and their affiliations were from universities and industries. The contents of the talks were mainly focused on quantum theory and quantum experiments but were widely spread as mentioned above. We believe that these talks will help readers to discover interesting problems and find ideas to solve his/her problems to be considered.

We are very much grateful to IMI of Kyushu university for sponsoring this conference, and we also thank Grant-in-Aid for Science Research (B) 16H03942, JSPS for financial support. Finally we would like to thank not only speakers but also all attendees. We hope all participants enjoy this exciting event in Fukuoka and wish that Nobel laureates will appear from participants in this conference.

Program Co-Chairs:
Asao Arai
Izumi Ojima
Fumio Hiroshima
Mathematical quantum field theory and related topics

Date: 6/June/2016 - 8/June/2016

Place: IMI Kyushu University

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Mathematical quantum field theory
and related topics

Date: 6 June/2016 - 8 June/2016
Place: West 1 building D-413, IMI Kyushu University, Fukuoka, Japan
http://www.imi.kyushu-u.ac.jp/pages/joint_research_auditorium.html

Monday 6 June

10:00-10:50  Tadahiro Miyao (Hokkaido)
  Long-range charge order in the two-dimensional ionic Hubbard model

11:00-11:50  Asao Arai (Hokkaido)
  Inequivalence of quantum Dirac fields of different masses and a general structure behind it

13:30-14:20  Daniel Braak (Augsburg)
  Integrable and non-integrable models in quantum optics

14:30-15:20  Motoichi Ohtsu (Tokyo, emeritus)
  Dressed Photons
  ---Concepts of off-shell photon and applications to light-matter fusion technology---

16:00-16:50  Masato Wakayama (Kyushu IMI)
  Representation theoretic approach to the spectrum of quantum Rabi or its generalized models

17:00-17:50  Marco Falconi (Rome)
  Bohr's correspondence principle in the Nelson model

Tuesday 7 June

10:00-10:50  Tomohiro Kanda (Kyushu)
  A KMS state on the resolvent CCR algebra

11:00-11:50  Fumio Hiroshima (Kyushu)
  Semi-relativistic QED

13:30-14:20  Oliver Matte (Aarhus)
  Differentiability properties of stochastic flows in non-relativistic QED

14:30-15:20  Zied Ammari (Rennes)
  On the relationship between non-linear Schroedinger dynamics,
  Gross-Pitaevskii hierarchy and Liouville's equation
16:00-16:50 Nobuhiro Asai (Aichi)
The radial Bargmann measure for the Fock space of type B

**Wednesday 8 June**

10:00-10:50 Kazuya Okamura (Nagoya)
On a mathematical treatment of measurement correlations

11:00-11:50 Masao Hirokawa (Hiroshima)
How is the ground state of the quantum Rabi model dressed with a real photon?

13:00-14:00 Akira Sakai (Hokkaido)
Self-avoiding walk on random conductors

14:00-15:00 Hiroshi Ando (Chiba)
Descriptive analysis of self-adjoint operators and the Weyl-von Neumann equivalence relation

15:00-16:00 Masao Hirokawa (Hiroshima)
Introduction of a mathematical approach to sum-frequency generation in non-linear optics

16:00-17:45 Tomihiro Hashizume (Hitachi/Tokyo Inst. Tech.) IMI-colloquium
Scanning non-linear optical probe microscopy utilizing tip-enhanced near field optics
# Mathematical quantum field theory and related topics

6.6[MON]-6.8[WED]2016

Kyushu University, Japan

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Abstract

We rigorously investigated the charge-charge correlation function of the ionic Hubbard model in two dimensions by reflection positivity. We prove the existence of charge-density waves for large staggered potential $\Delta$ (i.e., $\frac{\Delta}{2} + 2V > U$) at low temperatures, where $U$ and $V$ are the on-site and nearest-neighbor Coulomb repulsions, respectively. The results are consistent with previous numerical simulation results. We argue that the absence of charge-density waves for $\Delta = 0$ and $U$ are large enough (i.e., $U > \frac{\Delta}{2} + 2V$).

1. Background

The (extended) ionic Hubbard model is

$$H = (-t) \sum_{\langle i, j \rangle \sigma = \uparrow, \downarrow} (c_{i\sigma}^* c_{j\sigma} + c_{j\sigma}^* c_{i\sigma}) + U \sum_{j \in \Lambda} (n_j - \frac{1}{2})^2 + V \sum_{\langle i, j \rangle} (n_i - \frac{1}{2})(n_j - \frac{1}{2}) + \frac{\Delta}{2} \sum_{j \in \Lambda} (-1)^{|j|}(n_j - \frac{1}{2}).$$

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1. Background
   - The ionic Hubbard model
   - 1D system
2. Main result
   - Existence of the CDW
3. Method
   - Reflection positivity
4. Outline of proof of the main theorem
   - The Peierls argument
   - A modified chessboard estimate
   - The exponential localization of eigenvectors
\[ \Lambda = [-L, L]^2 \cap \mathbb{Z}^2 \text{ with } L \in \mathbb{N}. \]

- Hilbert space: the fermion Fock space

\[ \mathcal{H} = \mathcal{F}(\mathbb{C}^2(\Lambda)) \oplus \mathbb{C}, \]

where \( \mathcal{F}(\mathcal{H}) = \bigoplus_{n \geq 0} \wedge^n \mathcal{H}. \) Here, \( \wedge^n \mathcal{H} \) is the \( n \)-fold antisymmetric tensor product of \( \mathcal{H} \) with \( \wedge^0 \mathcal{H} = \mathbb{C}. \)

- \( c_{\sigma}(c_{\sigma}^*) : \) the fermion annihilation(creation) operator on site \( j = (j_1, j_2) \in \Lambda \) with spin \( \sigma. \)

- The number operator \( n_j \) is defined by

\[ n_j = n_{j1} + n_{j1} \]

with \( n_{j\sigma} = c_{\sigma}^* c_{\sigma}. \)

- Periodic boundary condition: \( L \equiv -L. \)

- \( U > 0: \) strength of the on-site Coulomb repulsion.

- \( V \geq 0: \) strength of the nearest neighbour Coulomb repulsion.

- \( \Delta \geq 0: \) staggered chemical potential.

- \( t > 0: \) hopping amplitude between nearest neighbor.

### Origin of the ionic Hubbard model

- The ionic Hubbard (IH) model was originally suggested to describe the charge-transfer organic salt [HT, NT].

- The IH model has been applied to analysis of ferroelectric perovskites [FG, RS].


### Expected results in 1D systems (\( V = 0 \))

- The IH model has two quantum critical points as \( U \) is varied for fixed \( \Delta: \) A conjecture by numerical simulation.

- \( U < U_1: \) the system is a band insulator (CDW state).

- \( U > U_2: \) the system is a Mott insulator (antiferromagnetic state).

- \( U_1 < U < U_2: \) the system has a bond-order.

**Remark 1.1**

Let \( b_j = (-1)^j \sum_{\sigma=1}^2 \left( c_{j+1\sigma}^c c_{j\sigma} + c_{j\sigma}^c c_{j+1\sigma} \right). \) The system is a bond-order if

\[ \lim_{j \to \infty} \langle b_j \rangle > 0. \]
The 2D ionic Hubbard model

Theoretical studies:

- Existence of the CDW is expected for $\frac{\Delta^2}{4} + 2V > U$.
- Existence of the AF order is expected for $U > \frac{\Delta^2}{2} + 2V$.
- Existence of the bond order is unclear.

Experimental studies [Me]:

- Recent experimental techniques make it possible to realize the ionic Hubbard model in an optical honeycomb lattice.
- For large $\Delta$, the charge density wave is reported.
- For large $U$, a strong suppression of charge density wave is detected.


Open problem:

Prove the phase diagram.

In this talk, I will focus on the existence of the CDW.
**Theorem 2.2**

Assume that $2V - U + \frac{\Delta}{2} > 0$. For sufficiently large $\beta$ and small $t$, we have

$$\lim_{|j| \to \infty} (-1)^{|j|} \langle q_{0j}q_j \rangle_{\beta,H} > 0,$$

where $|j| = |j_1| + |j_2|$ for each $j = (j_1, j_2) \in \Lambda$. Thus, there exists a long-range charge order (CDW).

**Remark 2.3**

If $V = 0$, then our result is consistent with phase diagrams suggested by numerical simulation [1,2,4]. On the other hand, if $\Delta = 0$, then Theorem 2.2 agrees with results expected by numerical simulation as well [3].

**Remark 2.4**

In three or more dimensions, we can prove the existence of long-range charge order (CDW) by the method established by Dyson, Lieb and Simon.


**Remark 2.5**

Assume $\Delta = 0$. If $2V - U < 0$, then we already know the following:

(i) For all $\beta$ and $t$,

$$\lim_{|j| \to \infty} (-1)^{|j|} \langle q_{0j}q_j \rangle_{\beta,H} = 0.$$

Thus, there is no long-range charge order [5, 7].

(ii) If $L$ finite, the ground state of $H$ is unique and antiferromagnetic [6].


### 3. Method

**Reflection positivity**

Cf. Quantum Pirogov-Sinai theory
Reflection positivity


Achievements:

- Justification of (a part of) the phase diagram expected by numerical simulation.
- The first application of the Fröhlich-Lieb method to the ionic Hubbard model.
- An extension of the chessboard estimate.

4. Outline of Proof

Let $\Lambda_{\text{e}} = \{ j \in \Lambda \mid j \text{ is even} \}$ and let $\Lambda_{\text{o}} = \{ j \in \Lambda \mid j \text{ is odd} \}$.

We can construct a unitary operator $V$ such that

$$Vc_jV^{-1} = \begin{cases} c_j^\sigma & \text{if } j \in \Lambda_{\text{e}}, \\ c_j^\sigma & \text{if } j \in \Lambda_{\text{o}}, \end{cases} \quad Vq_jV^{-1} = (-1)^{|j|}q_j.$$ 

Lemma 1:

Let $\hat{H} = VH^\dagger V^{-1}$. We have $\hat{H} = T + W$, where

$$T = \sum_{(i,j), \sigma} (-t) (c_i^\sigma c_j^\sigma + c_j^\sigma c_i^\sigma),$$

$$W = U \sum_{j \in \Lambda} q_j^2 - V \sum_{(i,j)} q_i q_j + \Delta \sum_{j \in \Lambda} q_j.$$ 

By Lemma 1, we know that

$$(-1)^{|j|} \langle q_i q_j \rangle_{\beta, \Lambda, \hat{H}} = \langle q_i q_j \rangle_{\Lambda}.$$ 

where $\langle \cdot \rangle_{\beta, \Lambda, \hat{H}}$ is abbreviated as $\langle \cdot \rangle_{\Lambda}$. Thus,

Theorem 4.1

Theorem 2.2 is equivalent to

$$\lim_{|j| \to \infty} \langle q_i q_j \rangle > 0$$

for sufficiently large $\beta$ and small $t$, where $\langle \cdot \rangle = \lim_{L \to \infty} \langle \cdot \rangle_{\Lambda}$.
Let $E_j(\cdot)$ be the spectral measure of $q_j$. We set

$E_j^{(0)} = E_j(\{0\}), \quad E_j^{(\pm)} = E_j(\{0, \pm 1\}), \quad E_j^{(-)} = E_j(\{-1\}).$

**Theorem 4.2**

For all $j \in \Lambda$, $\beta > 0$ and $\Lambda \subset \mathbb{Z}^2$, we have

$$\langle \eta \nu q_j \rangle_{\Lambda} \geq 1 - 3 \left\langle \sum_{a} F^{(a)}_\alpha \right\rangle_{\Lambda} - 2 \left\langle \sum_{a} F^{(a)}_\alpha F^{(a)}_\beta \right\rangle_{\Lambda} - 2 \left\langle \sum_{a} F^{(a)}_\beta F^{(a)}_\alpha \right\rangle_{\Lambda}.$$  

**Proof of Theorem 4.3 (A)**

Proof of Theorem 4.3 (A) consists of the following steps:

- **Step A-1:** A key inequality.
- **Step A-2:** The modified chessboard estimate.
- **Step A-3:** Estimates of $R^{\pm}_{\text{Low}}$ and $R^{\pm}_{\text{High}}$.
- **Step A-4:** The exponential localization of eigenvectors.
- **Step A-5:** Putting it all together.

Thus, to prove (1), it suffices to show the following:

**Theorem 4.3**

For arbitrary $\epsilon > 0$, there exist $\Lambda_0 \subset \mathbb{Z}^2$, $\beta_0 > 0$ and $t_0 \in (0,1)$ such that if $\Lambda \supset \Lambda_0$, $\beta > \beta_0$ and $0 < t < t_0$, then it holds that

(A) $\left\langle \sum_{a} F^{(a)}_\alpha F^{(a)}_\beta \right\rangle_{\Lambda} \leq \epsilon$,  

(B) $\left\langle \sum_{a} F^{(a)}_\alpha \right\rangle_{\Lambda} \leq \epsilon$.

I will explain a strategy of proof of Theorem 4.3 (A).

---

**Step A-1: A key inequality**

**Definition 4.4**

We regard $\Lambda$ as a 2-dimensional torus.

- The set of all connected sets$^1$ in $\Lambda$ is denoted by $S_\Lambda$:
  
  $S_\Lambda = \{ \gamma \subseteq \Lambda \mid \gamma: \text{connected} \}$.

- By a contour, we mean the set $\partial \gamma$ of nearest neighbour pairs associated with boundary of a set $\gamma \in S_\Lambda$ such that

  $\partial \gamma = \{ (i_1, j_1), \ldots, (i_n, j_n) \mid i_k \in \gamma, j_k \notin \gamma \}$.

---

$^1$We say that a subset $\gamma$ of $\Lambda$ is connected if any of its sites are linked by a path in $\gamma$.  

Theorem 4.5

We have

\[ \langle \mathcal{P}^{(+)\, P}^{(-)} \rangle_{\Lambda} \leq \sum_{m,n \in \mathbb{Z}} \left( \prod_{(i,j) \in \partial \gamma} \mathcal{P}_{i}^{(+)\, P} \mathcal{P}_{j}^{(-)} \right)_{\Lambda}. \]

Step A-2: The modified chessboard estimate

Recall that \( L = 2M + 1 \). We define projections \( \mathcal{P}^{(+)}_{\Lambda} \) and \( \mathcal{P}^{(-)}_{\Lambda} \) by

\[
\mathcal{P}^{(+)\, P}_{\Lambda} = \prod_{m=1}^{M} \prod_{n=-L}^{L-1} \begin{bmatrix} p_{m,n}^{(+)\, P} \backslash \backslash p_{m,n}^{(-)\, P} \end{bmatrix} \prod_{i=0}^{\infty} \mathcal{P}_{i}^{(+)\, P} \mathcal{P}_{i}^{(-)}. 
\]

\[
\mathcal{P}^{(-)\, P}_{\Lambda} = \prod_{m=1}^{M} \prod_{n=-L}^{L-1} \begin{bmatrix} p_{m,n}^{(-)\, P} \backslash \backslash p_{m,n}^{(+)\, P} \end{bmatrix} \prod_{i=0}^{\infty} \mathcal{P}_{i}^{(-)\, P} \mathcal{P}_{i}^{(+)}. 
\]

where

\[
\partial \mathcal{P}^{(\omega)} = \prod_{i=-L}^{L-1} \begin{bmatrix} p_{i}^{(\omega)} \backslash \backslash p_{i}^{(\omega)} \end{bmatrix}, \quad \omega = +, -.
\]
**Theorem 4.6**

Let \( P_\Lambda = \max \{ \langle P_\Lambda^{(+)} \rangle_\Lambda, \langle P_\Lambda^{(-)} \rangle_\Lambda \} \). We have

\[
\left\langle \prod_{(i,j) \in \partial \gamma} P_i^{(+)} P_j^{(-)} \right\rangle_\Lambda \leq P_{\partial \gamma} / 2 |\Lambda|.
\]

**Corollary 4.7 (The Peierls argument)**

There exists a \( C > 0 \) such that

\[
\langle P_m^{(+)} P_n^{(-)} \rangle_\Lambda \leq C \sum_{\epsilon=4}^{\infty} \epsilon^3 |\Lambda|^{1/2}.
\]

To prove Theorem 4.6, we need

- A modified chessboard estimate;
- Reflection positivity.

---

**Step A-3: Estimates of \( R_{\text{Low}}^{\pm} \) and \( R_{\text{High}}^{\pm} \)**

To prove Theorem 4.3 (A), it suffices to show that RHS of (2) is bounded by \( \epsilon \).

Let \( \hat{H} \) be the spectral measure of \( \hat{H} \). For each \( \delta > 0 \), we set

\[
E_\delta = E_\hat{H}(\mathbb{E}[\mathbb{E} + \delta |\Lambda|]), \quad E_\delta^+ = 1 - E_\delta,
\]

where \( \mathbb{E} = \min \text{spec}(\hat{H}) \), the ground state energy. \( \delta \) will be chosen later.

- We divide \( \langle P_\Lambda^{(\pm)} \rangle_\Lambda \) into two pieces:

\[
\langle P_\Lambda^{(\pm)} \rangle_\Lambda = R_{\text{Low}}^{(\pm)} + R_{\text{High}}^{(\pm)}.
\]

where

\[
R_{\text{Low}}^{(\pm)} = \left\langle E_\delta P_\Lambda^{(\pm)} \right\rangle_\Lambda, \quad R_{\text{High}}^{(\pm)} = \left\langle E_\delta P_\Lambda^{(\pm)} \right\rangle_\Lambda.
\]

---

**Theorem 4.8**

We have the following:

(i) \( |R_{\text{Low}}^{(\omega)}| \leq 2 |\Lambda| \left( \text{Tr}_\omega [P_\Lambda^0 E_\delta] \right)^1/2 \) for each \( \omega = +, - \).
(ii) \( |R_{\text{High}}^{(\omega)}| \leq 4 |\Lambda| e^{-\delta |\Lambda|} \) for each \( \omega = +, - \).
Step A-4: The exponential localization of eigenvectors

Theorem 4.9

We choose \( \delta = \beta - \xi \) with \( \xi \in (0, 1) \). If \( \frac{\Delta}{2} + 2V - U > 0 \), then we have

\[
\text{Tr}_B \left[ \mathbf{P}_\Delta^{(\pm)} E_\delta \right] \leq 4^{\left| \Lambda \right|} \gamma^4,
\]

where \( \gamma \) and \( d \) satisfy

\[
\gamma = C_1 t + O(\beta^{-t}), \quad d = C_2 |\Lambda| + O(|\Lambda|^{1/2})
\]

with \( C_1, C_2 > 0 \).

Step A-5: Putting it all together

By Theorems 4.8 and 4.9, we have

\[
\left\langle \mathbf{P}_\Delta^{(\pm)} \right\rangle \leq 4^{\left| \Lambda \right|} \left( e^{-D|\Lambda|} + e^{-\beta D|\Lambda|} \right),
\]

where

\[
A = \text{Const.} \log t^{-1}.
\]

Let \( D = \min \{ A, \beta \delta \} \). Recall that we have chosen \( \delta = \beta - \xi \).

We obtain

\[
\mathcal{P}_\Lambda \leq 2 \cdot 4^{\left| \Lambda \right|} e^{-D|\Lambda|}.
\]

Hence, by Corollary 4.7, we have

\[
\left\langle \mathbf{P}_\Lambda^{(\pm)} \right\rangle \leq C \sum_{\ell=1}^\infty \ell^2 (24)^\ell \exp \left\{ -\frac{D}{2} \ell t \right\}.
\]

Thus, by choosing \( \beta \) sufficiently large and \( t \) sufficiently small, we obtain Theorem 4.3 (A).

Open problems

- Existence of AF orders.
- Prove or disprove existence of bond-orders.

Summary

- We prove existence of the CDW by reflection positivity in the two-dimensional ionic Hubbard model.
- We extend the chessboard estimate.
- We argue the absence of the CDW for large \( U \).
- We argue that the unique ground state is antiferromagnetic if \( U \) is large enough.

These results agree with the phase diagram obtained by numerical simulation.
The (standard) chessboard estimate

Let \( \mathfrak{A} \) be a vector space with anilineral involution \( J \). Let \( \omega \) be a multilinear functional on \( \mathfrak{A}^{2L} \). Assume the following:

(i) \( \omega(A_1, \ldots, A_{2L}) = \omega(A_{2L}, \ldots, A_1) \).

(ii) \( \omega(A_1, \ldots, A_L, JA_L, \ldots, JA_1) \geq 0 \).

(iii) 
\[
|\omega(A_1, \ldots, A_{2L})| \leq \omega(A_1, \ldots, A_L, JA_L, \ldots, JA_1)^{1/2} \\
\times \omega(JA_{2L}, \ldots, JA_{L+1}, A_{L+1}, \ldots, A_{2L})^{1/2}.
\]

Then we have
\[
|\omega(A_1, \ldots, A_{2L})| \leq 2^{2L} \prod_{j=1}^{2L} \omega(JA_j, A_j, JA_j, A_j)^{1/2L}.
\]

A modified chessboard estimate

Let \( \mathfrak{A} \) be a vector space with anilineral involution \( J \). Let \( \omega \) be a multilinear functional on \( \mathfrak{A}^{2L+1} \) with \( M \) even. Assume the following:

(i) \( \omega(A_1, \ldots, A_{2M+1}) = \omega(A_2, \ldots, A_{2M+1}, A_1) \).

(ii) There exist real linear maps \( T_\alpha \) and \( T_\beta \) from \( \mathfrak{A} \) to \( \mathfrak{A} \) such that

\[ T_\alpha(T_\beta(A)) = T_\beta(A), \quad \alpha, \beta = +, - \]

(a) For each \( A \) in \( \mathfrak{A} \),

\[ T_\alpha(T_\beta(A)) = T_\beta(A), \quad \alpha, \beta = +, - \]

(b) \( \omega(A_1, \ldots, A_M, T_\alpha(A_{M+1}), JA_{M+1}, \ldots, JA_1) \geq 0 \).

(c) 
\[
|\omega(A_1, \ldots, A_{2M+1})| \leq \omega(A_1, \ldots, A_M, T_\alpha(A_{M+1}), JA_{M+1}, \ldots, JA_1)^{1/2} \\
\times \omega(JA_{2M+1}, \ldots, JA_{2M+1}, T_\beta(T_\alpha(A_{M+1})), A_{M+2}, \ldots, A_{2M+1})^{1/2}.
\]

Reflection positivity

- \( X_L, X_R \): complex Hilbert spaces
- \( \vartheta \) be an antiunitary transformation from \( X_L \) onto \( X_R \).
- \( A, B_1, \ldots, B_n \) are linear operators in \( X_L \).
- A is self-adjoint and bounded from below.
- \( B_j \) is bounded.
- Hamiltonian:

\[
H = H_0 - V,
\]

\[
H_0 = A \otimes I + I \otimes \vartheta A^{-1},
\]

\[
V = \sum_{j=1}^{n} (B_j \otimes \vartheta B_j - \vartheta B_j \otimes B_j^{-1}).
\]

\( H \) is a self-adjoint operator bounded from below and acts in \( X_L \otimes X_R \).

\( \vartheta \) is a bijective antilinear map which satisfies \( \langle \vartheta \psi | \vartheta \phi \rangle = (\langle \vartheta \psi | \vartheta \phi \rangle)^* \) for all \( \vartheta, \psi \in X_L \).
Theorem 4.10
Let \( C, D \in \mathcal{B}(X_L) \). We have
(i) \( (C \otimes aC^{-1}) \geq 0 \),
(ii) \( \left| (C \otimes aD^{-1}) \right| \leq (C \otimes aC^{-1})(D \otimes aD^{-1}) \).
Inequivalence of Quantum Dirac Fields of Different Masses and a General Structure Behind It

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1 Introduction

In the canonical formalism of quantum field theory (QFT), a Bose field theory, which is a QFT describing bosons, is based on a representation $\left(\mathcal{F}, \mathcal{D}, \{\phi(f), \pi(f) \mid f \in \mathcal{V}\}\right)$ of canonical commutation relations (CCR), where $\mathcal{F}$ is a complex Hilbert space, $\mathcal{D}$ is a dense subspace of $\mathcal{F}$, $\mathcal{V}$ is a real inner product space, and $\phi(f), \pi(f)$ are self-adjoint operators acting in $\mathcal{F}$ satisfying the following (i) and (ii): (i) (linearity in test vectors $f$) for all $f, g \in \mathcal{V}$ and $a, b \in \mathbb{R}$,

$$\phi(af + bg) = a\phi(f) + b\phi(g), \quad \pi(af + bg) = a\pi(f) + b\pi(g)$$

on $\mathcal{D}$ and (ii) (the CCR over $\mathcal{V}$) for all $f \in \mathcal{V}$,

$$\mathcal{D} \subset D(\phi(f)) \cap D(\pi(f))$$

(for a linear operator $A$, $D(A)$ denotes the domain of $A$), $\phi(f)\mathcal{D} \subset \mathcal{D}$, $\pi(f)\mathcal{D} \subset \mathcal{D}$ and

$$[\phi(f), \pi(g)] = i \langle f, g \rangle_{\mathcal{V}}, \quad [\phi(f), \phi(g)] = 0, \quad [\pi(f), \pi(g)] = 0, \quad f, g \in \mathcal{V},$$

on $\mathcal{D}$, where $[X, Y] := XY - YX$ and $\langle , \rangle_{\mathcal{V}}$ denotes the inner product of $\mathcal{V}$.

Two representations $\left(\mathcal{F}, \mathcal{D}, \{\phi(f), \pi(f) \mid f \in \mathcal{V}\}\right)$ and $\left(\mathcal{F}', \mathcal{D}', \{\phi'(f), \pi'(f) \mid f \in \mathcal{V}\}\right)$ of the CCR over $\mathcal{V}$ are said to be equivalent if there exists a unitary operator $U : \mathcal{F} \to \mathcal{F}'$ such that, for all $f \in \mathcal{V}$,

$$U\phi(f)U^{-1} = \phi'(f), \quad U\pi(f)U^{-1} = \pi'(f)$$

(operator equalities). In this case we write

$$\{\phi(f), \pi(f) \mid f \in \mathcal{V}\} \cong \{\phi'(f), \pi'(f) \mid f \in \mathcal{V}\}.$$

On the other hand, a Fermi field theory, which is a QFT describing fermions, is based on a representation $\left(\mathcal{F}, \{\psi(f), \psi(f)^* \mid f \in \mathcal{H}\}\right)$ of canonical anti-commutation relations (CAR), where $\mathcal{F}$ and $\mathcal{H}$ are complex Hilbert spaces, $\psi(f)$ is a bounded linear operator on $\mathcal{F}$ satisfying the following (i) and (ii) ($\psi(f)^*$ denotes the adjoint of $\psi(f)$): (i) (anti-linearity in test vectors $f$) for all $f, g \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$,

$$\psi(\alpha f + \beta g) = \alpha^* \psi(f) + \beta^* \psi(g),$$

where, for a complex number...
$z \in \mathbb{C}$, $z^*$ denotes the complex conjugate of $z$, and (ii) (the CAR over $\mathcal{H}$):

$$\{\psi(f), \psi(g)^*\} = \langle f, g \rangle_{\mathcal{H}},$$

$$\{\psi(f), \psi(g)\} = 0, \quad f, g \in \mathcal{H},$$

where $\{X, Y\} := XY + YX$ ($\langle f, g \rangle_{\mathcal{H}}$ is anti-linear in $f$ and linear in $g$).

In the construction of a Bose field model, a representation of the CCR over a real inner product space $\mathcal{V}$ is used to define the time-zero field $\phi_\xi(t, f)$ and its canonical conjugate momentum $\pi_\xi(t, f)$ ($f \in \mathcal{V}$), where $\xi$ is a physical parameter contained in the model such as mass, charge and coupling constant, in such a way that $(\mathfrak{F}, \mathcal{D}, \{\phi_\xi(t, f), \pi_\xi(t, f) | f \in \mathcal{V}\})$ is a representation of the CCR over $\mathcal{V}$, where $\mathfrak{F}$ denotes the Hilbert space of state vectors for the model. The time-$t$ fields are given by

$$\phi_\xi(t, f) := e^{itH\xi} \phi_\xi(0) e^{-itH\xi}, \quad \pi_\xi(t, f) := e^{itH\xi} \pi_\xi(0) e^{-itH\xi}, \quad t \in \mathbb{R},$$

where $H\xi$ is the Hamiltonian of the model.\(^1\) Note that, if $e^{itH \mathcal{D}} \subset \mathcal{D}$ for all $t \in \mathbb{R}$, then $(\mathfrak{F}, \mathcal{D}, \{\phi_\xi(t, f), \pi_\xi(t, f) | f \in \mathcal{V}\})$ also is a representation of the CCR over $\mathcal{V}$.

The same scheme applies to the construction of a Fermi field model with $(\mathfrak{F}, \mathcal{D}, \{\phi_\xi(f), \pi_\xi(f) | f \in \mathcal{V}\})$ replaced by a representation of the CAR over a Hilbert space.

The following problem may be interesting to know meaning of the parameter $\xi$ from representation theoretic point of view:

**Problem:** Is $(\phi_\xi(t, f), \pi_\xi(t, f))_{f \in \mathcal{V}}$ equivalent or inequivalent to $(\phi_\xi'(t, f), \pi_\xi'(t, f))_{f \in \mathcal{V}}$ for $\xi \neq \xi'$?

It is easy to see that, for $\xi \neq \xi'$, $(\phi_\xi(f), \pi_\xi(f))_{f \in \mathcal{V}} \cong (\phi_\xi'(f), \pi_\xi'(f))_{f \in \mathcal{V}}$ if and only if $(\phi_\xi(t, f), \pi_\xi(t, f))_{f \in \mathcal{V}} \cong (\phi_\xi'(t, f), \pi_\xi'(t, f))_{f \in \mathcal{V}}$. Hence, as for the above problem, it is sufficient to consider the family $(\phi_\xi(f), \pi_\xi(f))_{f \in \mathcal{V}}$ of representations of the CCR over $\mathcal{V}$.

**Example 1.1** Let $\mathcal{V} = \mathcal{S}_b(\mathbb{R}^3)$, the space of real rapidly decreasing $C^\infty$-functions on $\mathbb{R}^3$ and $(\varphi_m(f), \pi_m(f))_{f \in \mathcal{S}_b(\mathbb{R}^3)}$ be the time zero-fields in the standard free relativistic quantum Hermitian scalar field of mass $m > 0$ acting in the boson Fock space $\mathcal{F}_b(L^2(\mathbb{R}^3))$ over $L^2(\mathbb{R}^3)$ (see, e.g., [8, p.216]). Then $(\mathcal{F}_b(L^2(\mathbb{R}^3)), \mathcal{D}_0, \{\varphi_m(f), \pi_m(f) | f \in \mathcal{S}_b(\mathbb{R}^3)\})$ is an irreducible representation of the CCR over $\mathcal{S}_b(\mathbb{R}^3)$, where $\mathcal{D}_0$ is a dense subspace of $\mathcal{S}_b(L^2(\mathbb{R}^3))$.\(^2\) One has the following theorem:

**Theorem** ([8, Theorem X.46]). Let $m_1 \neq m_2$ ($m_1, m_2 > 0$). Then the representation

$$(\varphi_m(f), \pi_m(f))_{f \in \mathcal{S}_b(\mathbb{R}^3)}$$

is inequivalent to $(\varphi_{m_2}(f), \pi_{m_2}(f))_{f \in \mathcal{S}_b(\mathbb{R}^3)}$, i.e., there exists no unitary operator $U$ such that, for all $f \in \mathcal{S}_b(\mathbb{R}^3)$, $U\varphi_m(f)U^{-1} = \varphi_{m_2}(f)$ and $U\pi_m(f)U^{-1} = \pi_{m_2}(f)$.

\(^1\)We use the physical unit system where the reduced Planck constant (Dirac constant) $\hbar$ and the light speed $c$ are equal to $1$.

\(^2\)In fact, the representation $(\mathcal{F}_b(L^2(\mathbb{R}^3)), \mathcal{D}_0, \{\varphi_m(f), \pi_m(f) | f \in \mathcal{S}_b(\mathbb{R}^3)\})$ is a Weyl representation of the CCR over $\mathcal{S}_b(\mathbb{R}^3)$, a stronger concept of representation of CCR.
This theorem is very interesting in the sense that it clarifies a meaning of boson mass \( m \) from representation theoretic point of view, i.e., the boson masses form an index set for a family of irreducible representations of the CCR over \( \mathcal{F}_R(\mathbb{R}^3) \). It gives a conceptual cognition of boson masses.\(^3\)

There is a general structure behind this theorem in fact. It is shown that there exists a family of irreducible representations of CCR indexed by a family of self-adjoint operators, which, as a special case, yields the family \( \{ (\mathcal{F}_b(L^2(\mathbb{R}^3)), \mathcal{D}_0, \{ \varphi_m(f), \pi_m(f) | f \in \mathcal{F}_R(\mathbb{R}^3) \}) | m > 0 \} \). In this general and abstract level, one can see an essential origin of the inequivalence stated in the above theorem. See [2] for details.

It would be natural to ask if a similar structure exists in the case of a Fermi field model. The answer is affirmative. In this paper, we report some results on this aspect in the case of quantum Dirac fields.

### 2 Fermion Fock Space and Representations of CAR

#### 2.1 Irreducibility and equivalence of representations of CAR

A representation \( (\mathcal{F}, \{ \psi(f), \psi(f)^* | f \in \mathcal{H} \}) \) of the CAR over a complex Hilbert space \( \mathcal{H} \) is said to be irreducible if a closed subspace \( \mathcal{M} \) of \( \mathcal{H} \) such that \( \psi(f)^* \mathcal{M} \subset \mathcal{M} \), \( \forall f \in \mathcal{H} \) is \( \{ 0 \} \) or \( \mathcal{H} \).

Let \( (\mathcal{F}', \{ \psi'(f), \psi'(f)^* | f \in \mathcal{H} \}) \) be another representation of the CAR over \( \mathcal{H} \). Then the two representations \( (\mathcal{F}', \{ \psi'(f), \psi'(f)^* | f \in \mathcal{H} \}) \) and \( (\mathcal{F}, \{ \psi(f), \psi(f)^* | f \in \mathcal{H} \}) \) are said to be equivalent if there exists a unitary operator \( U : \mathcal{F} \to \mathcal{F}' \) such that, \( \forall f \in \mathcal{H}, \psi'(f) = U \psi(f) U^* \).

#### 2.2 Fock representation

Let \( \mathcal{H} \) be a complex Hilbert space. The fermion Fock space \( \mathcal{F}(\mathcal{H}) \) over \( \mathcal{H} \) is defined by

\[
\mathcal{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \wedge^n \mathcal{H},
\]

where \( \wedge^n \mathcal{H} \) is the \( n \)-fold antisymmetric tensor product Hilbert space of \( \mathcal{H} \) with \( \wedge^0 \mathcal{H} := \mathbb{C} \).

For each \( f \in \mathcal{H} \), the fermion annihilation operator \( A(f) \) with test vector \( f \) is defined to be the bounded linear operator on \( \mathcal{F}(\mathcal{H}) \) such that the adjoint \( A(f)^* \) is given in the following form:

\[
(A(f)^* \Psi)(0) = 0, \quad (A(f)^* \Psi)(n) = \sqrt{n} A_n(f \otimes \Psi^{(n-1)}), \quad n \geq 1, \Psi \in \mathcal{F}(\mathcal{H}),
\]

where \( A_n \) denotes the anti-symmetrization operator on the \( n \)-fold tensor product Hilbert space \( \otimes^n \mathcal{H} \) of \( \mathcal{H} \). It is easy to see that \( \{ A(f), A(f)^* | f \in \mathcal{H} \} \) obeys the CAR over \( \mathcal{H} \):

\[
\{ A(f), A(g) \} = 0, \quad \{ A(f), A(g)^* \} = \langle f, g \rangle_{\mathcal{H}}, \quad f, g \in \mathcal{H}.
\]

\(^3\)Masses of elementary particles appear also as a part of the parameters which label the irreducible unitary representations of the Poincaré group in the four-dimensional Minkowski space (see, e.g., [4, Chapter 2, §3] or [10, p.29]).
Hence $(\mathcal{F}(\mathcal{H}),\{A(f),A(f)^*|f \in \mathcal{H}\})$ is a representation of the CAR over $\mathcal{H}$. This representation is called the Fock representation of the CAR over $\mathcal{H}$. It is shown that the Fock representation $(\mathcal{F}(\mathcal{H}),\{A(f),A(f)^*|f \in \mathcal{H}\})$ is irreducible (see, e.g., [5, Proposition 5.2.2(3)] or [1, proof of Proposition 4.6]).

2.3 A family of irreducible representations of the CAR over $\mathcal{H}$

Let $\mathcal{H}_+$ be a closed subspace of $\mathcal{H}$. Then we have the orthogonal decomposition:

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-, \quad \mathcal{H}_- := \mathcal{H}_+^\perp.$$ 

Let $C : \mathcal{H} \to \mathcal{H}$ be a conjugation on $\mathcal{H}$, i.e., $C$ is an anti-linear mapping on $\mathcal{H}$ satisfying $C^2 = I$ (identity) and $\|C f\|_\mathcal{H} = \|f\|_\mathcal{H}$, $f \in \mathcal{H}$.

For two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we denote by $\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$ the Banach space of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. We introduce

$$\mathcal{B}_\pm := \mathcal{B}(\mathcal{H},\mathcal{H}_\pm)$$

and

$$\mathfrak{T}(\mathcal{H}) := \{T = \langle T_+, T_- \rangle | T_+ \in \mathcal{B}_+, T_-^*T_+ + T_+^* T_- = I\},$$

where

$$\tilde{T}_- := CT_- C.$$ 

Each $T \in \mathfrak{T}(\mathcal{H})$ defines an element of

$$\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H},\mathcal{H})$$

by

$$T f := \langle T_+, T_- f \rangle \in \mathcal{H}_+ \oplus \mathcal{H}_-, \quad f \in \mathcal{H}.$$ 

For each $T \in \mathfrak{T}(\mathcal{H})$, we define an anti-linear mapping $\psi_T : \mathcal{H} \to \mathcal{B}(\mathfrak{T}(\mathcal{H}))$ by

$$\psi_T(f) := A(T_+ f, 0) + A(0, T_- C f)^*, \quad f \in \mathcal{H}.$$ 

**Lemma 2.1** The pair $(\mathfrak{T}(\mathcal{H}),\{\psi_T(f),\psi_T^*(f)|f \in \mathcal{H}\})$ is a representation of the CAR over $\mathcal{H}$.

**Remark 2.2** A standard choice of $T = \langle T_+, T_- \rangle$ in the literature is given by

$$T_+ = P_+, \quad T_- = C P_- C,$$

where $P_\pm$ are the orthogonal projections onto $\mathcal{H}_\pm$ (e.g., [11, Chapter 10], [7, 9]). In this case, the representation is called a quasi-free representation (see, e.g., [6]). Hence the representation $(\mathfrak{T}(\mathcal{H}),\{\psi_T(f),\psi_T^*(f)|f \in \mathcal{H}\})$ gives a generalization of quasi-free representations. Note that the case where $\mathcal{H}_+ = \mathcal{H}$ and $T_+ = I$ (hence $\mathcal{H}_- = \{0\}$ and $T_- = 0$) yields the Fock representation.
For further developments, we need the following additional conditions for \( T = (T_+, T_-) \in \mathfrak{T}(\mathcal{H}) \):

\[
\begin{align*}
&T_+ T_+^* = I. \quad (T.1) \\
&T_- T_-^* = I. \quad (T.2) \\
&T_- T_+^* = 0. \quad (T.3)
\end{align*}
\]

Let

\[
\mathcal{T}_*(\mathcal{H}) := \{ T \in \mathfrak{T}(\mathcal{H}) | \text{(T.1)–(T.3) hold} \}.
\]

**Theorem 2.3** For all \( T \in \mathcal{T}_*(\mathcal{H}) \),

\[
\pi_T := (\mathfrak{F}(\mathcal{H}), \{ \psi_T(f), \psi_T(f)^* | f \in \mathcal{H} \})
\]

is an irreducible representation of the CAR over \( \mathcal{H} \).

Thus we have a family

\[
\{ \pi_T | T \in \mathcal{T}_*(\mathcal{H}) \}
\]

of irreducible representations of the CAR over \( \mathcal{H} \), indexed by a class of bounded linear operators on \( \mathcal{H} \).

A natural question is: when \( \pi_S \) and \( \pi_T \) with \( S \neq T \) are equivalent or inequivalent? The answer is given as follows:

**Theorem 2.4** Let \( T \) and \( S \) be in \( \mathcal{T}_*(\mathcal{H}) \) with \( T \neq S \). Then the two representations \( \pi_T \) and \( \pi_S \) are equivalent if and only if \( S_+ T_+^* \) and \( S_- T_+^* \) are Hilbert-Schmidt.

For proof of this theorem, see [3].

### 3 Application to the Free Quantum Dirac Fields

#### 3.1 The \( d \)-dimensional free Dirac operator

For each \( d \in \mathbb{N} \), we define an even number \( \nu \) as follows:

\[

u := \begin{cases} 
2(d+1)/2 & \text{if } d \text{ is odd} \\
2d & \text{if } d \text{ is even} 
\end{cases}
\]

There exist \( \nu \times \nu \) Hermitian matrices \( \{ \alpha_j \}_{j=1}^d \) and \( \beta \) satisfying the following anti-commutation relations (a representation of the Clifford algebra associated with the Euclidean vector space \( \mathbb{R}^{1+d} \)):

\[
\{ \alpha_j, \alpha_k \} = 2\delta_{jk} I_\nu, \quad \{ \alpha_j, \beta \} = 0, \quad \beta^2 = I_\nu \quad (j,k = 1, \ldots, d),
\]

where \( \delta_{jk} \) is the Kronecker delta and \( I_\nu \) is the \( \nu \times \nu \) unit matrix. The free Dirac operator with mass \( m \geq 0 \), which represents the free Hamiltonian of a Dirac particle—a relativistic charged quantum particle with spin \( 1/2 \)—of mass \( m \) is defined by

\[
H_m := \sum_{j=1}^d \alpha_j p_j + m\beta
\]
with \( p_j := -iD_j \) (\( D_j \) is the generalized partial differential operator in the \( j \)th component \( x_j \) of point \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \)), acting in the Hilbert space

\[
\mathcal{H}_d := L^2(\mathbb{R}^d; \mathbb{C}^\nu) = \{ f = (f_r)_{r=1}^\nu | f_r \in L^2(\mathbb{R}^d), r = 1, \ldots, \nu \}.
\]

We denote the dual space of \( \mathbb{R}^d \) by \( \mathbb{R}^{d*} := \{ k = (k_1, \ldots, k_d) | k_j \in \mathbb{R}, j = 1, \ldots, d \} \).

Let \( \mathcal{F}_d : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^{d*}) \) be the Fourier transform:

\[
(\mathcal{F}_df)(k) := \frac{1}{(2\pi)^d/2} \int_{\mathbb{R}^d} f(x)e^{-ikx} \, dx, \quad f \in L^2(\mathbb{R}^d), \text{ a.e.} \, k \in \mathbb{R}^{d*}
\]

in the \( L^2 \)-sense, where \( kx := \sum_{j=1}^d k_jx_j \). The Hilbert space for a Dirac particle in the momentum representation is given by

\[
\mathcal{H}_D := \mathcal{F}_d \mathcal{H}_d = L^2(\mathbb{R}^{d*}; \mathbb{C}^\nu).
\]

For each \( k \in \mathbb{R}^{d*} \), we define a \( \nu \times \nu \) Hermitian matrix \( h_m(k) \) by

\[
h_m(k) := \alpha k + m\beta,
\]

where \( \alpha k := \sum_{j=1}^d \alpha_j k_j \), and denote the multiplication operator by the matrix-valued function \( h_m(\cdot) \) on \( \mathcal{H}_D \) by \( H_m \):

\[
D(\hat{H}_m) := \left\{ f \in \mathcal{H}_D \mid \int_{\mathbb{R}^{d*}} \| h_m(k)f(k) \|_{\mathbb{C}^\nu}^2 \, dk < \infty \right\},
\]

\[
\hat{H}_m f(k) := h_m(k)f(k), \quad f \in D(\hat{H}_m), \text{ a.e.} \, k \in \mathbb{R}^{d*}.
\]

**Lemma 3.1**

\[
\mathcal{F}_d H_m \mathcal{F}_d^{-1} = \hat{H}_m.
\]

Let

\[
E_m(k) := \sqrt{k^2 + m^2}, \quad k \in \mathbb{R}^{d*},
\]

the energy of the free Dirac particle with momentum \( k \), and

\[
d_m(k) := \frac{m + E_m(k) + \beta \alpha k}{\sqrt{2E_m(k)(m + E_m(k))}} \quad \text{(the case } m > 0),
\]

\[
d_0(k) := \begin{cases} \frac{1}{\sqrt{2}} \left( 1 + \frac{\beta \alpha k}{|k|} \right) & \text{for } k \neq 0 \\ I_\nu & \text{for } k = 0 \end{cases} \quad \text{(the case } m = 0) .
\]

**Lemma 3.2** The matrix \( d_m(k) \) is unitary and

\[
d_m(k)h_m(k)d_m(k)^{-1} = E_m(k)\beta, \quad k \in \mathbb{R}^{d*}.
\]

We denote by \( \hat{D}_m \) the multiplication operator by \( d_m(\cdot) \). The operator

\[
U_m := \hat{D}_m \hat{F}_d
\]

is a unitary operator from \( \mathcal{H}_D \) to \( \hat{H}_D \). We have

\[
U_m H_m U_m^{-1} = E_m\beta.
\]

**Proposition 3.3** The free Dirac operator \( H_m \) is self-adjoint and absolutely continuous with

\[
\sigma(H_m) = (-\infty, -m] \cup [m, \infty).
\]
3.2 Eigenvectors of $h_m(k)$ and some operators

It is easy to see that

$$\dim \ker(\beta \pm 1) = \frac{\nu}{2}.$$ 

Hence, by diagonalization (if necessary), we can assume without loss of generality that

$$\beta = \begin{pmatrix} I_{\nu/2} & 0 \\ 0 & -I_{\nu/2} \end{pmatrix}.$$ 

We denote by $\{e_r\}^\nu_{r=1}$ the standard basis of $\mathbb{C}^\nu$:

$$e_r = (\delta_{rr'})^\nu_{r'=1}.$$ 

For all $k \in \mathbb{R}^{d*}$ and $s = 1, \ldots, \nu/2$, we define the following vectors in $\mathbb{C}^\nu$:

$$u_m(k, s) := d_m(k)^{-1}e_s \in \mathbb{C}^\nu, \quad v_m(k, s) := d_m(k)^{-1}e_{s+(\nu/2)} \in \mathbb{C}^\nu.$$ 

We have

$$h_m(k)u_m(k, s) = E_m(k)u_m(k, s), \quad h_m(k)v_m(k, s) = -E_m(k)v_m(k, s).$$

Namely $u_m(k, s)$ (resp. $v_m(k, s)$) is an eigenvector of $h_m(k)$ with positive (resp. negative) energy $E_m(k)$ (resp. $-E_m(k)$). It follows that the set $\{u_m(k, s), v_m(k, s) | s = 1, \ldots, \nu/2\}$ is a complete orthonormal basis of $\mathbb{C}^\nu$.

The Hilbert space $\mathcal{D}$ has the orthogonal decomposition

$$\mathcal{D} = \mathcal{D}_+ \oplus \mathcal{D}_-$$

with

$$\mathcal{D}_\pm := L^2(\mathbb{R}^{d*}; \mathbb{C}^{\nu/2}).$$

We define linear operators $T_{m\pm} : \mathcal{D}_- \to \mathcal{D}_\pm$ by

$$T_{m+} f := (u_m(\cdot, s)^* f)^{\nu/2}_{s=1} \in \mathcal{D}_+, \quad T_{m-} f := (v_m(\cdot, s)^* f)^{\nu/2}_{s=1} \in \mathcal{D}_-, \quad f \in \mathcal{D},$$

where, for $w = (w_r)_{r=1}^\nu : \mathbb{R}^{d*} \to \mathbb{C}^\nu$,

$$(wf)(k) := \sum_{r=1}^\nu w_r(k)f_r(k), \quad f = (f_r)_{r=1}^\nu \in \mathcal{D}, \quad k \in \mathbb{R}^{d*}$$

and

$$\tilde{w}(k) := w(-k), \quad k \in \mathbb{R}^{d*}.$$ 

It is easy to see that $T_{m\pm}$ are bounded with $||T_{m\pm}|| \leq \sqrt{\nu/2}$.
The adjoint of $T_{m\pm}$ are given as follows:

$$(T_{m+}^* f_+)_r = \sum_{s=1}^{\nu/2} u_{mr}(\cdot, s) f_{s+}, \quad f_+ = (f_+)^{\nu/2}_s \in \mathcal{H}_D,$$

$$(T_{m-}^* f_-)_r = \sum_{s=1}^{\nu/2} v_{mr}(\cdot, s) \tilde{f}_{s-}, \quad f_- = (f_-)^{\nu/2}_s \in \mathcal{H}_D, \ r = 1, \ldots, \nu.$$ 

We denote by $C$ the complex conjugation on $\mathcal{H}_D$:

$$Cf := (f_r)^{\nu/2}_r = 1.$$ 

For a linear operator $A$ on $\mathcal{H}_D$, we define $\tilde{A}$ by

$$\tilde{A} := CAC.$$ 

**Lemma 3.4** For all $m \geq 0,$

$$T_m := (T_{m+}, T_{m-})$$

is an element of $\mathfrak{T}_s(\mathcal{H}_D)$.

### 3.3 A free quantum Dirac field

We work with momentum representation. We denote by $a(f)$ ($f \in \mathcal{H}_D$) the annihilation operator on the fermion Fock space $\mathfrak{F}(\mathcal{H}_D)$ over $\mathcal{H}_D$.

For each $f \in \mathcal{H}_D$, we define

$$\hat{\psi}_m(f) := a(T_{m+} f, 0) + a(0, T_{m-} f)^*,$$

and set

$$\hat{\rho}_m := (\mathfrak{F}(\mathcal{H}_D), \{\hat{\psi}_m(f), \hat{\psi}_m(f)^* | f \in \mathcal{H}_D\}).$$

**Proposition 3.5** For each $m \geq 0$, $\hat{\rho}_m$ is an irreducible representation of the CAR over $\mathcal{H}_D$.

**Proof.** Apply Lemma 3.4 and Theorem 2.3.

Let

$$\psi_m(t, f) := \hat{\psi}_m(e^{itH_m} \hat{f}), \quad t \in \mathbb{R}, \ f \in \mathcal{H}_D,$$

and

$$\rho_m(t) := (\mathfrak{F}(\mathcal{H}_D), \{\psi_m(t, f), \psi_m(t, f)^* | f \in \mathcal{H}_D\}).$$

**Proposition 3.6** For each $m \geq 0$ and $t \in \mathbb{R}$, $\rho_m(t)$ is an irreducible representation of the CAR over $\mathcal{H}_D$.

**Proposition 3.7** Let $m \geq 0$ and $f \in D(H_m)$. Then the operator-valued function: $t \mapsto \psi_m(t, f) \in \mathcal{B}(\mathfrak{F}(\mathcal{H}_D))$ on $\mathbb{R}$ is differentiable in the operator norm topology and obeys the free functional Dirac equation

$$i \frac{d\psi_m(t, f)}{dt} = \psi_m(t, H_m f).$$
For a self-adjoint operator $L$ on $\mathcal{H}$, we denote by $d\Gamma_t(L)$ the fermion second quantization of $L$ on $\mathfrak{F}(\mathcal{H})$:

$$d\Gamma_t(L) := \oplus_{n=0}^{\infty} L^{(n)},$$

where $L^{(0)} := 0$,

$$L^{(n)} := \sum_{j=1}^{n} (I \otimes \cdots \otimes L \otimes I \cdots \otimes I) \uparrow \otimes_{\text{as}} L^{(n)} D(L),$$

where $\otimes_{\text{as}}$ denotes $n$-fold anti-symmetric algebraic tensor product and, for a closable operator $A$, $A$ denotes the closure of $A$.

**Proposition 3.8** For all $f \in \mathcal{H}_{D}$,

$$\psi_{m}(t, f) = e^{itd\Gamma_t(E_{m})} \tilde{\psi}_{m}(f) e^{-itd\Gamma(E_{m})}, \quad t \in \mathbb{R}.$$ 

By this proposition, we have

$$\rho_{m}(t) \cong \rho_{m}(0) \cong \tilde{\rho}_{m}, \quad \forall t \in \mathbb{R}.$$ 

We now state a main result:

**Theorem 3.9** Let $m_{1} \neq m_{2}$ ($m_{1}, m_{2} \geq 0$). Then $\rho_{m_{1}}$ and $\rho_{m_{2}}$ are inequivalent.

**Proof.** By Theorem 2.4, we need only to prove that, if $m_{1} \neq m_{2}$, then $T_{m_{1}+T^{*}_{m_{2}}}^{-}$ or $T_{m_{2}+T^{*}_{m_{2}}}^{-}$ is not Hilbert-Schmidt. See [3] for the details.

**Remark 3.10** One can construct a general class of inequivalent representations of CAR on $\mathfrak{F}(\mathcal{H}_{D})$, which includes $\{\tilde{\rho}_{m}\}_{m \geq 0}$ as a special case [3].

**Remark 3.11** One can consider also free quantum Dirac fields on the bounded region $M := [-L_{1}/2, L_{1}/2] \times \cdots \times [-L_{d}/2, L_{d}/2]$ ($L_{j} > 0$, $j = 1, \ldots, d$). In this case, we have a family $\{\rho_{m}^{(M)}\}_{m \geq 0}$ of irreducible representations of the CAR over $L^{2}(M; \mathbb{C}^{n})$ corresponding to $\{\rho_{m}\}_{m \geq 0}$. It is shown that, if $m_{1} \neq m_{2}$, then $\rho_{m_{1}}^{(M)}$ is equivalent to $\rho_{m_{2}}^{(M)}$ if and only if $d = 1$. See [3] for the details.

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**References**


Acknowledgment

This work is supported by JSPS KAKENHI No. 15K04888.

References


Integrable systems in classical mechanics:

Liouville: \( H(q_i, p_i), i = 1, \ldots, N \)

If there exist \( N \) functions \( L_j(q_i, p_i) \) with \( \{L_j, L_k\} = 0 \) the system is integrable

canonical transformation to action-angle variables:

\[
(q, p) \rightarrow (I, \phi) \quad H(q, p) \rightarrow \tilde{H}(I_1, \ldots I_f)
\]

canonical equations:

\[
\dot{\phi}_j = \frac{\partial \tilde{H}}{\partial I_j} = \omega_j \quad \dot{I}_j = -\frac{\partial \tilde{H}}{\partial \phi_j} = 0
\]

Non-integrable systems possible for \( N \geq 2 \)

Is this notion sensible in linear quantum mechanics?
The quantum Rabi model (QRM) can be considered integrable due to a discrete $Z_2$-symmetry. 

\[ H_{Q\text{bit}} = \Delta \sigma_z \quad H_{\text{int}} = g \sigma_x (a^\dagger + a) \quad H_{\text{rad}} = \omega a^\dagger a \]

Qbit or two-level atom dipole coupling single mode of radiation field

\[ H_R = \Delta \sigma_z + g \sigma_x (a^\dagger + a) + \omega a^\dagger a \]

Quantum Rabi model (single-mode spin-boson model)

Rabi 1936
Jaynes and Cummings 1963

Eigenvalue problem equivalent to coupled ordinary differential equations:

\[ (z + g) \frac{d}{dz} \phi_1(z) + (gz - E) \phi_1(z) + \Delta \phi_2(z) = 0 \]

\[ \phi_2(z) = \phi_1(-z) \]

\[ (z - g) \frac{d}{dz} \phi_2(z) - (gz + E) \phi_2(z) + \Delta \phi_1(z) = 0 \]

(\( \omega = 1 \))

The quantum Rabi model (QRM) can be considered integrable due to a discrete $Z_2$-symmetry.

Symmetry-induced level crossings in spectral graph degenerate eigenvalues are part of the exceptional spectrum.

$G$-function

$$G_\pm(x) = \sum_{m=0}^{\infty} K_m(x) \left[ 1 + \Delta \frac{x}{x - m} \right] g^m$$

$$mK_m = f_{m-1}(x)K_{m-1} - K_{m-2} \quad K_{-1} = 0 \quad K_1 = 1$$

$$f_m(x) = 2g + \frac{1}{2g} \left( m - x + \frac{\Delta^2}{x - m} \right)$$

$$x_n^+ - g^2 \in \text{spec}(H_\pm) \quad \leftrightarrow \quad G_\pm(x_n^+) = 0$$

$G_\pm(x; z)$ linear combination of formal solutions of a confluent N=4 Fuchsian ODE

$G_\pm(x; 0) = \left( 1 + \frac{\Delta}{x} \right) H_c(\alpha \ldots \eta; 1/2) - \frac{1}{2x} H'_c(\alpha \ldots \eta; 1/2)$

exceptional spectrum: $x \in \mathbb{N}_0 \quad \leftrightarrow \quad E = n - g^2$

lifting of the pole at $x = n$

sufficient condition: $K_n(n) = 0$ pole lifted for both parities solution is doubly degenerate
some exceptional solutions are **non-degenerate**

**explanation within G-function formalism:** \( \phi_{1,2}(z) \) holomorphic in \( \mathbb{C} \)

\[
y \frac{d}{dy} \psi_1 = x \psi_1 - \Delta \psi_2 \quad y = z + g \quad x = E + g^2 \quad \phi_{1,2} = e^{-gy + g^2} \psi_{1,2}
\]

\[
(y - 2g) \frac{d}{dy} \psi_2 = (x - 4g^2 + 2gy) \psi_2 - \Delta \psi_1
\]

\[
\tilde{\psi}_1 = cy^x - \Delta y^x \int_0^y \tilde{\psi}_2(y) \, dy
\]

\[
x = n \in \mathbb{N}_0 \quad c \neq 0 \quad \tilde{\psi}_2(y) = \sum_{m = n+1}^{\infty} K_m y^m
\]

\[
\tilde{\psi}_1(y) = cy^n - \Delta \sum_{m = n+1}^{\infty} K_m \frac{y^m}{m - n}
\]

\[
K_{n+1} = 1 \quad \text{exceptional} \quad G\text{-function} \quad G^{(n)}_{\pm}(g, \Delta)
\]

\[
G^{(n)}_{\pm}(g, \Delta) = 0 \quad \leftrightarrow \quad n - g^2 \in \text{spec}(H_{\pm})
\]

\[
K_n(n) = 0 \quad n \geq 1
\]

**determines parity degenerate exceptional spectrum**

**states with no fixed parity are possible**

\[
c = \frac{2gK_{n-1}}{\Delta} \quad \tilde{\psi}_2(y) = \sum_{m = 0}^{n-1} K_m y^n
\]

\[
\tilde{\psi}_1(y) = \Delta \sum_{m = 0}^{n-1} K_m \frac{y^m}{n - m} + \frac{2gK_{n-1}}{\Delta} y^n
\]

\( \tilde{\psi}_j(y) \) are polynomials in \( y = z + g \)

**solutions are quasi-exact**

**explanation with representation theory of \( \mathfrak{sl}_2 \)**

two-photon Quantum Rabi Model

\[ H_{2p} = \omega a^\dagger a + g(a^\dagger^2 + a^2)\sigma_x + \Delta \sigma_z \]

\text{coupling is non-linear in the bosonic operators}

same degrees of freedom as in QRM but larger discrete symmetry

\[ \hat{P}_1 = e^{i\pi a^\dagger a} \quad \hat{P}_2 = e^{i\frac{\pi}{2} a^\dagger a} \otimes \sigma_z \quad \hat{P}_1^2 = \mathbb{1}, \quad \hat{P}_2^2 = \hat{P}_1 \]

\[ [H_{2p}, \hat{P}_1] = [H_{2p}, \hat{P}_2] = 0 \]

\[ \{\hat{P}_1, \hat{P}_2\} \text{ generate } \mathbb{Z}_4\text{-symmetry of } H_{2p} \]

\[ H_{2p} \text{ is integrable} \]

\[ \text{but exact solution is more intricate than for the QRM} \]

Bargmann space of analytic functions

\[ f(z, \bar{z}) \in \mathcal{B} \quad \iff \quad \partial_{\bar{z}} f(z, \bar{z}) = 0, \quad \langle f | f \rangle < \infty \]

\( z \) is adjoint to \( \partial_z \) under the scalar product

\[ \langle f | g \rangle = \frac{1}{\pi} \int dz d\bar{z} e^{-|z|^2} \bar{f}(\bar{z}) g(z) \]

\text{isometry between } L^2(\mathbb{R}) \text{ and } \mathcal{B}
in $B$: $\hat{P}_1[\phi](z) = \phi(-z)$  
$\hat{P}_2 = \hat{T} \otimes \sigma_z$  
$\hat{T}[\phi](z) = \phi(iz)$

even and odd functions in $B$  
$B_\pm = \{ \phi(z) | \phi(z) = \pm \phi(-z) \}$

$\mathbb{Z}_4$-symmetry leads to four invariant subspaces

$\mathcal{H} = \mathcal{H}_+^+ \oplus \mathcal{H}_-^- \oplus \mathcal{H}_+^- \oplus \mathcal{H}_-^+$

$\mathcal{H}_+^+$ and $\mathcal{H}_-^-$ are isomorphic to $B_+$ and $B_-$

Eigenvalue equation in $\mathcal{H}_+^+$

$$\left[ \frac{d^2}{dz^2} + \omega z \frac{d}{dz} + z^2 - E \right] \psi(z^2) + \Delta \psi(-z^2) = 0 \quad (g = 1)$$

Non-local 2nd order ODE in $B_+$ (even analytic functions)

equivalent system

$$\phi_1'' + \omega z \phi_1' + (z^2 - E)\phi_1 = -\Delta \phi_2$$

$$\phi_2'' - \omega z \phi_2' + (z^2 + E)\phi_2 = \Delta \phi_1$$

$\phi_2(z) = \phi_1(iz)$

in contrast to QRM, no singular points except at $z = \infty$

single irregular singular point has s-rank 3

$\longrightarrow$ asymptotic behavior of normal solutions

$$\psi(z) = e^{\frac{1}{2}z^2 + \alpha z} z^\rho (c_0 + c_1 z^{-1} + c_2 z^{-2} + \ldots)$$

only normalizable if $|\gamma| < 1$ plane waves $e^{ipz}$ in $B$

$$\phi_p(z) = \frac{1}{\sqrt{2\pi}^{3/2}} \int_{-\infty}^\infty dx e^{ipx} e^{-\frac{1}{2}(z^2 + x^2) + \sqrt{2}zx} = \frac{e^{-p^2/2}}{\pi^{1/4}} e^{\frac{1}{2}z^2 + i\sqrt{2}pz}$$
asymptotic behavior of normal solutions

\[ \psi(z) = e^{\frac{2}{\omega} z^2 + \alpha z} \left( c_0 + c_1 z^{-1} + c_2 z^{-2} + \ldots \right) \]

expansion with fixed \( \gamma, \alpha, \rho \) only valid for single Stokes sector

\[ \omega > 2 : \]

\[ \gamma_{1,2} = \pm \left( \frac{\omega}{2} - \sqrt{\frac{\omega^2}{4} - 1} \right) \]

\[ |\gamma_{1,2}| < 1 \quad \text{admissible} \]

\[ \gamma_{3,4} = \pm \left( \frac{\omega}{2} + \sqrt{\frac{\omega^2}{4} - 1} \right) \]

\[ |\gamma_{3,4}| > 1 \quad \text{not admissible} \]

how to find the solution containing only components with \( \gamma_{1,2} \) ?

scale transformation in \( L^2(\mathbb{R}) \)

\[ I_\theta [\phi](x) = \phi(e^{\theta} x) \]

\( -\infty < \theta < \infty \)

\( I_\theta \) not unitary: \[ \langle I_\theta[\phi]|I_\theta[\psi] \rangle = e^{-\theta} \langle \phi|\psi \rangle \]

\[ a = \frac{1}{\sqrt{2}} (x + \partial_x) \quad \rightarrow \quad \text{ch}(\theta)a + \text{sh}(\theta)a^\dagger \]

\[ a^\dagger = \frac{1}{\sqrt{2}} (x - \partial_x) \quad \rightarrow \quad \text{ch}(\theta)a^\dagger + \text{sh}(\theta)a \]

bosonic Bogoliubov (squeezing) transformation in \( \mathcal{B} \)

\[ I_\theta = e^{-\theta/2} \exp \frac{\theta}{2}(\partial_x^2 - z^2) \]
transformation of two-photon problem with $t_{11}(\theta) = -\frac{2}{\omega}$

$$\omega_1 z \phi'_1 - E_1 \phi_1 = -\Delta \phi_2$$

$$2 \text{ch}(2\theta) \phi''_2 + \omega_2 z \phi'_2 + [2 \text{ch}(2\theta) z^2 + E_2] \phi_2 = \Delta \phi_1$$

$$E_{1,2} = E \mp \text{sh}(2\theta) - \omega \text{sh}^2(\theta)$$

$$\omega_{1,2} = \pm \text{sh}(2|\theta|)(\omega^2/2 \mp 2)$$

regular singular point at $z = 0$ besides irregular singular point at $\infty$

s-rank at $\infty$ is still 3: $\phi_1(z) = e^{\frac{\gamma}{2} z^2 + \alpha z^p (c_0 + c_1 z^{-1} + \ldots)}$

$$\gamma_1 = \frac{2}{\omega} < 1 \quad \text{admissible}$$

$$\gamma_2 = \frac{\omega}{2} > 1 \quad \text{not admissible}$$

apply $\mathbb{Z}_4$-symmetry $\quad \hat{T} = \exp i \frac{\pi}{2} z \partial_z$

$$\phi_2(z) = \hat{T}_\theta [\phi_1](z) \quad \hat{T}_\theta = I_\theta \hat{T} I_\theta^{-1}$$

$$I_\theta = e^{-\theta/2} \exp \frac{\theta}{2}(\partial_z^2 - z^2)$$

$$z \partial_z - \mathbb{I}/2, \quad z^2/2, \quad \partial_z^2/2 \quad \text{furnish representation of} \quad \mathfrak{su}(1, 1)$$

computation in defining representation of $SU(1, 1)$

$$\phi_1(z) \sim A_1 \exp \frac{z^2}{\omega} + B_1 \exp \frac{\omega z^2}{4} \quad \hat{T}_\theta \left[ \exp \gamma \frac{z^2}{2} \right](z) = \exp \eta(\gamma) \frac{z^2}{2}$$

$$\phi_2(z) \sim A_2 \exp \frac{z^2}{\omega} + B_2 \exp \frac{\omega z^2}{4} \quad \eta(\gamma) = -\frac{\gamma + \text{th}(2\theta)}{\text{th}(2\theta) \gamma + 1}$$
\[ \gamma_1 = \frac{2}{\omega} < 1 \quad \rightarrow \quad \eta(\gamma_1) = 0 \]
\[ \gamma_2 = \frac{\omega}{2} > 1 \quad \rightarrow \quad \eta(\gamma_2) = \infty \]

only normalizable functions are mapped by \( \hat{T}_\theta \) onto functions with asymptotics allowed by the ODE

spectral condition \( \phi_2(z) = \hat{T}_\theta[\phi_1](z) \)

may be evaluated using local Frobenius expansion around \( z = 0 \)

\[ \phi_2(z) = \sum_{n=0}^{\infty} a_n(E)z^{2n} \]
\[ a_0 = 1 \]
\[ G^{++}_{2p}(E) = 1 - \sum_{n=0}^{\infty} \frac{\Delta}{E_1(E) - 2n\omega_1} a_n(E) \frac{(2n)!}{\omega^n n!} \]

\[ G^{++}_{2p}(E) = 1 - \sum_{n=0}^{\infty} \frac{\Delta}{E_1(E) - 2n\omega_1} a_n(E) \frac{(2n)!}{\omega^n n!} \]

\[ E_1(E) = E - \text{sh}(2\theta) - \omega \text{sh}^2(\theta) \]

distance between consecutive poles of \( G^{++}_{2p}(E) \): \( 2\sqrt{\omega^2 - 4} \)
conclusions and outlook

- simple description of exceptional spectrum of the QRM within G-function formalism
- discrete symmetry relates global and local properties of ODEs in $\mathbb{C}$
- derivation of $G$-function for two-photon QRM
- explanation of spectral collapse
- extension to general connection problems with only irregular singularities?

spectral collapse if critical coupling $g = \omega/2$ is approached

LW Duan, YF Xie, DB and QH Chen, (2016) arXiv:1603.04503
However, the dressed photon is a virtual photon[1]: Who has seen it? [1] M. Ohtsu, *Dressed Photons* (Springer, 2014).

To answer this question,

Collaboration with,

1) Prof. I. Ojima (formerly Univ. Kyoto)  
   Prof. H. Saigoh (Nagahama Bio Univ.)  
   Dr. K. Okamura (Nagoya Univ.)  
2) Prof. M. Katori (Chuo UNiv.)
[Question 1]
Is any light generated at the probe apex?

NO!

[Question 2]
Are light emitting diodes and lasers fabricated by using a silicon (Si) crystal?

Indirect transition-type semiconductor

NO!
Consider two electronic energy levels, and assume that the parities of their state functions are same between each other. Now, the question is: Is the transition between these energy levels taken place when they are illuminated by light?

\[ \langle \psi_f | e \mathbf{r} \cdot \mathbf{E} | \psi_i \rangle = 0 \]

Electric dipole-forbidden transition

Answer to Question 1 Yes!

Spectrometer beyond diffraction limit
Answer to Question 2

Yes!

Si LED

Blue Green Red

Wavelength; 1.3μm
Optical power; 1W(@R.T.)

SiC LED

UV-Violet Bluish-white Blue Green

Si laser

Ridge waveguide

SiO₂ film

Al electrode

Active layer

Si substrate

- Optical and electrical relaxation oscillator (Si)
- Infrared photodetector with optical amplification (Si)
- Polarization rotator (ZnO, SiC)
- UV-Violet LED (ZnO)
- Yellow LED (GaP)

Answer to Question 3

Yes!

AND logic gate

Input 1
Input 2
Output

NOT logic gate

Energy source
Input
Output

Incident propagating light spot

Position (nm)

Light intensity (a.u.)

0 10 10²

Time (ns)

0 2
Off shell photon?

\[ \Delta E \cdot \Delta t \geq \hbar \]

\[ \Delta E: \text{large} \]

\[ \Delta t: \text{short} \]

Virtual photon

\[ \Delta p: \text{large} \quad (\Delta k \gg k) \]

\[ \Delta p \cdot \Delta x \geq \hbar \]

\[ \Delta x: \text{small} \quad (\Delta x \ll \lambda) \]

Optical near field

Physical picture of “optical near field + virtual photon” = dressed photon
Physical picture of dressed photons

Dressed photon
≠ Collection of corpuscles (by I. Newton),
Free photon (by A. Einstein)

Problems!

A virtual cavity, for deriving the quantum number operator, cannot be defined because the DP exists in a nanometric space (a sub-wavelength-size).

The electromagnetic mode cannot be defined.

Photons of the infinite modes and electron-hole pairs of the infinite energy levels have to be considered in the nanometric space.

\[
\hat{H} = \sum_{k,\lambda} \hbar \omega_k \hat{a}^\dagger_k \hat{a}_k + \sum_{\alpha > F, \beta < F} (E_\alpha - E_\beta) \hat{b}^\dagger_{\alpha \beta} \hat{b}_{\alpha \beta} - \int \psi^\dagger (r) \rho (r) \psi (r) \cdot \hat{D}^\perp (r) dv
\]

Diagonalization

\[
\hat{a}^\dagger = \sum_{k,\lambda} \hat{a}^\dagger_{k,\lambda} + iN_k \sum_{\alpha > F, \beta < F} \left( \rho_{\alpha \beta} (k) \hat{b}^\dagger_{\alpha \beta} + \rho_{\beta \alpha} (k) \hat{b}_{\alpha \beta} \right)
\]

\[
\hat{a} = \sum_{k,\lambda} \hat{a}_{k,\lambda} - iN_k \sum_{\alpha > F, \beta < F} \left( \rho^*_{\alpha \beta} (k) \hat{b}_{\alpha \beta} + \rho^*_{\beta \alpha} (k) \hat{b}^\dagger_{\alpha \beta} \right)
\]

\[
V_{eff} (r) \propto \exp \left( - \frac{r}{a} \right)
\]
Further possibilities of dressing: Interaction between dressed photon and phonon


Displacement operator function for phonon (Generating multi-mode coherent phonons)

\[ \hat{H} = \sum_{i=1}^{N} \hbar \omega \hat{a}^\dagger_i \hat{a}_i + \sum_{p=1}^{N} \hbar \Omega_p \hat{c}^\dagger_p \hat{c}_p + \sum_{i=1}^{N} \hbar \chi_p \hat{a}^\dagger_i (\hat{c}^\dagger_p + \hat{c}_p) + \sum_{i=1}^{N-1} \hbar J (\hat{a}^\dagger_i \hat{a}_{i+1} + \hat{a}^\dagger_{i+1} \hat{a}_i) \]

Dressed photon

Diagonalization

Dressed-photon—phonon (DPP)

Impurity


Impurity atom

Nanometric material tip

Dressed photon

Dressed  photon

Phonon

DP-phonon interaction

DP Hopping

Edge

Further possibilities of dressing: Interaction between dressed photon and phonon


Displacement operator function for phonon (Generating multi-mode coherent phonons)

\[ \hat{\alpha}_i^\dagger = \hat{a}_i^\dagger \exp \left\{ -\sum_{p=1}^{N} \frac{\chi_p}{\Omega_p} (\hat{c}^\dagger_p - \hat{c}_p) \right\} \]

Dressed photon

Dressed photon

Incident light

Macroscopic system

Electronic state

Phonon state

Electronic state

Phonon state

Two-step transition

Incident light

Macroscopic system

Electronic state

Phonon state

Electronic state

Phonon state

Two-step transition

Incident light

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Macroscopic system

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Two-step transition

Incident light

Macroscopic system

Electronic state

Phonon state

Electric dipole-allowed transition

\[ |E_{ex};el> \otimes |E_{ex};ph> \]

\[ E_{ex} + n\hbar \nu_{phonon} \]

Electric dipole-forbidden transition

\[ |E_{y};el> \otimes |E_{y};ph> \]

\[ E_{y} + n\hbar \nu_{phonon} \]

Two-step transition

\[ |E_{ex};el> \otimes |E_{ex};ph> \]
Advantages:
[Si] nontoxic, abundant,…
[Device] low energy consumption, integrated with Si electronic devices,…

However:
Indirect transition-type semiconductor
• For electron–hole recombination
  → A phonon is required to satisfy the momentum conservation law.

DPP: accompanied with multi-mode coherent phonons

\[ \hat{\alpha}^i = \hat{a}^i \exp \left\{ - \sum_{\nu=1}^{\mathcal{K}} \frac{\mathcal{X}_\nu}{\Omega} \left( \hat{c}^\dagger_\nu - \hat{c}_\nu \right) \right\} \]

Multi-mode coherent phonons
Light emission process via DPP

Two-step (For IR PBD)

Conduction band

$E_{ex} + n\hbar\nu_{phonon}$

Valence band

$E_g + n\hbar\nu_{phonon}$

Fabrication: DPP-assisted annealing

Boron (B) atom

(ion implantation: $2 \times 10^{19} \text{cm}^{-3}, 0.04\%$, depth 2 μm)

Si substrate

DPP

Stimulated emission

(Energy dissipation to the outside)

Cooling

Annealing by Joule energy

Varying the spatial Distribution of B atoms

Joule heat

Forward current

n-type

p-type

$p-n$ homojunction

Light

($\lambda=1.3 \mu m: \hbar\nu_{anneal}<E_g$)

Impurity atom

DPP

DPP

DPP
Si PBD


- Two-photons are emitted from a single electron
  → Total external quantum efficiency = 150%

- $\lambda_{em} = 1.3\mu m$
- Emitted optical power = 1W (at R.T.)
- External quantum efficiency = 15% (@1.32 ± 0.15 μm)

Packaged PDB

Direct transition-type (on shell technology)

Commercial InGaAs LED
- 3mW (L12771*)
- 2-5% (L10822*)
*Hamamatsu Photonics K.K.

- Temperature of the device surface
  - By stimulated emission
  - Temperature (°C)
  - Time (min.)
  - Emission intensity
  - Stationary state
  - Time

- Commercial InGaAs LED
  - 3mW (L12771*)
  - 2-5% (L10822*)
*Hamamatsu Photonics K.K.
Emission spectra

Serves as a breeder that creates photons of $h\nu_{em} = h\nu_{anneal}$

By an IR camera
By a visible camera

Driving a radiometer

Burning a plastic tape

Demonstration

Imaging

Photon breeding with respect to photon energy
Photon breeding with respect to photon energy


Degree of polarization

\[
\frac{I_{//} - I_{\perp}}{I_{//} + I_{\perp}}
\]

Annealing time (min.)

Photon breeding with respect to photon spin


Degree of polarization

\[
\text{Annealing time (min.)}
\]

Light intensity (a.u.)

Wavelength (μm)

Photon energy (eV)

Degree of polarization
Origin of PB: Controlled spatial distribution of B atoms

Length and orientation of B atom pair

By SIMS

By atom probe field ion microscopy

Irradiation light

No change by the DPP-assisted annealing?
However, ----

Origin of PB: Controlled spatial distribution of B atoms

By SIMS

By atom probe field ion microscopy

Irradiation light

No change by the DPP-assisted annealing?
However, ----

Length and orientation of B atom pair

Non-annealed

After the DPP-assisted annealing

Irradiation light
Localized phonons

\[ p = \frac{h}{3a} \]

\( n = 3 \) (\( \propto d \)):

1. Satisfies the momentum conservation law
2. Compensates for the difference between \( h\nu_{\text{anneal}} \) and \( E_g \).

\( h\nu_{\text{em}} = E_g - 3E_{\text{phonon}} \)

Opt. mode phonon

\( h\nu_{\text{anneal}} = 0.924\text{eV} \)

\( 3E_{\text{phonon}} = 1.12\text{eV} \)

B atom pair serves as a phonon localization center for creating a DPP.

B atom pair:
One B atom and its nearest neighbor


Photon breeding with respect to photon energy

\[ p_{X \rightarrow \Gamma} = \frac{h}{a} \]

\( n = 3 \) (\( \propto d \)):

1. Satisfies the momentum conservation law
2. Compensates for the difference between \( h\nu_{\text{anneal}} \) and \( E_g \).
Photon breeding with respect to photon spin

Annealing by a linearly polarized light

Si PB oscillator (Si laser)


Lasing spectra (CW, @R.T.)
Near-infrared lasing

Total power > 200mW

Conventional laser: 2mW

Direct transition-type (on shell technology)

Dressed photon technology: Generic technology

Logic gate device
Optical router system
Optical pulse shape measurement system
Optical security system
Photon breeding device (PBD)
Green LED
Si waveguide array
X-ray zone plate
Lithography system
Surface polishing system
HDD system (1Tb/inch²)
Spectrometer

Device
System
Fabrication

Example
Laser mirror, HDD, optical disk.

Examples

Limit by current source

12.5A/cm²
Where the dressed photon is created?

The DP is a virtual photon[1].

Nobody can see it without detecting the energy transfer from the nano-system to the surrounding macro-system through energy dissipation.

- Micro-macro duality in quantum physics[2]
- Category theory[3]
- Non-equilibrium statistical mechanics of the open system.

**Optimum condition for the DPP-assisted annealing**

![Diagram of irradiation light and emitted light](image)

**Non-equilibrium statistical mechanics**

Simulation by stochastic model (by M. Katori, Chuo-Univ., 2016)

1. Diffusive motion of doped B atoms: interacting random-walk model on a lattice (aggregation process to make B atom-pairs with specified separations)

2. Coupling of electrons and phonons:
   - Poisson processes with mean-field-type couplings

Basic properties of photon breeding in fabrication and operation processes:
- Reproduced
Results of simulation(1)


**Temperature**

**Emission intensity**

**The optimum condition**

**Degree of polarization**

Results of simulation(2)


**Diffusion and pairing by DPP-assisted annealing**

**Poisson point process vs Ginibre point process** (by M. Katori)

**Snapshot of the position of gaseous molecules**

B atom pairs: isolated (Ginibre point process)

Origin of a high emission power!

similar to solid state lasers or gas lasers (with isolated ions or molecules)

**Spatial distribution of B atoms:** Coding the optical properties of Si LED
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Teijin, Sodick, V-Tech., Nitto Kouki, Toyo Senko, Sematech (USA)

For more details

(3) 大津元一、「ドレスト光子」（朝倉書店、2013）
(4) 大津元一、「ドレスト光子はやわかり」（丸善、2014）
I. Bohr’s correspondence principle in mathematics. When we talk about the correspondence principle, we mean the following quantum-classical dictionary.

<table>
<thead>
<tr>
<th></th>
<th>Quantum (Non-Commutative)</th>
<th>$\hbar \to 0$</th>
<th>Classical (Commutative)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>States</strong></td>
<td>Non-Comm. probabilities</td>
<td>$\to$</td>
<td>Class. probabilities</td>
</tr>
<tr>
<td><strong>Observables</strong></td>
<td>Non-Comm. Random Variables</td>
<td>$\to$</td>
<td>Class. Random Variables</td>
</tr>
<tr>
<td><strong>Dynamics</strong></td>
<td>Unitary linear evolution</td>
<td>$\to$</td>
<td>Nonlinear Hamiltonian flow</td>
</tr>
</tbody>
</table>

Bohr’s correspondence principle is necessary for a quantum theory to be in agreement with observation (since at macroscopic scales systems behave commutatively). For Quantum Field Theories however, even at the formal level it is not clear whether the correspondence principle should hold or not, especially when a renormalization procedure is involved.

In these notes we concisely discuss the correspondence principle for the renormalized model introduced by E. Nelson, that describes non-relativistic bosons in interaction with a scalar relativistic bosonic field with Yukawa coupling. We omit references throughout these notes; the interested reader may consult [1].

II. The classical system. The classical motion is described by a system of two coupled equations: one is Schrödinger and the other Klein-Gordon, with non-linear Yukawa coupling.

\[
\begin{cases}
i\hat{\mathcal{C}}t u = -\Delta u + Au \\
(\Box + 1) A = -|u|^2 \\
u(0) = u_0 \\
A(0) = A_0, \quad \hat{\mathcal{C}}t A(0) = \hat{\mathcal{A}}(0)
\end{cases}
\]

(S-KG[Y])

Date: June 6, 2016. “Mathematical Quantum Field Theory and Related Topics”; IMI Kyushu University, Fukuoka; Japan.
In these notes we set $M_u = \frac{1}{2}$, $m_A = 1$, and no external potential acting on $u$; but the results hold in a more general situation.

The system $(S\text{-KG}[Y])$ is known to be globally well-posed on suitable Sobolev spaces, e.g. on $H^1(\mathbb{R}^3, \mathbb{C}) \oplus H^1(\mathbb{R}^3, \mathbb{R}) \oplus L^2(\mathbb{R}^3, \mathbb{R})$. However it is convenient to make a change of variables from the real-valued $(A, \partial_t A)$ to the complex valued $\alpha$ given by $A = \sqrt{2}\text{Re} \mathcal{F}^{-1}(\omega^{-1/2} \alpha)$, $\partial_t A = \sqrt{2}\text{Im} \mathcal{F}^{-1}(\omega^{-1/2} \alpha)$. Therefore we obtain

$$(S\text{-KG}_\alpha[Y])$$

$$\begin{cases}
    i\partial_t u = -\Delta u + A(\alpha)u \\
    i\partial_t \alpha = \omega \alpha + \frac{1}{\sqrt{2}\omega} \mathcal{F}(|u|^2) \\
    u(0) = u_0 \\
    \alpha(0) = \alpha_0
\end{cases}$$

**Proposition II.1.** $S\text{-KG}_\alpha[Y]$ is globally well-posed on the energy space $H^1(\mathbb{R}^3, \mathbb{C}) \oplus \mathcal{F} H^{1/2}(\mathbb{R}^3, \mathbb{C})$ and on $L^2(\mathbb{R}^3, \mathbb{C}) \oplus L^2(\mathbb{R}^3, \mathbb{C})$.

In addition, $S\text{-KG}_\alpha[Y]$ can be viewed as an Hamiltonian system, with energy functional

$$(1)$$

$$\mathcal{E}(u, \alpha) = \langle u, -\Delta u \rangle_2 + \langle \alpha, \omega \alpha \rangle_2 + \frac{1}{\sqrt{2}\omega} \int_{\mathbb{R}^6} \frac{1}{\sqrt{2}\omega} (\bar{\alpha} e^{-ik \cdot x} + \alpha e^{ik \cdot x}) |u|^2 \mathrm{d}x \mathrm{d}k$$

densely defined on $D(\mathcal{E}) \supseteq H^1 \oplus \mathcal{F} H^{1/2}$.

**III. The quantum system.** The quantum dynamics should be characterized by the following formal operator on $\mathcal{H} = \Gamma_s(L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3))$:

$$H = \int_{\mathbb{R}^3} \frac{1}{\sqrt{2}\omega(k)} \psi^\#(x) (a^\#(k)e^{-ik \cdot x} + a(k)e^{ik \cdot x}) \psi(x) \mathrm{d}x \mathrm{d}k$$

$$+ \int_{\mathbb{R}^3} \frac{1}{\sqrt{2}\omega(k)} \psi^\#(x)(a^\#(k)e^{-ik \cdot x} + a(k)e^{ik \cdot x}) \psi(x) \mathrm{d}x \mathrm{d}k$$

where $\psi^\#$ and $a^\#$ are the $\hbar$-dependent annihilation/creation operators corresponding to the first and second $L^2$-space respectively. More precisely, we have $[\psi(x), \psi^\#(x')] = \hbar \delta(x-x')$; and $[a(k), a^\#(k')] = \hbar \delta(k-k')$.

$H$ is not defined as an operator because of the $a^\#$-creation term in the interaction, but $\langle \cdot, H \cdot \rangle_{\Gamma_s}$ is a densely defined quadratic form.

To rigorously define the dynamics it is possible to perform a self-energy renormalization. We introduce the fibration $\mathcal{H} = \bigoplus_{n=0}^\infty \mathcal{H}_n$ with $\mathcal{H}_n = L^2_\sigma(\mathbb{R}^{3n}) \otimes \Gamma_s(L^2(\mathbb{R}^3))$, and the self-adjoint operator $H_\sigma$, $\sigma \in \mathbb{R}^+$, with regularized interaction. Then we perform a dressing transformation in
order to single out the divergent self-energy. Define the dressing “group” 
\((e^{-\frac{i}{\hbar}T_\sigma(\sigma_0)})_{\theta \in \mathbb{R}}\), with \(\sigma_0 \in \mathbb{R}^+\) and 
\[T_\sigma(\sigma_0) = \int_{\mathbb{R}^6} \psi^*(x) (\alpha^*(k) g_\sigma(\sigma_0, k) e^{-ik \cdot x} + a(k) \bar{g}_\sigma(\sigma_0, k)e^{ik \cdot x}) \psi(x) dx dk,\]
g_\sigma(\sigma_0, k) = 1_{\{|\sigma_0| \leq |\cdot| \leq \sigma_0\}}(k) g_0(k)\) where \(g_0 \in L^2(\mathbb{R}^3)\) is suitably chosen.

**Proposition III.1.** \(0 \leq \sigma_0 \leq \sigma \leq \infty \Rightarrow T_\sigma(\sigma_0)\) self-adjoint.

Finally, we define the dressed Hamiltonian 
\[\hat{H}_\sigma(\sigma_0) = e^{\frac{i}{\hbar}T_\sigma(\sigma_0)} H_\sigma e^{-\frac{i}{\hbar}T_\sigma(\sigma_0)} - \hbar E_\sigma(\sigma_0)\int_{\mathbb{R}^3} \psi^*(x) \psi(x) dx;\]
where \(E_\sigma(\sigma_0) \to -\infty\) is the divergent self-energy.

**Theorem III.2.** \(\forall n \in \mathbb{N}, \exists \sigma_0(n, \hbar), \forall \sigma_0 < \sigma \leq \infty:\)
- \(\hat{H}_\sigma(\sigma_0)|_{\mathcal{H}_n}\) self-adjoint with domain \(\hat{D}_{\sigma,n} \subset Q(H_0|_{\mathcal{H}_n})\);
- \(\hat{H}_\sigma(\sigma_0)|_{\mathcal{H}_n}\) \(\sigma\to\infty\)-limit \(\hat{H}(\sigma_0)|_{\mathcal{H}_n}\) self-adjoint; and the corresponding unitary groups converge strongly.

We want to extend the definition of \(\hat{H}(\sigma_0)|_{\mathcal{H}_n}\) to the whole Fock space \(\Gamma_s(L^2 \oplus L^2);\) however this can be done in many ways. We choose the following that is most suited for the limit \(\hbar \to 0\).

**Theorem/Definition III.1** (Renormalized Hamiltonians). \(\forall \sigma_0 \in \mathbb{R}^+, \exists \mathcal{R}(\sigma_0, \hbar)\) such that 
\[
\hat{H}(\sigma_0) := \begin{cases} 
\hat{H}(\sigma_0)|_{\mathcal{H}_n} & n \leq \mathcal{R}(\sigma_0, \hbar) \\
0 & n > \mathcal{R}(\sigma_0, \hbar)
\end{cases},
\]
\[
H_{\text{ren}}(\sigma_0) := e^{-\frac{i}{\hbar}T_\sigma(\sigma_0)} \hat{H}(\sigma_0) e^{\frac{i}{\hbar}T_\sigma(\sigma_0)},
\]
are self-adjoint on \(\mathcal{H}\). Given \(\sigma_0 \in \mathbb{R}^+\) and \(\hbar \in \mathbb{R}^+\), we say that the renormalized dynamics is non-trivial in any sector with at most \(\mathcal{R}(\sigma_0, \hbar)\) non-relativistic bosons. The number \(\mathcal{R}(\sigma_0, \hbar)\) can be explicitly computed; in particular it is proportional to \(\sigma_0\), and inversely proportional to \(\hbar\).

**IV. S-KG_\alpha[Y] revisited: classical dressing.** S-KG_\alpha[Y] is the Hamiltonian equation corresponding to the energy functional \(\mathcal{E}\) defined in (1). We denote by \(E(\cdot) : \mathbb{R} \times (H^1 \oplus \mathcal{F} H^{1/2}) \to H^1 \oplus \mathcal{F} H^{1/2}\) the corresponding Hamiltonian flow in the energy space. In other words, \(E(t)(u_0, \alpha_0)\) is the solution at time \(t\) of S-KG_\alpha[Y].

Now we introduce a group of nonlinear symplectic transformation on the energy space, called classical dressing. Let the functional \(\partial_{g_x}(\sigma_0) :\)
$L^2 \oplus L^2 \to \mathbb{R}$ be defined as follows:

$$D_{g_x(\sigma_0)}(u, \alpha) = \int_{\mathbb{R}^d} \left( g_x(\sigma_0, k)x + \tilde{g}_x(\sigma_0, k)\alpha(k) \right)e^{-ik \cdot x} |u(x)|^2 \, dx \, dk.$$ 

Then the corresponding Hamiltonian flow $D_{g_x(\sigma_0)}(\theta) : H^1 \oplus \mathfrak{F} H^{1/2} \to H^1$ for any $\theta \in \mathbb{R}$, and it has an explicit and easy form whenever $g$ has fixed parity.

**Remark IV.1.** Using the standard Wick quantization, we obtain the following very interesting results:

- $(\mathcal{E})^{\text{Wick}} = \langle \cdot, H \cdot \rangle$ (not well-defined);
- $(\hat{\mathcal{E}}(\sigma_0))^{\text{Wick}} = (\mathcal{E} \circ D_{g_x(\sigma_0)}(-1))^{\text{Wick}} = \langle \cdot, \hat{H}(\sigma_0) \cdot \rangle$ (renormalized and well-defined on any sector with at most $\mathcal{N}(\sigma_0, \hbar)$ non-relativistic bosons);
- $E(t) = D_{g_x(\sigma_0)}(1) \circ \hat{E}(\sigma_0, t) \circ D_{g_x(\sigma_0)}(-1) \overset{\text{Quant}}{\underset{\hbar \to 0?}{\longrightarrow}} e^{-\frac{i}{\hbar} H_{\text{ren}}(\sigma_0)} = e^{-\frac{i}{\hbar} T_x(\sigma_0)}e^{-\frac{i}{\hbar} \hat{H}(\sigma_0)}e^{\frac{i}{\hbar} T_x(\sigma_0)}$.

Therefore $\hat{\mathcal{E}}(\sigma_0)$ seems to be the form of the energy most suitable for quantization.

**V. The “classical” meaning of $\sigma_0$.** \[ \inf_{(u, \alpha) \in D(\mathcal{E})} \mathcal{E}(u, \alpha) = -\infty; \] on the other hand \[ \inf_{\|u\|_2 \leq \sqrt{\mathcal{C}}} (u, \alpha) > -\infty. \]

Since $E(t)$ preserves the $L^2$-norm (mass) of Schrödinger’s equation, the constraint $\|u\|_2 \leq \sqrt{\mathcal{C}}$ that makes the energy bounded below is a natural assumption. It is also natural to look for quantum configurations that make the classical energy bounded from below, *i.e.* we consider to be admissible families of quantum states only those families whose classical limits are probability measures in $\mathcal{M}(L^2 \oplus L^2)$, concentrated inside the “ball”

$$B_\mathcal{E}(u) = D(\mathcal{E}) \cap \left\{ (u, \alpha) \in L^2 \oplus L^2, \|u\|_2 \leq \sqrt{\mathcal{C}} \right\}.$$ 

We remark that this is only a necessary condition, since there may be families of quantum states whose limits are all concentrated inside $B_\mathcal{E}(u)$, but have an unbounded from below or undefined quantum energy.

**Proposition V.1.** Any state $\varrho_\hbar$ on $\mathcal{H}$ with at most $[\mathcal{E}/\hbar] \in \mathbb{N}$ non-relativistic particles can be written as a linear combination:

$$\varrho_\hbar = \sum_{i \in \mathbb{N}} \lambda_i(\hbar) |\psi_i(\hbar)\rangle \langle \psi_i(\hbar)|,$$ 

55
where each $\psi_i(h) \in \mathcal{H}$ has non-zero components only on $\bigoplus_{n=0}^{[\mathcal{E}/h]} \mathcal{H}_n$. In addition, if $\varrho_h$ has at most $[\mathcal{E}/h] \in \mathbb{N}$ non-relativistic particles, then

$$\varrho_{h_k} \xrightarrow{h_k \to 0} \mu \in \mathcal{M}(L^2 \oplus L^2) \Rightarrow \mu \text{ is concentrated inside } B_{\mathcal{E}}(u).$$

Finally, $(\varrho_h)_{h \in (0,1)}$ satisfies

$$(A_0) \quad (\forall k \in \mathbb{N}) \, \text{Tr} \left( \varrho_h \left( \int_{\mathbb{R}^3} \psi^*(x)\psi(x)dx \right)^k \right) \leq \mathcal{C}^k.$$

In the light of the above, it would be suitable to have a way of defining – for any $C \in (0,1)$ – the quantum dynamics on the relevant sector $\mathcal{H}$. This is possible, uniformly in $h$, exploiting the freedom of choice of $\sigma_0$: it is sufficient to choose a $\sigma_0$ satisfying

$$\left( \frac{\sigma_0 - 2M}{2h} - 1 \right) \mathcal{M}(\sigma_0, h) \geq [\mathcal{E}/h].$$

Here $M$ is a constant that depends only on the parameters of $\mathcal{E}$ (masses and coupling constant, that are all fixed in these notes). Therefore the choice of $\sigma_0$ is in some sense constrained by the physical requirement that the classical energy should be bounded from below.

VI. Bohr’s correspondence principle. We are now ready to give a precise meaning to the quantum-classical dictionary of Section I. We make the following assumptions on quantum states: the first is assumption $(A_0)$ above, the second is the following

$$(A_\varnothing) \quad (\exists K > 0) \, (\forall h \in (0,1)) \, \text{Tr} \left( \varrho_h \left( \int_{\mathbb{R}^3} \psi^*(x)\psi(x)dx + \int_{\mathbb{R}^3} a^*(k)a(k)dk \right. \\
\left. + e^{-\frac{i}{\hbar}T_x(\sigma_0)H_0} e^{\frac{i}{\hbar}T_x(\sigma_0)} \right) \right) \leq K.$$

The latter assumption means, roughly speaking, that the family of states has uniformly bounded mass and dressed free energy density.

**Theorem VI.1** (Ammari - F. 2016). Let $\mathcal{E} > 0$, and let $\sigma_0(\mathcal{E})$ be such that $e^{-\frac{i}{\hbar}tH_{\text{ren}}(\sigma_0)}$ is non-trivial on any state with at most $[\mathcal{E}/h]$ non-relativistic bosons. If $(\varrho_h)_{h \in (0,1)}$ is a family of quantum states satisfying $(A_0)$ and $(A_\varnothing)$, then the correspondence principle holds for evolved states:

$$\varrho_{h_k} \xrightarrow{h_k \to 0} \mu \Leftrightarrow e^{-\frac{i}{\hbar}tH_{\text{ren}}(\sigma_0)} \varrho_{h_k} e^{\frac{i}{\hbar}tH_{\text{ren}}(\sigma_0)} \xrightarrow{h_k \to 0} \varrho_{h_k(t)} \in \mathcal{M}(L^2 \oplus L^2) \Rightarrow \mu(t), \forall t \in \mathbb{R}.$$

**Corollary VI.2** (Informal). For suitably regular densely defined classical observables $b : L^2 \oplus L^2 \supset D(b) \to \mathbb{R}$, and suitable quantization procedures
Quant_k, the correspondence principle holds (weakly) for observables:

\[ q_{\hbar_k} \xrightarrow{\hbar_k \to 0} \mu \iff \text{Tr}\left( q_{\hbar_k}(t) b^{\text{Quant}_k} \right) \xrightarrow{\hbar_k \to 0} \int_{D(b)} b(u, \alpha) d(\mathcal{E}(t) \# \mu)(u, \alpha), \forall t \in \mathbb{R}. \]

**Remark VI.3.** With the notation \( q_{\hbar_k} \to \mu \) it is meant that the generating functional \( \mathcal{G}_{q_{\hbar_k}} : L^2(\mathbb{R}) \to \mathbb{R} \) of \( q_{\hbar_k} \) converges to the Fourier transform \( \mathcal{F} \mu : L^2(\mathbb{R}) \to \mathbb{C} \) of a unique probability measure \( \mu \).

**VII. Outline of the proof.** The idea is to exploit the classical identity

\[ \mathcal{E}(t) = D_{g_x(\sigma_0)}(1) \circ \hat{E}(\sigma_0, t) \circ D_{g_x(\sigma_0)}(-1) \]

to relate the dressed and undressed evolution. This is of crucial importance since we have an explicit form only for \( \hat{H}(\sigma_0) \) (as a quadratic form).

The core of the proof is to prove the convergence:

\[ q_{\hbar_k} \xrightarrow{\hbar_k \to 0} \mu \iff e^{-\frac{i}{\hbar_k} \hat{t} \hat{H}(\sigma_0)} \xrightarrow{\hbar_k \to 0} \hat{E}(t) \# \mu, \forall t \in \mathbb{R}. \]

The other steps are a simple combination of the following results:

- \( q_{\hbar_k} \xrightarrow{\hbar_k \to 0} \mu \iff e^{-\frac{i}{\hbar_k} \theta T_x(\sigma_0)} \xrightarrow{\hbar_k \to 0} D_{g_x(\sigma_0)}(\theta) \# \mu \), for any \( \theta \in \mathbb{R} \) and \( \sigma_0 \in \mathbb{R}^+ \);
- \( q_{\hbar_k}(t) = e^{-\frac{i}{\hbar_k} \hat{t} H(\sigma_0)} e^{-\frac{i}{\hbar_k} \hat{t} \hat{t} \hat{H}(\sigma_0)} e^{\frac{i}{\hbar_k} \hat{t} T_x(\sigma_0)} \xrightarrow{\hbar_k \to 0} D_{g_x(\sigma_0)}(\sigma_0) e^{-\frac{i}{\hbar_k} \hat{t} T_x(\sigma_0)}; \)
- \( \mathcal{E}(t) = D_{g_x(\sigma_0)}(1) \circ \hat{E}(\sigma_0, t) \circ D_{g_x(\sigma_0)}(-1) \).

The proof of (2) is obtained as follows. With a term-by-term analysis, we identify the classical limit of the interaction picture integral equation:

\[ \text{Tr}\left( \tilde{q}_{\hbar_k}(t) W_{\hbar_k}(\xi) \right) = \text{Tr}\left( \tilde{q}_{\hbar_k} W_{\hbar_k}(\xi) \right) + \frac{i}{\hbar_k} \int_0^t \text{Tr}\left( \tilde{q}_{\hbar_k} \left[ (\hat{H}(\sigma_0) - H_0), W_{\hbar_k}(\xi_s) \right] \right) ds. \]

We thus obtain a transport equation for a classical measure \( \tilde{\mu}_t \):

\[ \partial_t \tilde{\mu}_t + \nabla^T (V(t) \tilde{\mu}_t) = 0. \]

This equation is solved by \( \tilde{\mu}_t = E_0(-t) \# \hat{E}(t) \# \mu_0 \). In addition, exploiting the regularity properties of \( \mu_0 \) (inherited by those of \( q(0) \)) it is also possible to prove that the aforementioned solution is unique, using optimal transportation techniques.

**References**

The resolvent CCR algebra and KMS states

Tomohiro Kanda

1 Introduction

In 2008, D. Buchholz and H. Grundling defined the resolvent CCR algebra (c.f. [6], [7]). The resolvent CCR algebra is the universal C*-algebra generated by the family of resolvents of operators satisfying canonical commutation relations (CCR). On the resolvent CCR algebra, we considered one-parameter groups of *-automorphisms and their KMS states, which correspond to the equilibrium states of weakly coupled anharmonic quantum oscillators on \( \mathbb{Z} \). We obtained the existence and the uniqueness of regular KMS states [13]. In this paper, we present some properties of the resolvent CCR algebra and our results.

Let \((X, \sigma)\) be a non-degenerate symplectic space. Let \(\Psi(f)\), \(f \in X\), be a self-adjoint operator on a Hilbert space \(H\) such that

\[
[\Psi(f), \Psi(g)] = i\sigma(f, g)\|, \quad f, g \in X
\]  

(1.1)

on some domain \(D \subset H\). The operator \(\Psi(f)\) is called a field operator. The equation (1.1) is called CCR. It is well known that the relation (1.1) cannot be realized by bounded operators. We present an example of field operators.

Example 1.1. Let \(\sigma\) be the bilinear form on \(\mathbb{R}^{2n} \times \mathbb{R}^{2n}\) defined by

\[
\sigma(q_k, p_l) = \delta_{kl} = -\sigma(p_l, q_k), \quad \sigma(q_k, q_l) = \sigma(p_k, p_l) = 0, \quad k, l \in \{1, \cdots, n\}
\]  

(1.2)

where \(\{q_k, p_k \mid k = 1, \cdots, n\}\) is a basis on \(\mathbb{R}^{2n}\). Then \(\{q_k, p_k \mid k = 1, \cdots, n\}\) is called a symplectic basis. The space \((\mathbb{R}^{2n}, \sigma)\) is a non-degenerate symplectic space. Let \(\Psi(\sum_{k=1}^{n}(a_kq_k + b_kp_k))\), \(a_k, b_k \in \mathbb{R}\), be the essentially self-adjoint operator on \(S(\mathbb{R}^n)\) defined by

\[
(\Psi(\sum_{k=1}^{n}(a_kq_k + b_kp_k))f)(x) = \sum_{k=1}^{n}(a_k(\Psi(q_k)f)(x) + b_k(\Psi(p_k)f)(x)) = \sum_{k=1}^{n}(a_kx_kf(x) + b_k(-i\frac{\partial f}{\partial x_k})(x)),
\]  

(1.3)

for \(f \in S(\mathbb{R}^n)\), where \(S(\mathbb{R}^n)\) is the set of all rapidly decreasing smooth functions on \(\mathbb{R}^n\). The operator \(\Psi(\sum_{k=1}^{n}(a_kq_k + b_kp_k))\) satisfy (1.1). This representation is called the Schrödinger representation.

Let \(\mathcal{A}(X, \sigma)\) be the *-algebra generated by field operators \(\Psi(f)\), where \((X, \sigma)\) is a non-degenerate symplectic space. For a quantum observable \(Q \in \mathcal{A}(X, \sigma)\), the Heisenberg time evolution of \(Q\) associated with a Hamiltonian \(H\) has the formal form

\[
\alpha_t(Q) = e^{iHt}Qe^{-iHt}.
\]  

(1.4)

However, the operators \(Q \in \mathcal{A}(X, \sigma)\) and \(H\) are unbounded. Thus, we have to specify the domain of these operators. Specialists considered the unitaries generated by \(\Psi(f)\), \(W(f) = \exp(\Psi(f))\). The C*-algebra generated by \(W(f)\) is called the Weyl CCR algebra. It was used in the study of Bose–Einstein
condensation of ideal gas [5]. Let \( X = \mathbb{R}^2 \) with a symplectic basis \( \{ q, p \} \). We consider the Schrödinger representation of the Weyl CCR algebra and

\[
H = -\frac{d^2}{dx^2} + V(x),
\]

(1.5)

where \( V \) is real valued continuous function vanishing at infinity. Then the one-parameter group of \(*\)-automorphisms \( \alpha \) on \( \mathcal{B}(L^2(\mathbb{R}, dx)) \) defined in the same manner as (1.4), does not leave the Weyl CCR algebra invariant ([7] and [12]).

In 2007, D. Buchholz and H. Grundling considered the resolvent CCR algebra for study the supersymmetric quantum field theory [6]. The resolvent CCR algebra is a \( C^* \)-algebra generated by \( \mathcal{R}(\lambda, f) = (i\lambda \mathbb{1} - \Psi(f))^{-1} \), where \( \lambda \in \mathbb{R}\setminus\{0\} \) and \( f \in X \). In [7], [9], and [11], they have shown that a wider class of interacting Hamiltonians of physical interests give rise to one parameter groups of \(*\)-automorphisms for the resolvent CCR algebra. For example, let \( H_L \) be the self-adjoint operator on \( L^2(\mathbb{R}^{2L}) \) defined by

\[
H_L = \sum_{-L \leq k \leq L} \left( -\frac{\partial^2}{\partial x_k^2} + \omega^2 x_k^2 + V(x_k) \right) + \sum_{-L \leq k < L} \varphi(x_k - x_{k+1}),
\]

(1.6)

where \( V \) and \( \varphi \) are real valued rapidly decreasing smooth functions on \( \mathbb{R} \). The function \( V \) corresponds to the potential and \( \varphi \) corresponds to the nearest neighbor interaction. Then, the one-parameter group of \(*\)-automorphisms \( \alpha^L \) defined in (3.27) leaves \( \mathcal{R}_L \), where \( \mathcal{R}_L \) is defined in (3.24) [13, Theorem 1.1.].

On the resolvent CCR algebra \( \mathcal{R} \) defined in (3.24), we considered the thermodynamic limit \( \alpha \) of \( \alpha^L \) [13, Theorem 1.2.], where \( \alpha \) defined in (3.28). We present the details in section 3 and section 5. For the one-parameter group of \(*\)-automorphisms \( \alpha \), we obtained the following theorem.

**Theorem 1.2.** [13, Theorem 1.3.]

There exists a unique regular \((\alpha, \beta)\)-KMS state for any \( \beta > 0 \).

In this paper, we define the Weyl CCR algebra and the resolvent CCR algebra and present some results in [7] in section 2. In section 3, we construct one-parameter groups of \(*\)-automorphisms on the resolvent CCR algebra. In section 4, we construct KMS states associated with one-parameter groups of \(*\)-automorphisms defined in the section 3 and present our main results [13]. In [11], D. Buchholz proved more general result. We explain some results of [11] in section 5.

## 2 CCR algebras

In this section, we present the Weyl CCR algebra and the resolvent CCR algebra.

### 2.1 The Weyl CCR algebra

First, we define the Weyl CCR algebra.

**Definition 2.3.** Let \( (X, \sigma) \) be a non-degenerate symplectic space. The Weyl CCR algebra \( \mathcal{W}(X, \sigma) \) is the \( C^* \)-algebra generated by the set \( \{ W(f) \mid f \in X \} \), where the elements \( W(f), f \in X \), satisfy the following relations: For \( f, g \in X \),

\[
\begin{align*}
W(0) &= \mathbb{1}, \quad (2.7) \\
W(f)^* &= W(-f), \quad (2.8) \\
W(f)W(g) &= \exp \left( -i\frac{\sigma(f, g)}{2} \right) W(f + g). \quad (2.9)
\end{align*}
\]
We introduce the notion of regular representations on the Weyl CCR algebra as follows.

**Definition 2.4.** Let \((X,\sigma)\) be a non-degenerate symplectic space. A representation \((\mathfrak{B},\pi)\) of the Weyl CCR algebra \(\mathcal{W}(X,\sigma)\) is regular if the unitary groups \(\pi(t)\) are strongly continuous for any \(f \in X\).

A state \(\omega\) on the Weyl CCR algebra \(\mathcal{W}(X,\sigma)\) is regular if the GNS representation \((\mathfrak{B}_\omega,\pi_\omega)\) associated with \(\omega\) is regular.

For example, the Schrödinger representation defined in Example 1.1 is a regular representation. If \((\mathfrak{B},\pi)\) is a regular representation of the Weyl CCR algebra \(\mathcal{W}(X,\sigma)\), then due to Stone’s theorem there exists a self-adjoint operator \(\Psi_\pi(f)\) satisfying
\[
\pi(W(tf)) = \exp(it\Psi_\pi(f)).
\] (2.10)
The operator \(\Psi_\pi(f)\) satisfies CCR.

For a regular representation, we have the following theorem.

**Theorem 2.5.** (Stone–von Neumann uniqueness theorem, c.f. [5, Corollary 5.2.15.])

Let \(\mathcal{W}(\mathfrak{B},\sigma)\) be the Weyl CCR algebra over a finite-dimensional Hilbert space \(\mathfrak{B}\) with the symplectic form \(\sigma(f,g) = \text{Im}(f,g)\mathfrak{B}\).

It follows that each regular state on \(\mathcal{W}(\mathfrak{B},\sigma)\) is normal with respect to the Schrödinger representation.

The Weyl CCR algebra is a simple C*-algebra (c.f. [5, Theorem 5.2.8.]). It leads to the following fact.

**Theorem 2.6.** (M. Fannes, A Verbeure, 1974, [12] and D. Buchholz, H. Grundling, 2008, [7])

Let \(\pi_0\) be the Schrödinger representation of the Weyl CCR algebra \(\mathcal{W}(\mathbb{R}^2,\sigma)\) on \(L^2(\mathbb{R})\). Let \(H_1\) be the self-adjoint operator on \(S(\mathbb{R})\) defined by
\[
(H_1f)(x) = -\frac{d^2f}{dx^2}(x) + V(x)f(x), \quad f \in S(\mathbb{R})
\] (2.11)
where \(V\) is a real valued rapidly decreasing smooth function on \(\mathbb{R}\). Let \(\alpha^{(1)}\) be the one-parameter group of \(*\)-automorphisms on \(\mathcal{B}(L^2(\mathbb{R}))\) defined by
\[
\alpha^1_t(A) = e^{it\lambda}Ae^{-it\lambda}, \quad A \in \mathcal{B}(L^2(\mathbb{R})).
\] (2.12)
Then \(\alpha^{(1)}\) does not leave \(\pi_0(\mathcal{W}(\mathbb{R}^2,\sigma))\) invariant unless \(V = 0\).

### 2.2 The resolvent CCR algebra

**Definition 2.7.** (D. Buchholz and H. Grundling, 2008, [7])

Let \((X,\sigma)\) be a non-degenerate symplectic space. The resolvent CCR algebra \(\mathcal{R}(X,\sigma)\) is the universal C*-algebra generated by the set \(\{R(\lambda, f) \mid \lambda \in \mathbb{R}\setminus\{0\}, f \in X\}\), where the elements \(R(\lambda, f)\) \((\lambda \in \mathbb{R}\setminus\{0\}, f \in X)\) satisfy the following relations: For \(\lambda, \mu \in \mathbb{R}\setminus\{0\}, f, g \in X\),
\[
R(\lambda, f) = -i\frac{1}{\lambda},
\] (2.13)
\[
\mu R(\mu, f) = R(\lambda, f),
\] (2.14)
\[
R(\lambda, f)^* = R(-\lambda, f),
\] (2.15)
\[
R(\lambda, f) - R(\mu, f) = i(\mu - \lambda)R(\lambda, f)R(\mu, f),
\] (2.16)
\[
[R(\lambda, f), R(\mu, g)] = i\sigma(f,g)R(\lambda, f)R(\mu, g)^2R(\lambda, f),
\] (2.17)
\[
R(\lambda, f)R(\mu, g) = R(\lambda + \mu, f + g)[R(\lambda, f) + R(\mu, g) + i\sigma(f,g)R(\lambda, f)^2R(\mu, g)] \quad (\lambda + \mu \neq 0).
\] (2.18)
**Definition 2.8.** Let \((X, \sigma)\) be a non-degenerate symplectic space. A representation \((\mathcal{S}, \pi)\) of the resolvent CCR algebra \(\mathcal{R}(X, \sigma)\) is regular if
\[
\ker \pi(R(\lambda, f)) = \{0\}
\]
for any \(f \in X\) and some \(\lambda \in \mathbb{R}\setminus\{0\}\).

A state \(\omega\) on the resolvent CCR algebra \(\mathcal{R}(X, \sigma)\) is regular if the GNS representation \((\mathcal{S}_\omega, \pi_\omega)\) associated with \(\omega\) is regular.

Due to the discussion of the pseudo-resolvents (see e.g. [22, Theorem 1, p. 216]), if \((\mathcal{S}, \pi)\) is a regular representation of \(\mathcal{R}(X, \sigma)\), then there exists a self-adjoint operator \(\Psi_\pi(f)\) on \(\mathcal{S}\) satisfying
\[
\pi(R(\lambda, f)) = \frac{1}{i\lambda\mathbb{I} - \Psi_\pi(f)}.
\]

The operator \(\Psi_\pi(f)\) satisfies CCR [7, Theorem 4.2.].

There exists a one-to-one correspondence between a regular representation of the Weyl CCR algebra and of the resolvent CCR algebra [7, Corollary 4.4.].

**Theorem 2.9.** [7, Theorem 4.9.]
Let \((X, \sigma)\) be an arbitrary dimensional non-degenerate symplectic space. A regular representation \((\mathcal{S}, \pi)\) of the resolvent CCR algebra \(\mathcal{R}(X, \sigma)\) is faithful, i.e.
\[
\ker \pi = \{0\}.
\]

**Theorem 2.10.** [7, Proposition 4.7.]

Let \((X, \sigma)\) be a finite dimensional non-degenerate symplectic space. Let \((\mathcal{S}, \pi)\) be an irreducible representation of \(\mathcal{R}(X, \sigma)\). Put \(X_R = \{ f \in X \mid \ker \pi(R(\lambda, f)) = \{0\} \}, X_T = \{ f \in X_R \mid \pi(R(\lambda, f))^{-1} \in \mathcal{B}(\mathcal{S}) \}\) and \(X_S = X \setminus X_R\). Let \(\{q_1, \ldots, q_n\}\) be a basis for \(X_T\). Then we can augment this basis of \(X_T\) by \(\{p_1, \ldots, p_n\}\) which is not contained in \(X_S\) into a symplectic basis of \(Q := \text{Span}\{q_1, p_1; \ldots; q_n, p_n\}\), i.e. \(\sigma(q_i, p_j) = \delta_{i,j}, 0 = \sigma(q_i, q_j) = \sigma(p_i, p_j)\). Then we have the decomposition
\[
X = Q \oplus (Q^+ \cap X_R) \oplus (Q^+ \cap X_S)
\]
into non-degenerate spaces such that \(Q^+ \cap X_R \subset \{0\} \cup (X_R \setminus X_T)\) and \(Q^+ \cap X_S \subset \{0\} \cup X_S\), where \(S^+\) is symplectic complement of subspace \(S \subset X\).

In contrast theorem 2.6, D. Buchholz and H. Grundling showed that the following theorem.

**Theorem 2.11.** (D. Buchholz, H. Grundling, 2008, [7])

Let \(\pi_0\) be the Schrödinger representation of the resolvent CCR algebra \(\mathcal{R}(\mathbb{R}^2, \sigma)\) on \(L^2(\mathbb{R})\). Let \(H_1\) be the self-adjoint operator on \(L^2(\mathbb{R})\) defined in (2.11) and \(\alpha^{(1)}_t\) be the one-parameter group of *-automorphisms on \(\mathcal{B}(L^2(\mathbb{R}))\) defined in (2.12). Then \(\alpha^{(1)}_t\) leaves \(\pi_0(\mathcal{R}(\mathbb{R}, \sigma))\).

Moreover, let \(H_2\) be the essentially self-adjoint operator on \(S(\mathbb{R}^2)\) defined by
\[
(H_2 f)(x_1, x_2) = -\frac{\partial^2 f}{\partial x_1^2}(x_1, x_2) - \frac{\partial^2 f}{\partial x_2^2}(x_1, x_2) + \frac{1}{2} f(x_1, x_2) + \frac{1}{2} f(x_1, x_2) + \varphi(x_1 - x_2) f(x_1, x_2), \quad f \in S(\mathbb{R}^2)
\]
(2.22)
where \(\varphi\) is a real valued rapidly decreasing smooth function on \(\mathbb{R}\) and \(\alpha^{(2)}_t\) be the one-parameter group of *-automorphisms on \(\mathcal{B}(L^2(\mathbb{R}^2))\) defined by
\[
\alpha^{(2)}_t(A) = e^{itH_2} A e^{-itH_2}, \quad A \in \mathcal{B}(L^2(\mathbb{R}^2)).
\]
(2.23)
Then \(\alpha^{(2)}_t\) leaves \(\pi_0(\mathcal{R}(\mathbb{R}^2, \sigma))\).

We omit the details in this paper. More detail discussion is in [7], [9] and [10].
3 One-parameter groups of ∗-automorphisms

In this section, we construct one-parameter groups of ∗-automorphisms on the resolvent CCR algebra, which corresponds to weakly coupled anharmonic crystals. To explain our construction, we introduce some symbols.

For any subset Λ ⊂ \( \mathbb{Z} \), we denote \( c_c(\Lambda) \) by the space of all finitely supported functions \( f : \Lambda \rightarrow \mathbb{C} \).

We define the symplectic form \( \sigma \) on \( c_c(\Lambda) \) by \( \sigma(f, g) = \text{Im} \langle f, g \rangle \) for \( f, g \in c_c(\Lambda) \), where \( \langle \cdot, \cdot \rangle \) is the canonical inner product on \( \ell^2(\mathbb{Z}) \). For \( L \in \mathbb{N} \), we put \( \Lambda_L := (-L, L] \subset \mathbb{Z} \). Then we denote \( R_L = R(c_c(\Lambda_L), \sigma) \).

By theorem 2.11, we can prove the following.

**Proposition 3.12.** [13, Theorem 1.1.]

Let \( H_L, L \in \mathbb{N} \), be the self-adjoint operator on \( L^2(\mathbb{R}^2 L) \) defined by

\[
H_L = \sum_{-L < k \leq L} \left( -\frac{\partial^2}{\partial x_k^2} + \omega^2 x_k^2 + V(x_k) \right) + \sum_{-L < k < L} \varphi(x_k - x_{k+1})
\]

where \( \omega > 0 \) and \( V, \varphi \) are real valued rapidly decreasing smooth functions. Let \( \tilde{\alpha}_L \) be the one-parameter group of ∗-automorphisms on \( B(L^2(\mathbb{R}^2 L)) \) defined by

\[
\tilde{\alpha}_L^t(A) := e^{itH_L} A e^{-itH_L}, \quad A \in B(L^2(\mathbb{R}^2 L)).
\]

Then \( \tilde{\alpha}_L^t \) leaves \( \pi_0(\mathcal{R}_L) \) where \( \pi_0 \) is the Schrödinger representation.

By theorem 2.9, the Schrödinger representation \( \pi_0 \) is faithful. We define the one-parameter group of ∗-automorphisms \( \alpha_L^t \) on \( \mathcal{R}_L \) by

\[
\alpha_L^t(A) := \pi_0^{-1} (\tilde{\alpha}_L^t(\pi_0(A))), \quad A \in \mathcal{R}_L.
\]

Next, we consider the thermodynamic limit of \( \alpha_L \). In [16], [17], [18], [19] and [20], B. Nachtergaele and R. Sims considered the Lieb–Robinson bounds for one-parameter groups of ∗-automorphisms and the existence of the thermodynamic limit. D. Buchholz considered the thermodynamic limit in the resolvent CCR algebra as well [11, Proposition 3.4.]. Using their techniques, we have the following.

**Proposition 3.13.** [13, Theorem 1.1.]

There exists a unique one-parameter group of ∗-automorphisms \( \alpha \) on \( \mathcal{R} \) such that

\[
\alpha(A) = \lim_{L \to \infty} \alpha_L^t(A), \quad (A \in \mathcal{R}),
\]

where the limit is norm limit in \( \mathcal{R} \).

4 KMS states

In this section, we consider KMS states associated with the one-parameter group of ∗-automorphisms \( \alpha \) defined in (3.28) for inverse temperature \( \beta > 0 \). Note that \( \alpha_t(A) \) is not norm continuous as a function of \( t \in \mathbb{R} \) for certain \( A \) in the resolvent CCR algebra. However, the set of elements \( A \) for which \( \alpha_t(A) \) have analytic extension as function of \( t \) is weakly dense in regular representations. We define KMS states as follows.
Definition 4.14. Let $\alpha$ be a (not necessary continuous) one-parameter group of $\ast$-automorphisms on a unital C*-algebra $A$. A state $\psi$ on $A$ is an $(\alpha, \beta)$-KMS state, $\beta > 0$, if $\psi$ is an $\alpha$-invariant state, i.e. $\psi(\alpha_t(A)) = \psi(A)$, $A \in A$, and $\psi(Qa_0(R))$ is a continuous function in $t \in \mathbb{R}$ for any $Q, R \in A$ satisfying the KMS boundary condition, namely, there exists a function $F_{Q,R}(t)$ holomorphic in $\Im \beta > 0$, bounded continuous on the closure $\overline{T_\beta}$ of $I_\beta$ such that

$$F_{Q,R}(t) = \psi(Q\alpha_t(R)), \quad F_{Q,R}(t + i\beta) = \psi(\alpha_t(R)Q), \quad t \in \mathbb{R},$$

for any $Q, R \in A$. 

First, we construct an $(\alpha^0 \otimes \alpha^{0,L}, \beta)$-KMS state, $\beta > 0$. Note that $R$ is not equal to $R_L \otimes R_{L'}$ in general [7, Theorem 5.1]. However, for any regular representation $(\mathcal{S}, \pi)$ of $R$ and any positive integer $L$, $\pi(R_L) \otimes \pi(R_{L'})$ is a weakly dense subalgebra in $\pi(R)$ ([13, Lemma 4.3] and [7, Theorem 4.2. (v)]). Now, we construct a regular representation of $R_L \otimes R_{L'}$. We denote $\mathcal{S}_L = L^2(\mathbb{R}^{2L}) \otimes \mathcal{F}_c(L^2(\mathbb{R}, dx))$ and $\pi = \pi_0 \otimes \pi_F$, where $\mathcal{F}_c(L^2(\mathbb{R}, dx))$ is the Bose–Fock space over $L^2(\mathbb{R}, dx)$ (See e.g. [5]), $(L^2(\mathbb{R}^{2L}), \pi_0)$ is the Schrödinger representation of $R_L$, and $(\mathcal{F}_c(L^2(\mathbb{R}, dx)), \pi_F)$ is the Fock representation of $R_{L'}$. Then $(\mathcal{S}, \pi)$ is a representation of $R_L \otimes R_{L'}$. Since the representation $\pi$ is regular [7, Theorem 4.2. (v)], $(\mathcal{S}, \pi)$ is a regular representation of $R$ as well. Let $\alpha^{0,L}$ be the one-parameter group of $\ast$-automorphisms on $R_{L'}$ defined by

$$\alpha^{0,L}(A) = \pi_F^{-1}(e^{i\beta d\xi(A)} \pi_F(A) e^{-i\beta d\xi(A)})^L, \quad A \in R_{L'},$$

where $H^0 = -\frac{d^2}{dx^2} + \omega^2x^2$ and $d\xi(A)$ is the second quantization of $H^0$. By [5, Proposition 5.2.27.], $e^{-\beta d\xi(A)}$ is a trace class operator on $\mathcal{F}_c(L^2(\mathbb{R}, dx))$. Thus, we define the $(\alpha^0 \otimes \alpha^{0,L}, \beta)$-KMS state $\psi_L$ on $R$ by

$$\psi_L(\cdot) = \frac{\text{Tr}_{L'} \otimes \mathcal{F}_c(L^2)(e^{-\beta d\xi(A)} \pi_F(\cdot))}{\text{Tr}_{L'} \otimes \mathcal{F}_c(L^2)(e^{-\beta d\xi(A)} \pi_F(\cdot))}.$$  

Since $R$ is a unital C*-algebra, the set $\{\psi_L \mid L \in \mathbb{N}\}$ has a cluster point $\psi$. Using the technique of the functional analysis, we have the following theorem.

Theorem 4.15. [13, Theorem 5.8.]

The state $\psi$ constructed in the above is a regular $(\alpha, \beta)$-KMS state on $R$ for any $\beta$.

Next, we consider the uniqueness of regular $(\alpha, \beta)$-KMS states. We use the technique of the case of quantum spin systems [1]. H. Araki defined the relative entropy of normal states on a von Neumann algebra ([2] and [3]).

Let $\psi_1$ and $\psi_2$ be positive normal linear functionals on a von Neumann algebra $M$. By the theory of standard form of von Neumann algebra, there exists a Hilbert space $\mathcal{H}_1, \mathcal{H}_2 \subset \mathcal{H}$ such that $
abla \bar{\psi_1}(a) = \langle \xi_1, a \xi_1 \rangle$ and $\psi_2(a) = \langle \xi_2, a \xi_2 \rangle$ for all $a \in M$, where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{H}$. Let $S_{\xi_1,\xi_2}$ be densely defined closable operator defined by

$$S_{\xi_1,\xi_2}a\xi_1 = a^*\xi_2.$$  

Then, we define the relative modular operator $\Delta_{\xi_1,\xi_2}$ by

$$\Delta_{\xi_1,\xi_2} = S_{\xi_1,\xi_2}^* S_{\xi_1,\xi_2},$$

where $S_{\xi_1,\xi_2}^*$ is the operator closure of $S_{\xi_1,\xi_2}$. We denote the orthonormal projection from $\mathcal{H}$ onto $M'\xi$, by $s(\psi), i = 1, 2$, where $M'$ is commutant of $M$ and $\overline{M'\xi}$ is the closure of $M'\xi, \xi \in \mathcal{H}$. On a von Neumann algebra, we can define the relative entropy as follows.
Definition 4.16. Let \( \psi_1 \) and \( \psi_2 \) be normal states on a von Neumann algebra \( \mathcal{M} \). Then we define the relative entropy \( S \) of states \( \psi_1 \) and \( \psi_2 \) by the following equation:

\[
S(\psi_1, \psi_2) = \begin{cases} 
- \int_0^\infty \log(\lambda) dE(\lambda) (s(\psi_1) \leq s(\psi_2)), \\
\infty & \text{otherwise}
\end{cases} \tag{4.34}
\]

where \( \int_0^\infty \lambda dE(\lambda) \) is the spectral decomposition of the relative modular operator \( \Delta_{\xi_1, \xi_2} \).

Let \( \mathcal{A}_x, x \in \mathbb{Z} \), be the matrix algebra \( M_n(\mathbb{C}) \). For a finite \( \Lambda \subset \mathbb{Z} \), we put

\[
\mathcal{A}_\Lambda = \bigotimes_{a \in \Lambda} \mathcal{A}_a, \quad \mathcal{A} = \bigcup_{\Lambda \subset \mathbb{Z}} \mathcal{A}_\Lambda. \tag{4.35}
\]

Using the relative entropy, H. Araki showed that lemma 4.17 and lemma 4.18. Before we present the lemmas, we define the notion of quasi-containment.

Let \( (\tilde{\xi}_i, \pi_i), i = 1, 2 \), be representations of a \( C^* \)-algebra \( \mathcal{A} \). If the kernels of \( \pi_1 \) and \( \pi_2 \) coincide and the mapping \( \pi_1(A) \to \pi_2(A), A \in \mathcal{A} \), extends to a \( * \)-isomorphism of weak closures, then \( \pi_1 \) and \( \pi_2 \) are said to be quasi-equivalent (See e.g. [4, Definition 2.4.25.] and [4, Theorem 2.4.26.]). If a subrepresentation of \( \pi_2 \) is quasi-equivalent to \( \pi_1 \), then \( \pi_2 \) is said to quasi-contain \( \pi_1 \).

Lemma 4.17. [1, Lemma 1.]

Let \( \psi_1 \) and \( \psi_2 \) be states on \( \mathcal{A} \) and \( (\tilde{\xi}_1, \pi_1) \) and \( (\tilde{\xi}_2, \pi_2) \) be the GNS representations associated with \( \psi_1 \) and \( \psi_2 \), respectively. If \( \pi_2 \) does not quasi-contain \( \pi_1 \), then there exists a sequence of projections \( \{ e_n \in \bigcup_{L \in \mathbb{N}} \mathcal{A}_{(L, L)} \mid n \in \mathbb{N} \} \) such that

\[
\lim_{n} \psi_1(e_n) = a > 0, \tag{4.36}
\]

\[
\lim_{n} \psi_2(e_n) = 0. \tag{4.37}
\]

Lemma 4.18. [1, Lemma 2.]

Let \( \psi_1 \) and \( \psi_2 \) be states on \( \mathcal{A} \). For \( L \in \mathbb{N} \), we denote \( \psi_j^L \) by the restriction of \( \psi_j \) to \( \mathcal{A}_{(L, L)} \). If

\[
\sup_L S(\psi_1^L, \psi_2^L) =: \lambda < \infty, \tag{4.38}
\]

then \( \pi_2 \) quasi-contains \( \pi_1 \), where \( \pi_j \) is the GNS representation of \( \mathcal{A} \) associated with \( \psi_j, j = 1, 2 \).

To prove the uniqueness of KMS states on the resolvent CCR algebra, we require the following lemmas for the resolvent CCR algebra. Note that there exists the minimal two-sided ideal \( \mathcal{K}_L \) in \( \mathcal{R}_L, L \in \mathbb{N} \). For a regular representation \( (\tilde{\xi}_i, \pi) \), \( \pi(\mathcal{K}_L) = C \), where \( C \) is the set of all compact operators on \( \mathcal{H} \) ([7, Theorem 5.4.] and [9, Theorem 4.5. (i)]). Moreover, \( \cup_{L \in \mathbb{N}} \pi(\mathcal{K}_L) \) is weakly dense in \( \cup_{L \in \mathbb{N}} \pi(\mathcal{R}_L) \) [13, Lemma 4.3.]. Thus, we have the following lemma.

Lemma 4.19. [13, Lemma 4.5.]

Let \( \psi_1 \) and \( \psi_2 \) be regular states on \( \mathcal{R} \) and \( (\tilde{\xi}_1, \pi_1) \) and \( (\tilde{\xi}_2, \pi_2) \) be the GNS representations associated with \( \psi_1 \) and \( \psi_2 \), respectively. If \( \pi_2 \) does not quasi-contain \( \pi_1 \), then there exists a sequence of projections \( \{ e_n \in \cup_{L \in \mathbb{N}} \mathcal{R}_L \mid n \in \mathbb{N} \} \) such that

\[
\lim_{n} \psi_1(e_n) = a > 0, \tag{4.39}
\]

\[
\lim_{n} \psi_2(e_n) = 0. \tag{4.40}
\]

By theorem 2.5, a regular representation on \( \mathcal{R}_L, L \in \mathbb{N} \), is quasi-equivalent to the Schrödinger representation. Thus, for regular states \( \psi_1 \) and \( \psi_2 \), we can consider the relative entropy \( S(\psi_1, \psi_2) \), where \( \psi_1 \) and \( \psi_2 \) are states on \( \pi_0(\mathcal{R}_L)' \), which are normal extensions of states \( \psi_1 \) and \( \psi_2 \), respectively. By the above discussion, we get the following lemma.
Lemma 4.20. [13, Lemma 4.9.]
Let \( \psi_1 \) and \( \psi_2 \) be regular states on \( \mathcal{R} \). If
\[
\sup_{L} S(\hat{\psi}_1 \| \rho_0(\mathcal{R}_L), \hat{\psi}_2 \| \rho_0(\mathcal{R}_L)) =: \mu < \infty,
\]
then \( \pi_2 \) quasi-contains \( \pi_1 \) where \( \pi_j \) is the GNS representation of \( \mathcal{R} \) associated with \( \psi_j \), \( j = 1, 2 \).

By using lemma 4.19, lemma 4.20, and von Neumann algebraic arguments, we can prove the following theorem.

Theorem 4.21. [13, Theorem 1.3.]
There exists a unique regular \((\alpha, \beta)\)-KMS state on \( \mathcal{R} \) for any \( \beta > 0 \).

Remark 4.22. Note that there exist non-regular primary \((\alpha, \beta)\)-KMS states. Let \( \pi_0 \) be the Schrödinger representation on \( L^2(\mathbb{R}^{2L}) \), \( L \in \mathbb{N} \). We define the representation of \( \mathcal{R} \) on \( L^2(\mathbb{R}^{2L}, dx) \) by
\[
\pi(R(\lambda, f)) = \begin{cases} \pi_0(R(\lambda, f)) & f \in c_c(\Lambda_L) \\ 0 & f \in c_c(\mathbb{Z}) \setminus c_c(\Lambda_L) \end{cases}, \quad \lambda \in \mathbb{R} \setminus \{0\}, f \in c_c(\mathbb{Z}).
\]

Let \( \omega \) be the state on \( \mathcal{R} \) defined by
\[
\omega(A) = \frac{\text{Tr}_L(e^{-\beta H_\mathcal{R}} \rho(A))}{\text{Tr}_L(e^{-\beta H_\mathcal{R}})}, \quad A \in \mathcal{R},
\]
where \( \text{Tr}_L \) is the trace on \( L^2(\mathbb{R}^{2L}) \). The state \( \omega \) is a non-regular primary \((\alpha, \beta)\)-KMS state. See also [11, Lemma 4.2.] and [13, Lemma 5.4.].

Remark 4.23. A. Uhlmann defined the relative entropy of positive linear functionals on any \(*\)-algebra [21]. Thus, we can consider the relative entropy for states on any \( C^* \)-algebra. F. Hiai, M. Ohy, and M. Tsukada showed that the relative entropy defined by H. Araki is equal to the relative entropy defined by A. Uhlmann, in the special case. We used this fact in [13].

5 Higher dimensional lattice systems

In this section, we consider a quantum lattice system over \( \mathbb{Z}^d \), \( d \in \mathbb{N} \). To each site \( \lambda \in \mathbb{Z}^d \), we associate the Hilbert space \( \mathcal{H}_\lambda = L^2(\mathbb{R}^d) \). For any subset \( \Lambda \subset \mathbb{Z}^d \), we denote \( c_c(\Lambda) \) by the space of all finitely supported function \( f : \Lambda \to \mathbb{C} \). We define the symplectic form \( \sigma \) on \( c_c(\Lambda) \) by \( \sigma(f, g) = \text{Im}(f, g)_\ell^2 \) for \( f, g \in c_c(\Lambda) \), where \((\cdot, \cdot)_\ell^2 \) is the canonical inner product on \( \ell^2(\mathbb{Z}^d) \).

Using lemma 2.2 and lemma 2.3 in [11], we can prove the following

Theorem 5.24. Let \( \Lambda \) be a finite subset of \( \mathbb{Z}^d \). Let \( H_\Lambda \) be the self-adjoint operator on \( L^2(\mathbb{R}^{d|\Lambda|}) \) defined by
\[
H_\Lambda = \sum_{\lambda \in \Lambda} (-\Delta_\lambda + x_\lambda^2 + V(x_\lambda)) + \sum_{|\lambda| = |\mu| = 1} \varphi(x_\lambda - x_\mu),
\]
where \( -\Delta_\lambda \) is the positive laplacian on \( \mathcal{H}_\lambda \) for each \( \lambda \in \Lambda \), \( || \cdot ||_\mathcal{H} \) is the canonical metric on \( \mathbb{Z}^d \), and \( V \) and \( \varphi \) are real valued continuous functions on \( \mathbb{R} \) vanishing at infinity. Then the one-parameter group of \(*\)-automorphisms \( \alpha^\lambda_t \) on \( \mathcal{B}(L^2(\mathbb{R}^{d|\Lambda|})) \) defined by
\[
\alpha^\lambda_t(A) := e^{itH_\Lambda} A e^{-itH_\Lambda}, \quad A \in \mathcal{B}(L^2(\mathbb{R}^{d|\Lambda|}))
\]
leaves \( \pi_0(\mathcal{R}(c_c(\Lambda), \sigma)) \) where \( \pi_0 \) is the Schrödinger representation on \( L^2(\mathbb{R}^{d|\Lambda|}) \).

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As in the case of section 3, we define the one-parameter group of \( \ast \)-automorphisms \( \alpha^\Lambda \) on \( \mathcal{R}(c_c(\Lambda), \sigma) \) defined by
\[
\alpha^\Lambda_t(A) := \pi_0^{-1}(\alpha^\Lambda_t(\pi_0(A))), \quad A \in \mathcal{R}(c_c(\Lambda), \sigma).
\]

By using the above theorem and lemma 3.1 and lemma 3.2 in [11], we get the following theorem.

**Theorem 5.25.** [11, Proposition 3.4.]

The thermodynamic limit
\[
\alpha_t = \lim_{\Lambda \to \mathbb{Z}^d} \alpha^\Lambda_t
\]

exists pointwise on \( \mathcal{R}(c_c(\mathbb{Z}^d), \sigma) \) in norm topology, \( t \in \mathbb{R} \), and defines a one-parameter group of \( \ast \)-automorphisms \( \alpha \) on \( \mathcal{R}(c_c(\mathbb{Z}^d), \sigma) \) with potential and nearest neighbor interactions all over \( \mathbb{Z}^d \).

D. Buchholz showed the existence of regular primary \((\alpha, \beta)\)-KMS state at high temperature [11, Proposition 4.7.]. Using the technique of the proof of [13, Theorem 5.8.] and [13, Theorem 1.3.], we have the following proposition.

**Proposition 5.26.** There exists a regular \((\alpha, \beta)\)-KMS state on \( \mathcal{R}(c_c(\mathbb{Z}^d), \sigma) \) for any \( \beta > 0 \).

**References**


References


1 Pauli-Fierz model

In this paper we are concerned with the so-called semi-relativistic Pauli-Fierz model in quantum electrodynamics, which is abbreviated as SRPF model. SRPF model is a relativistic version of the so-called Pauli-Fierz model which has been studied so far. Before explaining SRPF model we should review results obtained for the Pauli-Fierz model.

The Pauli-Fierz Hamiltonian is defined by

$$H_{PF} = \frac{1}{2}(p_x - A(x))^2 + V + H_{rad}. \quad (1.1)$$

Below we give definitions of notations appeared in (1.1). Operator $H_{PF}$ is a linear operator defined on the Hilbert space $\mathcal{H}$ given by the tensor product of Hilbert spaces:

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F},$$

where $\mathcal{F}$ denotes the boson Fock space over one particle state space $L^2(\mathbb{R}^3 \times \{1, 2\}) = W$, i.e,

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} [\otimes^n s W]$$

with symmetric $n$-fold tensor product $\otimes^n s , p_x = (-i\nabla_1, -i\nabla_2, -i\nabla_3)$ denotes the momentum operator for the matter (= electron) and $V : \mathbb{R}^3 \to \mathbb{R}$ an external potential. $H_{rad}$ is a self-adjoint operator acting on $\mathcal{F}$, which denotes the free field Hamiltonian defined by the second quantization of the multiplication operator by $|k| = \omega(k)$, i.e.,

$$H_{rad} = d\Gamma(\omega).$$

Finally in order to define the quantized radiation field $A_\mu(x)$, we introduce the annihilation operator and the creation operator. For $f \in W$, let $a^\dagger(f) : \mathcal{F} \to \mathcal{F}$ be the creation operator defined by

$$(a^\dagger(f)\Phi)^{(n+1)} = \sqrt{n + 1} S_{n+1} \left( f \otimes \Phi^{(n)} \right),$$

where $S_{n+1}$ is the symmetrizer on $\otimes^{n+1} W$. Then the annihilation operator $a(f)$ is given by the adjoint of $a^\dagger(f)$:

$$a(f) = (a^\dagger(f))^*.$$
It is satisfied that \([a(f), a^\dagger(g)] = (\bar{f}, g)\). Then for each \(x \in \mathbb{R}^3\), quantized radiation field \(A_\mu(x)\) is given by
\[
A_\mu(x) = a^\dagger(f_\mu^x) + a(\bar{f}_\mu^x),
\]
where \(f_\mu^x(k, j) = \frac{\varphi(k)}{\sqrt{\omega(k)}} e_{\mu}^j(k) e^{-ikx}\) and \((e^1(k), e^2(k), k/|k|)\) is a right-hand system in \(\mathbb{R}^3\) for each \(k \in \mathbb{R}^3\). Suppose that \(\hat{\varphi} = \hat{\varphi} = \hat{\varphi} = \hat{\varphi} = \hat{\varphi}\) and \(p_{\sqrt{\omega}}\varphi, \sqrt{\omega}\varphi, \varphi/\sqrt{\omega}, \varphi/\omega \in L^2(\mathbb{R}^3)\) throughout this paper. Let \(V\) be relatively bounded with respect to \(-\Delta\) with a relative bound strictly smaller than one. Then \(H_{PF}\) is self-adjoint on \(D(-\Delta) \cap D(H_{rad})\) and bounded from below. Moreover it is essentially self-adjoint on any core of \(-\Delta + H_{rad}\).

The spectral properties of \(H_{PF}\) have been studied in the last two decades, in particular special attentions have been payed for studying the so-called ground state. In general, eigenvectors associated with the bottom of the spectrum of self-adjoint operator \(K\) is called the ground state of \(K\). We note that the existence of ground states does not necessarily hold true. Under some condition it is proven that \(H_{PF}\) has the unique ground state. This fact is not trivial due to the zero spectral gap, i.e., the bottom of the spectrum of \(H_{PF}\) is the edge of the continuous spectrum.

2 Semi-relativistic Pauli-Fierz model and Feynman-Kac formula

As is seen above \(H_{PF}\) can be regarded as the minimal coupling of the decoupled Hamiltonian \(-\frac{1}{2}\Delta + V + H_{rad}\) by \(A(x)\). The SRPF Hamiltonian is defined by the Schrödinger operator \(-\frac{1}{2}\Delta + V\) replaced by the semi-relativistic Schrödinger operator \(\sqrt{-\Delta + M^2}\).

We give the definition of SRPF Hamiltonian. It is however not straightforward to define SRPF Hamiltonian as a self-adjoint operator due to non-local kinetic term. The lemma below is a key fact to define SRPF Hamiltonian.

Lemma 2.1 ([5]) Suppose that \(\omega^{3/2} \hat{\varphi}, \hat{\varphi}/\sqrt{\omega} \in L^2(\mathbb{R}^3)\). Then \((p_x - A(x))^2 + M^2\) is essentially self-adjoint on \(D(\Delta) \cap C^\infty(N)\), where \(N\) denotes the number operator and \(C^\infty(N) = \cap_{k=1}^\infty D(N^k)\).

We denote the closure of \((p_x - A(x))^2 + M^2\) by simply the same notation \((p_x - A(x))^2 + M^2\), which is self-adjoint. Hence we can define the self-adjoint operator
\[
T = \sqrt{(p_x - A(x))^2 + M^2}
\]
by the spectral resolution of \((p_x - A(x))^2 + M^2\).

Definition 2.2 Suppose that \(\omega^{3/2} \hat{\varphi}, \hat{\varphi}/\sqrt{\omega} \in L^2(\mathbb{R}^3)\). Then \(H_{SRPF}\) is defined by
\[
H_{SRPF} = T + H_f + V,
\]
where \(+\) denotes the quadratic form sum. From now on we write \(H\) for \(H_{SRPF}\) for notational convenience.

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In order to construct a functional integral representation (Feynman-Kac type formula) we prepare probabilistic notations. Let \((B_t)_{t \in \mathbb{R}}\) be 3-dimensional Brownian motion on a probability space \((\Omega_p, B_p, P^p)\) and \((T_t)_{t \geq 0}\) be a subordinator defined on a probability space \((\Omega, B_\nu, \nu)\) such that

\[
\mathbb{E} \left[ e^{-uT_t} \right] = e^{-t(\sqrt{2u+M^2} - M)} \text{ for } u \geq 0.
\]

We are concerned with \(H_{SRPF}\) by means of a functional integral. Let \(f, g \in L^2(\mathbb{R}^d)\) and \(X_t = B_{T_t}\) for \(t \geq 0\). Then it is well known that

\[
\int_{\mathbb{R}^3} dx \mathbb{E}_p \left[ f(X_0) g(X_t) \right] = (f, e^{-th} g),
\]

where \(h\) denotes the semi-relativistic Schrödinger operator:

\[
h = \sqrt{-\Delta + M^2} - M.
\]

Furthermore we can see that

\[
\int_{\mathbb{R}^3} dx \mathbb{E}_p \left[ f(X_0) g(X_t) e^{-\int_0^t V(B_s) ds} \right] = (f, e^{-t(h+V)} g)
\]

Let us consider the field part. Let \((Q, \mu)\) be a probability space and \((\phi(f), f \in \oplus^3 L^2(\mathbb{R}^3))\) a Gaussian random variable indexed by \(f \in \oplus^3 L^2(\mathbb{R}^3)\) such that the mean is zero and the covariance is given by

\[
\mathbb{E}_\mu [\phi(f)\phi(g)] = \frac{1}{2} (\hat{f}, D\hat{g}),
\]

where \(D = D(k) = \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2} \right)_{1 \leq \mu, \nu \leq 3}\) is a 3 \times 3 matrix. Also we define a probability space \((Q_E, \mu_E)\) and the Gaussian random variable \((\phi_E(f), f \in \oplus^3 L^2(\mathbb{R}^4))\) such that the mean is zero and the covariance is given by

\[
\mathbb{E}_{\mu_E} [\phi_E(f)\phi_E(g)] = \frac{1}{2} (\hat{f}, D \otimes 1\hat{g}).
\]

It is well-known that \(\mathcal{F} \cong L^2(Q, d\mu)\) and \(A_\mu(x) \cong \phi(\oplus_{\nu=1}^3 \delta_{\mu\nu} \bar{\varphi}(\cdot - x))\), where \(\bar{\varphi} = (\varphi / \sqrt{\omega})^\vee\). Moreover there exists a family of isometries \((J_t)_{t \in \mathbb{R}}\) such that \(J_t : L^2(Q_E) \rightarrow L^2(Q_E)\) with

\[
J_t^* J_s = e^{-|t-s| H_{rad}},
\]

where \(H_{rad}\) denotes the free field Hamiltonian in \(L^2(Q)\), which is unitary equivalent to \(H_{rad}\) in \(\mathcal{F}\). Then we define

\[
\tilde{H} = \sqrt{\left( p_x - \tilde{A}(x) \right)^2 + M^2 + V + H_{rad}}
\]

in \(L^2(\mathbb{R}^3) \otimes L^2(Q)\), where \(\tilde{A}_\mu(x) = \phi(\oplus_{\nu=1}^3 \delta_{\mu\nu} \bar{\varphi}(\cdot - x))\). It is seen that \(H \cong \tilde{H}\). Under this identification we consider \(\tilde{H}\) instead of \(H\) in what follows.
Theorem 2.3 ([7]) It follows that

\[ (F, e^{-t\tilde{H}} G) = \int_{\mathbb{R}^3} dx [e^{-i\tilde{\phi}_E(K_t)} J_t F(B_t), e^{-i\tilde{\phi}_E(K_t)} J_t G(B_t) e^{-\int_{0}^{t} V(B_s) ds}], \]

where

\[ K_t = \oplus_{t=1}^{3} \int_{0}^{T_t} \tilde{\phi}(\cdot - B_s) ds \]

with \( T_s = \inf\{ t \geq 0 | T_t = s \} \), and \( J_t : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^4) \) is defined by

\[ j_t \tilde{f}(k_0 k) = \frac{e^{-itk_0}}{\sqrt{\pi}} \frac{\sqrt{\omega(k)}}{\sqrt{\omega(k)^2 + |k_0|^2}} \hat{f}(k), \quad (k, k_0) \in \mathbb{R}^3 \times \mathbb{R}. \]

A crucial point of the functional integral representation is that an interaction term is put together as \( e^{-i\tilde{\phi}_E(K_t)} \). The immediate corollary is to specify the domain of \( e^{H} \).

Theorem 2.4 ([7, 2]) Suppose that \( \omega^{3/2} \tilde{\phi}, \tilde{\phi}/\sqrt{\omega} \in L^2(\mathbb{R}^3) \) and \( V \) is relatively bounded with respect to \( \sqrt{-\Delta} \) with a relative bound strictly smaller than one. Then \( \tilde{H} \) is self-adjoint on \( D(\sqrt{-\Delta}) \cap D(\tilde{H}_{\text{rad}}) \).

Proof We show the outline of the proof. Using the functional integral representation we can show that

\[ e^{-t\tilde{H}} D(\sqrt{-\Delta}) \cap D(\tilde{H}_{\text{rad}}) \subset D(\sqrt{-\Delta}) \cap D(\tilde{H}_{\text{rad}}) \]

which yields that \( \tilde{H}_0 \) is essentially self-adjoint on \( D(\sqrt{-\Delta}) \cap D(\tilde{H}_{\text{rad}}) \). Next we can show the bound

\[ \|\sqrt{-\Delta} F\| + \|\tilde{H}_{\text{rad}} F\| \leq C \|\tilde{H}_0 F\|, \quad (2.1) \]

where \( \tilde{H}_0 \) is \( \tilde{H} \) with \( V \) replaced by 0. From (2.1) it follows that \( \tilde{H}_0 \) is self-adjoint on \( D(\sqrt{-\Delta}) \cap D(\tilde{H}_{\text{rad}}) \). Furthermore we can see that \( V \) is relatively bounded with respect to \( \tilde{H}_0 \). Hence \( \tilde{H} \) is self-also adjoint on \( D(\sqrt{-\Delta}) \cap D(\tilde{H}_{\text{rad}}) \) by the Kato-Rellich theorem.

3 Existence and uniqueness of ground state

In this section we review the existence and the uniqueness of the ground state of \( \tilde{H} \). Let us assume that \( M > 0 \). In this case the existence of ground state has been shown in e.g., [8, 9] for \( M > 0 \) and \( m \geq 0 \). Now we suppose that \( M = 0 \). In this case Hamiltonian under consideration is of the form

\[ |p_x - A(x)| + V + \tilde{H}_{\text{rad}}. \]

Hence the kinetic term is not smooth function of \( \frac{1}{2} (p_x - A(x))^2 \), which is a serious disadvantage to show the existence of the ground state. In [2] it is shown that

\[ \sigma(H) = \{E\} \cup [E + m, \infty) \]
under the assumption $\omega(k) = \sqrt{|k|^2 + m^2}$. The most singular case is $m = M = 0$, but the existence of the ground state is established in $[3]$. 

Theorem 3.1 ([3]) Suppose that $\omega^{3/2} \varphi, \varphi/\sqrt{\omega} \in L^2(\mathbb{R}^3)$, $\lim_{|x| \to \infty} V(x) = \infty$ and $\|\nabla V\|_{\infty} < \infty$. Then $\tilde{H}$ has the ground state.

Proof We introduce an artificial boson mass $m > 0$. In this case the existence of normalized ground state $\Psi_m$ is established. It is enough to show that the weak limit of $\Psi_m$,

$$w - \lim_{m \to 0} \Psi_m = \Psi_0,$$

is non-zero. In order to avoid infrared divergence we introduce $\tilde{H}_R$ which is defined by $\tilde{H}$ with $\varphi(-x)$ replaced by $\varphi(-x) - \varphi(\cdot)$. Note that $\tilde{H}_R \cong \tilde{H}$. By an application of asymptotic annihilation operator:

$$a_\infty(f) = \lim_{t \to \infty} e^{i\tilde{H}_R} e^{-it\mathcal{N}} a(f) e^{it\mathcal{N}} e^{-it\tilde{H}_R},$$

we can see that $a_\infty(f) \Psi_m = 0$. By the Cook method argument, we then have

$$a(f) \Psi_m = -\int_{\mathbb{R}^3} f(k) \left( \tilde{H}_R - E + \omega(k) \right)^{-1} c_j(k) \langle x \rangle^2 \Psi_m dk,$$

where $c_j(k)$ is a bounded operator for each $k \in \mathbb{R}^3$, and $\langle x \rangle = \sqrt{|x|^2 + 1}$. Let $N$ be the number operator in $\mathcal{F}$. From this formula we can see that

1. $\|N \Psi_m\| \leq C \|\langle x \rangle^2 \Psi_m\|$, where $C$ is independent of $m$.
2. $\sup_m \|\Psi_m^{(n)}\|_{W^{1,p}(\Omega)} < \infty$ for $1 \leq p < 2$ and any compact set $\Omega \subset \mathbb{R}^{3n}$.
3. $\|e^{\epsilon |x|} \Psi_m\| < \infty$.

By (1) - (3) above, we can see that $\Phi_m$ strongly converges to $\Phi_0$. Thus $\Phi_0 \neq 0$ follows.

Finally we show the uniqueness of ground state.

Corollary 3.2 ([5, 7]) Let $t \geq 0$. Then $e^{i\frac{\epsilon}{2} N} e^{-t \tilde{H}} e^{-i\frac{\epsilon}{2} N}$ is positivity improving. In particular when $\tilde{H}$ has a ground state, it is unique up to multiple constants.

4 Decay of bound states

In this section we discuss a martingale property of $\tilde{H}$, which can be applied to spatial decay of bound states. Let $h = -\frac{\epsilon}{2} \Delta + V$ be a Schrödinger operator with an external potential $V$. Define

$$X_t(x) = e^{tE} e^{-\int_0^t V(B_s + x) ds} f(B_t + x),$$

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where \( f \) denotes a bound state such that \( hf = Ef \). Then it can be checked that \( (X_t(x))_{t \geq 0} \) is a martingale with respect to the natural filtration of Brownian motion. We would like to extend this to quantum field theory. Define

\[
H_t(x) = e^{tE}e^{-\int_0^t V(B_{Ts} + x)ds}e^{-i\phi(K_t(x))}J_t \Phi(B_{T_1} + x)
\]

where \( \tilde{H} \Phi = E \Phi \).

**Theorem 4.1** ([7]) Let \( V \) be relativistic Kato decomposable potential. Then there exists a filtration \( (M_t)_{t \geq 0} \) such that \( (H_t(x))_{t \geq 0} \) is a martingale, i.e.,

\[
E^0.0_E[H_t(x)|M_s] = H_s(x) \text{ for } t > s.
\]

An application of the martingale property is to show the spatial decay of bound state \( \Phi \).

**Corollary 4.2** Let \( \tau \) be a stopping time with respect to \((M_t)_{t \geq 0}\). Then

\[
\|\Phi_b(x)\|_{L^2(Q)} \leq \|\Phi_b\|_E \left[ e^{-\int_0^{\tau} (V(Z_t+x) - E)dt} \right]
\]

where \( Z_t = B_{T_t} \).

**Proof** By Theorem 4.1 we see that \( (J_0 \Phi \cdot H_t(x))_{t \geq 0} \) is an \( L^2(QE) \)-valued martingale. Hence \( (J_0 \Phi \cdot H_{t \wedge \tau}(x))_{t \geq 0} \) is also martingale which implies that

\[
E^0.0_E [J_0 \Phi \cdot H_t(x)] = E^0.0_E [J_0 \Phi \cdot H_{t \wedge \tau}(x)].
\]

We have

\[
\|\Phi_b(x)\|_{L^2(Q)} = \sup_{\|\Phi\| = 1} E^0.0_E [J_0 \Phi \cdot H_t(x)]
\leq \sup_{\|\Phi\| = 1} E^0.0_E [J_0 \Phi \cdot H_{t \wedge \tau}(x)] \leq \|\Phi\|_E \left[ e^{-\int_0^{\tau} (V(Z_t+x) - E)dt} \right].
\]

Spatial decay properties can be derived immediately from the lemma above.

**Corollary 4.3** ([7]) Let \( V \) be relativistic Kato decomposable.

1. Suppose that \( \lim_{|x| \to \infty} V(x) + E < 0 \).
2. \((m > 0)\) there exists constant \( C \) such that

\[
\|\Phi_b(x)\| \leq Ce^{-|x|}\|\Phi_b\|,
\]

\((m = 0)\) there exists constant \( C \) such that

\[
\|\Phi_b(x)\| \leq \frac{C}{1 + |x|^4}\|\Phi_b\|.
\]

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2. Suppose that $\lim_{|x| \to \infty} V(x) = \infty$. Then there exist constants $c$ and $C$ such that

$$\|\Phi_b(x)\| \leq C e^{-c|x|}.$$ 

**Proof** (1) We take $\tau_R = \inf \{s|z_s + x| < R\}$, which is a stopping time, and corollary follows from Corollary 4.2. (2) We take $\tau_R = \inf \{s|z_s| > R\}$ which is also a stopping time and the corollary follows from Corollary 4.2.

5 Path measures associated with the ground state

For SRPF Hamiltonian we can consider the path measure associated with the ground state. The path measure of this kind is useful to study ground state expectation with respect to observables for the Nelson model and spin-boson model \[1, 4\] in scalar quantum field theory. Unfortunately the path measure of SRPF Hamiltonian cannot be applied as those models do. In this section we only show the existence of the path measure for SRPF Hamiltonian. Let $\phi_2 \in L^2(\mathbb{R}^3)$ be positive. Define the family of probability measures $\mu_T, T > 0$, by

$$
\mu_T(A) = \frac{1}{Z_T} \int_{\mathbb{R}^3} dx \mathbb{E}_{\mathbb{P}_{T \times \nu}} \left[ \mathbb{1}_A \phi(B_{-T_t}) \phi(B_{T_t}) e^{-\int_t^t V(z_s) ds} \right],
$$

where $\xi = \frac{1}{2}(K_t, DK_t)$ and $A \in \mathcal{G}$. Here $\mathcal{G} = \cup_{s \geq 0} \mathcal{F}_{[-s, s]}$ and $\mathcal{F}_{[-s, s]} = \sigma (Z_r : r \in [-s, s])$.

$$
Z_r = B_{T_r} = \begin{cases} 
B_{T_r} & r > 0 \\
B_{-T_r} & r < 0.
\end{cases}
$$

$\xi$ plays a role of a pair interaction in a Gibbs measure, but we do not mention it here. Let $\mathfrak{X} = \Omega_p \times \Omega_{\nu}$.

**Theorem 5.1** There exists a probability measure $\mu_\infty(A)$ on $(\mathfrak{X}, \sigma(\mathcal{G}))$ such that

$$
\lim_{T \to \infty} \mu_T(A) = \mu_\infty(A) \quad \forall A \in \mathcal{G}.
$$

**Proof** The main idea of the proof is to show that

$$
\mu_T(A) = e^{2Es} \int_{\mathbb{R}^3} dx \mathbb{E}_{\mathbb{P}} \left[ \mathbb{1}_A \left( \frac{\phi_{T-s}(Z_s)}{\|\phi\|}, J_{[-s, s]} \frac{\phi_{T-s}(Z_s)}{\|\phi_T\|} \right) \right],
$$

where $\phi_s = e^{-s(\tilde{H} - E)} \phi \otimes \mathbb{1}$ and

$$
J_{[-t,t]} = J_{-t} e^{-\int_t^t V(z_s) ds} e^{-i \phi E(K_t)} J_t.
$$

Since $\tilde{H}$ has the ground state, we can see that $\frac{\phi_{T-s}}{\|\phi_T\|} \to e^{sE} \Psi_g$ as $T \to \infty$. Hence we have

$$
\lim_{T \to \infty} \mu_T(A) = e^{2Es} \int_{\mathbb{R}^3} dx \mathbb{E} \left[ \mathbb{1}_A \left( \Psi_g(Z_s), J_{[-s, s]} \Psi_g(Z_s) \right) \right]
$$

and the right hand side above has the extension to the probability measure in $(\mathfrak{X}, \sigma(\mathcal{G}))$.

\[ \square \]
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References


SMOOTHNESS OF STOCHASTIC FLOWS IN NON-RELATIVISTIC QED

OLIVER MATTE

Abstract. Recently, stochastic differential equations associated with models of non-relativistic matter interacting with quantized radiation fields have been analyzed by B. Güneysu, J.S. Möller, and the present author. Here we report on new results addressing the differentiability properties of the corresponding stochastic flow and present a Bismut-Elworthy-Li type formula for the derivatives of the associated semi-group.

1. Introduction

In this proceeding we report on some new results [11] in the stochastic analysis of a mathematical model for non-relativistic (NR) quantum mechanical matter particles interacting with a quantized relativistic radiation field. The prime example covered is the standard model of NR quantum electrodynamics (QED) describing NR electrons interacting with the quantized, ultra-violet cut-off electromagnetic field. We refer to, e.g., [9, 14] for a general introduction to the mathematical analysis of such systems and reference lists related to this extensively studied subject.

In what follows we shall first introduce the model we are interested in (Sect. 2) and recall an existence and uniqueness theorem for an associated stochastic differential equation (SDE) from our recent joint article with B. Güneysu and J.S. Möller [3] as well as a Feynman-Kac (FK) formula [3, 5] (Sect. 3). In Sect. 4, some representation formulas for the solutions of our SDE are recalled and employed to show their smooth dependence on initial conditions. We shall see in Sect. 5 that the derivatives of the solutions are themselves solutions of SDEs, which imply Burkholder-Davis-Gundy (BDG) type bounds on the derivatives. These results permit to study some mapping properties of the associated semi-group and to solve the corresponding heat equation (Sect. 6). In the final Sect. 7, we discuss a Bismut-Elworthy-Li (BEL) formula [1, 2] for the derivatives of the semi-group. If not specified otherwise, all results announced here stem from the forthcoming article [11].

2. Brief description of the model

Our model is given by a Hamiltonian, i.e., a self-adjoint operator generating the quantum mechanical time evolution of the matter-radiation system. The underlying Hilbert space is

$$\mathcal{H} := L^2(\mathbb{R}^\nu; \mathbb{C}^d \otimes \mathcal{F}),$$

for some $\nu, d \in \mathbb{N}$,

where the target space is the tensor product of a $d$-dimensional Hilbert space accounting for spin degrees of freedom (if any) and the state space $\mathcal{F}$ of the quantized
radiation field. Since the latter consists of an undetermined number of bosons,
\[ \mathcal{F} := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \bigotimes_{\ell=1}^{n} \mathfrak{h} \]
is the bosonic Fock space over the one-boson Hilbert space
\[ \mathfrak{h} := L^2(\mathcal{M}, \mathfrak{A}, \mu). \]
In the above formulas \( \oplus \) denotes the symmetric tensor product of Hilbert spaces and \( (\mathcal{M}, \mathfrak{A}, \mu) \) can be any \( \sigma \)-finite measure space such that \( \mathfrak{h} \) is separable.

The particular model treated here is determined by the following objects and hypotheses which are assumed to be fulfilled throughout this proceeding:
(a) The dispersion relation of the bosons, \( \omega : \mathcal{M} \to [0, \infty) \), is assumed to be measurable and \( \mu \)-a.e. strictly positive.
(b) \( \tilde{\nu} \in \mathbb{N} \) and \( \sigma_1, \ldots, \sigma_{\tilde{\nu}} \) are hermitian \( d \times d \)-matrices.
(c) We fix some \( \alpha \geq 1 \) and introduce the auxiliary one-boson Hilbert space
\[ \mathfrak{k}_\alpha := L^2(\mathcal{M}, \mathfrak{A}, (\omega^{-1} + 1 + \omega^{2\alpha})\mu) \subset \mathfrak{h}. \]
The maps \( \mathbb{R}^\nu \ni x \mapsto (G_{1,x}, \ldots, G_{\nu,x}) \in \mathfrak{k}_\alpha \) and \( \mathbb{R}^\nu \ni x \mapsto (F_{1,x}, \ldots, F_{\tilde{\nu},x}) \in \mathfrak{k}_\alpha \) define the matter-radiation coupling. They are both assumed to be smooth with bounded partial derivatives of any order. Furthermore, we assume the existence of a conjugate linear isometry \( C : \mathfrak{h} \to \mathfrak{h} \) commuting with \( \omega \) such that
\[ CG_{\ell,x} = G_{\ell,x} \quad \text{and} \quad CF_{j,x} = F_{j,x}, \]
for all \( \ell, j, \) and \( x \). We abbreviate
\[ q_x := \sum_{\ell=1}^{\nu} \partial_{x_\ell} G_{\ell,x}, \quad x \in \mathbb{R}^\nu. \]
(d) The electrostatic potential \( V : \mathbb{R}^\nu \to \mathbb{R} \) is smooth with bounded partial derivatives of any order.

See, e.g., [3, App. 1] for the precise choices of \( \omega \) and the \( \sigma_j, F_j, G_\ell \) in the standard model of NRQED. If the ultra-violet cut-off is implemented by a Schwartz function, then every value of \( \alpha \geq 1 \) is allowed for in applications to NRQED.

We shall further employ the following standard notation for operators in \( \mathcal{F} \); see, e.g., [9, 14]: The standard bosonic creation and annihilation operators in \( \mathcal{F} \) corresponding to some \( f \in \mathfrak{h} \) are denoted by \( a^\dagger(f) \) and \( a(f) \), respectively. Then \( \varphi(f) \), the closure of \( a^\dagger(f) + a(f) \), is the corresponding self-adjoint field operator. The symbol \( d\Gamma(\omega) \) denotes the differential second quantization of the maximal multiplication operator associated with \( \omega \).

For every fixed \( x \in \mathbb{R}^\nu \), we now define an operator in \( \mathbb{C}^d \otimes \mathcal{F} \) with domain of definition \( \mathcal{C}^d \otimes \mathcal{D}(d\Gamma(\omega)) \) by
\[ \hat{H}(x) := \mathbb{1}_{\mathcal{C}^d} \otimes \left\{ \frac{1}{2} \sum_{\ell=1}^{\nu} \varphi(G_{\ell,x})^2 - i \frac{1}{2} \varphi(q_x) + d\Gamma(\omega) + V(x) \mathbb{1}_\mathcal{F} \right\} \]
\[ - \sum_{j=1}^{\tilde{\nu}} \sigma_j \otimes \varphi(F_{j,x}). \]
(2.1)
This operator is well-defined and closed; its adjoint has the same domain and is given by the same formula with only the minus sign in front of the second term in the curly brackets in (2.1) flipped to a plus sign [3, App. 2]. Let \( \mathcal{D}(\hat{H}) \) be the vector space of all \( \Psi \in C^2(\mathbb{R}^\nu, \mathbb{C}^d \otimes \mathcal{F}) \cap C(\mathbb{R}^\nu, \mathbb{C}^d \otimes \mathcal{D}(d\Gamma(\omega))) \) such that
\[ \partial_{x_1} \Psi, \ldots, \partial_{x_\nu} \Psi \in C(\mathbb{R}^\nu, \mathcal{C}^d \otimes \mathcal{D}(d\Gamma(\omega)^{1/2})) \], where the domains of powers of \( d\Gamma(\omega) \) are equipped with their graph norms. Then we further set

\[
(\hat{H}\Psi)(x) := \frac{1}{2} \sum_{\ell=1}^\nu (\partial^2_{x_\ell} - 2i\varphi(G_{\ell,x})\partial_{x_\ell})\Psi(x) + \hat{H}(x)^*\Psi(x), \quad x \in \mathbb{R}^\nu,
\]

for all \( \Psi \in \mathcal{D}(\hat{H}) \). The restriction of \( \hat{H} \) to \( \mathcal{H} \) is essentially self-adjoint \([4, 6]\) and its closure,

\[
(2.2) \quad H := \hat{H}|_{\mathcal{H}},
\]

is the afore-mentioned Hamiltonian for our matter-radiation system.

3. Stochastic differential equations and Feynman-Kac formula

In what follows, the interval \( I \) is either \([0, \infty)\) or \([0, \tau]\), for some \( \tau > 0 \), and \( \mathcal{B} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in I}, \mathbb{P}) \) is a filtered probability space satisfying the usual assumptions. If \( \mathcal{H} \) is a separable Hilbert space, then the space of continuous \( \mathcal{H} \)-valued semimartingales with respect to \( \mathcal{B} \) is denoted by \( \mathcal{S}_I(\mathcal{H}) \). Moreover, \( \mathcal{B} = (B_1, \ldots, B_\nu) \in \mathcal{S}_{[0, \infty)}(\mathbb{R}^\nu) \) is a \( \nu \)-dimensional Brownian motion with respect to \( \mathcal{B} \) with covariance matrix \( \mathbb{1}_{\mathbb{R}^\nu} \) and the process \( X = (X_1, \ldots, X_\nu) \in \mathcal{S}_I(\mathbb{R}^\nu) \) is either identical to \( B \) or it is, up to indistinguishability, the unique solution of the stochastic differential equation

\[
X_t = q + B_t - \int_0^t \frac{X_s}{\tau - s} \, ds, \quad t \in [0, \tau], \quad X_\tau = 0,
\]

for a Brownian bridge from some \( \mathcal{F}_0 \)-measurable \( q : \Omega \to \mathbb{R}^\nu \) to zero. We abbreviate \( X^x := x + X \) and \( B^x := x + B \), for any \( x \in \mathbb{R}^\nu \).

In this situation the article \([3]\) provides the following existence and uniqueness result for a SDE associated with the model introduced in the previous section:

**Theorem 3.1.** Let \( x \in \mathbb{R}^\nu \). Then, for all \( t \in I \), there exist \( \mathcal{F}_t \)-\( \mathcal{B}(\mathcal{C}^d \otimes \mathcal{F}) \)-measurable maps

\[
W_t[X^x] : \Omega \to \mathcal{B}(\mathcal{C}^d \otimes \mathcal{F}),
\]

such that \((t, x, \gamma) \mapsto W_t[X^x](\gamma)\) is \( \mathcal{B}(I \times \mathbb{R}^\nu) \otimes \mathcal{F} \)-\( \mathcal{B}(\mathcal{C}^d \otimes \mathcal{F}) \)-measurable with a separable image in \( \mathcal{B}(\mathcal{C}^d \otimes \mathcal{F}) \) and such that, for every \( \mathcal{F}_0 \)-measurable \( \eta : \Omega \to \mathcal{C}^d \otimes \mathcal{D}(d\Gamma(\omega)) \), the process \((W_t[X^x]\eta)_{t \in I}\) has the following properties:

(i) it belongs to \( \mathcal{S}_I(\mathcal{C}^d \otimes \mathcal{F}) \);

(ii) its paths are \( \mathbb{P} \)-a.s. in \( \mathcal{C}(I, \mathcal{C}^d \otimes \mathcal{D}(d\Gamma(\omega))) \);

(iii) it \( \mathbb{P} \)-a.s. solves the following stochastic differential equation on \([0, \sup I]\),

\[
Y_s(x, \eta) = \eta - \int_0^s \hat{H}(X^x_s)Y_s(x, \eta) \, ds + \sum_{\ell=1}^\nu \int_0^s i\varphi(G_{\ell,x}^x_s)Y_s(x, \eta) \, dX_{\ell,s}.
\]

Up to indistinguishability, \((W_t[X^x]\eta)_{t \in I}\) is the only process satisfying (i)–(iii). Furthermore,

\[
\|W_t[X^x]\| \leq \exp \left( c \sup_{y \in \mathbb{R}^\nu} \sum_{j=1}^\nu \|\omega^{-1/2} F_{j,y}\|^2 - \int_0^t V(X^x_s) \, ds \right), \quad t \in I,
\]

with a constant \( c > 0 \) depending only on \( \max_{j=1}^\nu \|\sigma_j\| \).
In the case $X = B$ we shall abbreviate

\[(3.3) \quad W_t(x) := W_t(B^x), \quad t \geq 0, \; x \in \mathbb{R}^n.\]

In view of (3.2) the following $C^d \otimes \mathcal{F}$-valued expectations with respect to $\mathbb{P}$ are well-defined, for all $\Psi \in L^p(\mathbb{R}^n, C^d \otimes \mathcal{F})$ with $p \in [1, \infty]$,

\[(3.4) \quad (T_t \Psi)(x) := \mathbb{E}[W_t(x)^* \Psi(B^x)], \quad t \geq 0, \; x \in \mathbb{R}^n.\]

We then have the following FK formula [3, 5]:

**Theorem 3.2.** Let $t \geq 0$ and $\Psi \in \mathcal{H}$. Then $e^{-tH}\Psi = T_t \Psi$.

In the scalar case $d = 1$, a FK formula in NRQED has been derived first in [5] with the help of some ideas from [12]. For $d = 2$, a weaker version of the FK formula, representing matrix elements of the semi-group by limits of expectations of certain regularized FK integrands, has been derived in [7]. For $d > 1$, the FK formula as stated in Thm. 3.2 has been proved first in [3]. That the FK integrands actually give rise to solutions of (3.1) is a result of [3] as well.

Notice that the adjoint of $W_t(x)$ appears in (3.4) which at some later point causes a technical issue (cf. the first paragraphs of Sect. 6). We can, however, get rid of the adjoint by the following constructions:

Let $\tau > 0$ and consider the time-reversed process $\mathcal{R}_\tau B$ given by $(\mathcal{R}_\tau B)_t := B_{\tau-t}$, for all $t \in [0, \tau]$. Define the associated filtration $(\mathcal{F}_t)_{t \in [0, \tau]}$ as the standard completion of the filtration $(\mathcal{G}_t)_{t \in [0, \tau]}$ with $\mathcal{G}_t$ denoting the $\sigma$-algebra generated by $B_t$ and all increments $B_s - B_t$ with $0 \leq t < s \leq \tau$. Then $\mathcal{R}_\tau B$ is the up to indistinguishability unique solution of the following SDE for a Brownian bridge from the $\mathcal{G}_0$-measurable initial condition $B_\tau$ to zero,

\[b_t = B_\tau + \tilde{B}_t - \int_0^t \frac{b_s}{\tau - s} \, ds, \quad t \in [0, \tau], \quad b_\tau = 0.\]

Here $\tilde{B}$ is a new Brownian motion on $[0, \tau]$ with respect to the filtration $(\mathcal{F}_t)_{t \in [0, \tau]}$; see [13]. In particular, we may apply Thm. 3.1 to $\mathcal{R}_\tau B$. Another result of [3] then implies that

\[(3.5) \quad W_\tau(x)^* = W_\tau[(\mathcal{R}_\tau B)^x].\]

4. **Formulas for the solution processes**

The proof of Thm. 3.1 in [3] is based on some fairly explicit representation formulas for $W[X^x]$ that we again exploit in [11] to learn more about the pointwise (on $\Omega$) regularity of the map $x \mapsto W[X^x]$.

To explain this in more detail we start by recalling the definition of a family of isometries $j_t : \mathfrak{F}_\alpha \to \mathfrak{F}_\alpha$, $t \in \mathbb{R}$, introduced in [12], where

\[\mathfrak{F}_\alpha := L^2(\mathbb{R} \times \mathcal{M}, [\omega^{-1} + \omega^{2\alpha}] \lambda \otimes \mu),\]

with $\lambda$ denoting the Lebesgue-Borel measure on $\mathbb{R}$. These isometries are given by

\[(4.1) \quad j_t f(k_0, k) := \pi^{-1/2} e^{-ik_0 \omega(k)} (\omega(k)^2 + k_0^2)^{-1/2} f(k),\]

for all $t \in \mathbb{R}$, $f \in \mathfrak{F}_\alpha$, and a.e. $(k_0, k) \in \mathbb{R} \times \mathcal{M}$. They are defined such that $j_t^* j_s = e^{-|s-t|} \omega$, $s, t \in \mathbb{R}$, which indicates their usefulness in the present context. Both maps $\mathbb{R} \ni t \mapsto j_t$ and $\mathbb{R} \ni t \mapsto j_t^*$ are strongly continuous. We now define

\[K[X^x] \in S_1(\mathfrak{F}_\alpha), \quad U^- [X^x] \in S_1(\mathfrak{F}_\alpha), \quad U^+ [X^x] \in S_1(\mathfrak{F}_\alpha), \quad u[X^x] \in S_1(\mathbb{R}),\]
by setting, for all \( t \in I \) and \( x \in \mathbb{R}^\nu \),

\[
K_t[X^x] := \sum_{\ell=1}^{\nu} \int_0^t j_{s,\ell} G_{t,\ell} X^x_s \, ds + \frac{1}{2} \int_0^t j_{s,\ell} q X^x_s \, ds,
\]

\[
U_t^- [X^x] := j_{s,\ell} K_t[X^x], \quad U_t^+ [X^x] := j_{s,\ell} K_t[X^x],
\]

\[
u_t[X^x] := \frac{1}{2} || K_t[X^x] ||^2_{L^2(\mathbb{R}^\nu)} + \int_0^t V(X^x) \, ds.
\]

The non-obvious fact that \( U^+(x) \) is a semi-martingale is shown in [3]. By repeatedly applying the Kolmogorov test lemma to double difference quotients of the map \( x \mapsto K(x) \) (a method we learned from [8]) we derive the following result:

**Lemma 4.1.** For all \( x \in \mathbb{R}^\nu \), we can choose a modification of \( K[X^x] \) such that, for every elementary event \( \gamma \in \Omega \), the maps \( \mathbb{R}^\nu \ni x \mapsto K_t[X^x](\gamma) \in \hat{\mathcal{F}}_\alpha \) with \( t \in I \) are smooth and the maps \( I \times \mathbb{R}^\nu \ni (t, x) \mapsto \partial^0_{x} K_t[X^x](\gamma) \in \hat{\mathcal{F}}_\alpha \) with multi-indices \( \beta \in \mathbb{N}_0^\nu \) are continuous. In particular, the \( \hat{\mathcal{F}}_\alpha \)-valued processes \( U^\pm[X^x] \) and the real-valued process \( u[X^x] \) have analogous regularity properties.

For all \( t > 0 \), \( \ell \in \mathbb{N}_0 \), and \( f_1, \ldots, f_\ell, g \in \mathfrak{f}_0 := L^2(\mathcal{M}, (\omega^{-1} + 2)\mu) \), we further define

\[
F_{t,\ell}(f_1, \ldots, f_\ell, g) := \sum_{n=0}^{\infty} \frac{1}{n!} a^+(f_1) \cdots a^+(f_\ell) a^+(g)^n e^{-t\mu(\omega)}. 
\]

This series converges absolutely in the operator norm on \( \mathcal{B}(\hat{\mathcal{F}}) \). The map \( \mathfrak{f}_0 \ni g \mapsto F_{0,\ell}(g) \in \mathcal{B}(\hat{\mathcal{F}}) \) is analytic and its \( \ell \)-th derivative at \( g \) is precisely the \( \ell \)-linear map \( F_{t,\ell}(\cdot, g) \). All derivatives of \( F_{0,\ell} \) depend continuously on \( (t, g) \in (0, \infty) \times \mathfrak{f}_0 \) with respect to the natural norm on the set of \( \ell \)-linear maps from \( \mathfrak{f}_0 \) to \( \mathcal{B}(\hat{\mathcal{F}}) \).

In the case \( d = 1 \), the process \( W_t[X^x] \) is now given by [3]

\[
W_t[X^x] := e^{-u_t[X^x]} F_{0,1/2}(iU_t^-[X^x]) F_{0,1/2}(iU_t^+[X^x])^*, \quad t \in I \setminus \{0\}, \ x \in \mathbb{R}^\nu.
\]

Combining this formula with the above remarks on \( F_{0,1} \) and Lem. 4.1, we arrive at the following theorem, at least in the case \( d = 1 \):

**Theorem 4.2.** There exist modifications of \( W_t[X^x] \) such that, for every \( \gamma \in \Omega \), the maps \( \mathbb{R}^\nu \ni x \mapsto W_t[X^x](\gamma) \in \mathcal{B}(C^d \otimes \hat{\mathcal{F}}) \) with \( t \in I \) are smooth and the maps \( (I \setminus \{0\}) \times \mathbb{R}^\nu \ni (t, x) \mapsto \partial^0_{x} W_t[X^x](\gamma) \in \mathcal{B}(C^d \otimes \hat{\mathcal{F}}) \) with \( \beta \in \mathbb{N}_0^\nu \) are continuous.

In the matrix-valued case \( d > 1 \), the formula for \( W_t[X^x] \) is given by a norm convergent series of \( \mathcal{B}(C^d \otimes \hat{\mathcal{F}}) \)-valued, \( n \)-fold time-ordered integrals, whose integrands are given by processes similar to the one in (4.3) but involving \( n \) additional creation operators and additional \( \mathcal{B}(C^d) \)-valued processes; see [3, App. 6]. While the corresponding formula is too involved to be stated precisely in the present proceeding, its structure is still simple enough to allow for a pedestrian proof of Thm. 4.2 based on (a slight extension of) Lem. 4.1.

All constructions above can also be applied to the time shifted data

\[
\mathbb{B}_s := (\Omega, \mathfrak{F}, (\mathfrak{F}_{s+t})_{t \geq 0}, \mathbb{P}), \quad \mathbb{B}_t := \mathbb{B}_{s+t} - \mathbb{B}_s,
\]

for any \( s \geq 0 \). When the data \( (\mathbb{B}, \mathbb{B}) \) is replaced by \( (\mathbb{B}_s, \mathbb{B}_s) \) in the definition of \( W_t(x) \), then we denote the resulting random variable by \( W_{s,s+t}(x) \). In fact, we can modify all involved processes \( K \) in such a way that the flow relations

\[
W_{s,t}(\mathbb{B}_{t-s}^x) W_{r,s}(x), \quad t \geq s \geq r \geq 0, \ x \in \mathbb{R}^\nu,
\]

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are satisfied on $\Omega$.

5. Stochastic differential equations for derivatives of the flow

The results of the pointwise (on $\Omega$) regularity analysis of $W$ presented in the previous section ensure that the derivatives of the flow are again given by well-defined $\mathcal{B}(\mathbb{C}^d \otimes \mathcal{F})$-valued maps. The information obtained by the pointwise analysis is, however, not yet sufficient to study the regularity of the functions $T_t \Psi : \mathbb{R}^\nu \to \mathbb{C}^d \otimes \mathcal{F}$ defined in (3.4). In fact, applying triangle inequalities to the terms of the form (4.2) appearing in our formulas for $W$ and its derivatives leads to highly suboptimal bounds on operator norms. In the case $d = 1$, for instance, estimating the operator norm of $W_t$ trivially by means of (4.3) yields the bound $\|W_t\| \leq c e^{-\omega t} \prod_{j=1}^\infty \sum_{n=0}^\infty (n!)^{-1/2} c (1 + (\omega t)^{-1/2}) U_t^\infty \|\eta\|_n$, where we dropped all arguments $[X^x]$. In the example of NRQED, we do not even know whether its right hand side is $\mathbb{P}$-integrable. On the other hand, the bound (3.2) holds true, which is derived by applying Itô’s formula to the process $\|W_s[X^x]\eta\|^2$ and employing (3.1).

Therefore, to obtain useful moment bounds on the derivatives of $W$ we should inductively verify that the processes $\partial_s W[X^x] \eta$ are again solutions of SDEs and employ the latter to derive suitable BDG type bounds. In fact, the following holds:

**Theorem 5.1.** Let $\eta : \Omega \to \mathbb{C}^d \otimes \mathcal{D}(d\Gamma(\omega))$ be $\mathcal{F}_0$-measurable. Then all partial derivatives with respect to components of $x \in \mathbb{R}^\nu$ of the semi-martingales $(W_t[X^x] \eta)_{t \in I}$ solve the stochastic differential equations formally obtained by repeatedly differentiating (3.1). Furthermore, for all $\beta \in \mathbb{N}_0^p$, $p \in \mathbb{N}$, and $t \in I$,

$$
\sup_{x \in \mathbb{R}^\nu} \mathbb{E} \left[ \sup_{s \leq t} \| (1 + s \Gamma(\omega))^{\beta} \partial_x^\beta W_s[X^x] \eta \|_p^p \right] + \sup_{x \in \mathbb{R}^\nu} \mathbb{E} \left[ \left( \int_0^t \| d\Gamma(\omega) \|^{1/2} (1 + s \Gamma(\omega))^{\beta} \partial_x^\beta W_s[X^x] \eta \|_p^2 \, dr \right)^{1/2} \right] \leq c_{p,\alpha,\beta, t} \mathbb{E} \left[ \| \eta \|_p^p \right],
$$

where the constants $c_{p,\alpha,\beta, t} > 0$ also depend on the coefficients $\omega$, $G_t$, $F_j$, and $\sigma_j$.

We introduce the following notation for double difference quotients,

$$
D_{x,x'}^{h,h'} : [A] := \frac{1}{h} \left( A[X^x+he] - A[X^x] \right) - \frac{1}{h'} \left( A[X^x+h'e] - A[X^x] \right),
$$

where $e$ is a fixed canonical unit vector in $\mathbb{R}^\nu$. Then the crucial (and most tedious) step in the proof of the previous theorem is to derive the inequality

$$
\mathbb{E} \left[ \sup_{s \leq t} \| (1 + s \Gamma(\omega))^{\beta} D_{x,x'}^{h,h'} [\partial^\beta W_s] \eta \|_p^p \right] \leq c_{p,\alpha,\beta, t} (|h - h'| + |x - x'|)^{p\kappa} \mathbb{E} \left[ \| \eta \|_p^p \right],
$$

for all $p \in \mathbb{N}$ and some $\kappa \in (0, 1)$, using the SDEs for $(\partial^\beta W_t[X^x] \eta)_{t \in I}$; see [8] for similar bounds in a different situation. With the help of (5.1) we can then verify that the process $(\partial_x^\beta e^s W_t[X^x] \eta)_{t \in I}$ solves the SDE formally obtained by differentiating the equation for $(\partial^\beta W_t[X^x] \eta)_{t \in I}$ in the direction of $e$. The weights $(1 + s \Gamma(\omega))^\alpha$ are necessary in the above bounds, because the SDE (3.1) and its differentiated versions contain unbounded field operators and the unbounded operator $d\Gamma(\omega)$. (At least we need weights with $\alpha = 1$ and inverse weights like $(1 + (t - s) d\Gamma(\omega))^{-\alpha}$ are useful as well.)
6. The Heat Equation

Employing (3.5) and (5.1) as well as Thms. 4.2 and Thm. 5.1 we can argue that
the operators \( T_t^{(\beta)} \) given by
\[
(T_t^{(\beta)} \Psi)(x) := \mathbb{E}\{ [\partial_x^\alpha W_t(x)]^* \Psi(B^x_s) \}, \quad t \geq 0, \ x \in \mathbb{R}^\nu, \ \beta \in \mathbb{N}^\nu_0,
\]
map the spaces \( C^k_b(\mathbb{R}^\nu, C^d \otimes \mathcal{F}) \) of \( k \)-times continuously differentiable functions \( \Psi \) with bounded derivatives of order \( \leq k \) into themselves. (We have to use the identity \( (\partial_x^\alpha W_t(x))^* = \partial_x^\alpha W_t[(\mathcal{R}, B)^x] \) to ensure that the above expectations are well-defined \( C^d \otimes \mathcal{F} \)-valued Bochner-Lebesgue integrals; once this is settled one can continue working with the processes \( W(x) \).) Furthermore, the Leibniz rules
\[
(6.1) \quad \partial_x^\beta(T_t \Psi) = \sum_{\beta_1 + \beta_2 = \beta} \frac{\beta!}{\beta_1! \beta_2!} T_t^{(\beta_1)} \partial_x^{\beta_2} \Psi, \quad \beta \in \mathbb{N}^\nu_0, \ \Psi \in C^{|\beta|}_b(\mathbb{R}^\nu, C^d \otimes \mathcal{F}),
\]
are valid. These remarks are needed to solve the “heat equation” associated with the operator-valued partial differential operator \( \hat{H} \):

**Theorem 6.1.** Let \( \Phi_0 \in C^2_b(\mathbb{R}^\nu, C^d \otimes \mathcal{F}) \). Define \( \Phi : [0, \infty) \times \mathbb{R}^\nu \to C^d \otimes \mathcal{F} \) by setting \( \Phi(t, x) := \Phi_t(x) := (T_t \Phi_0)(x) \). Then \( \Phi_t \in \mathcal{D}(\hat{H}) \), \( t > 0 \), and
\[
\partial_t \Phi(t, x) = -(\hat{H} \Phi_t)(x), \quad t > 0, \ x \in \mathbb{R}^\nu.
\]
If \( d\Gamma(\omega) \Phi_0 \) and \( d\Gamma(\omega)^{1/2} \partial_x \Phi_0 \) belong to \( C_b(\mathbb{R}^\nu, C^d \otimes \mathcal{F}) \), then
\[
\lim_{t \downarrow 0} \frac{1}{t} \left( \Phi(t, x) - \Phi(0, x) \right) = -(\hat{H} \Phi_0)(x), \quad x \in \mathbb{R}^\nu.
\]

The proof of this theorem proceeds along well-known lines but is complicated by the fact that the coefficients of the operator \( \hat{H} \) are unbounded operators on Fock space. To show that the natural condition \( \alpha \geq 1 \) (see Hypothesis (c) in Sect. 2) is sufficient we actually need BDG type bounds on commutators of \( W_t(X^x) \) with functions of \( d\Gamma(\omega) \) as well.

7. A Bismut-Elworthy-Li Type Formula

In this final section we discuss a BEL type formula associated with our model that reveals the smoothing properties of the maps \( T_t \) and in particular of the semigroup \( e^{-tH} \) generated by the Hamiltonian defined in (2.2).

The integrands in our BEL formulas involve the following \( \mathcal{B}(C^d \otimes \mathcal{F}) \)-valued Bochner-Lebesgue integrals, where \( j \in \{1, \ldots, \nu \} \), \( t \geq 0 \), and \( x \in \mathbb{R}^\nu \),
\[
(7.1) \quad D_{j,t}(x) := \int_0^t W_{s,t}(B^x_s)(\partial_{x_j} W_s(x) - i\varphi(G_{j,B^x_s}) W_s(x)) ds;
\]
recall (4.4) and the notation introduced in the paragraph preceding it. To argue that the integral (7.1) is well-defined, at every fixed elementary event \( \gamma \in \Omega \), we actually have to insert our formulas for \( W_{s,t}, \partial_{x_j} W_s, \) and \( W_s \), normal order the so-obtained expression, and show by explicit estimations that the resulting integrand is absolutely integrable on \( (0, t] \). In fact, the following holds:

**Proposition 7.1.** Let \( j \in \{1, \ldots, \nu \} \). Then, for all \( \gamma \in \Omega \), the maps \( \mathbb{R}^\nu \ni x \mapsto D_{j,t}(x, \gamma) \in \mathcal{B}(C^d \otimes \mathcal{F}) \) with \( t \geq 0 \) are smooth and, for all \( \beta \in \mathbb{N}^\nu_0 \), the corresponding partial derivatives satisfy the following:
(i) the map \((t, x, \gamma) \mapsto \partial_k D_{j,t}(x, \gamma)\) is \(\mathcal{B}([0, \infty) \times \mathbb{R}^\nu) \otimes \mathfrak{F}(\mathcal{C}^d \otimes \mathcal{F})\)-measurable with a separable image;
(ii) for all \((t, x) \in [0, \infty) \times \mathbb{R}^\nu\), \(\partial_k D_{j,t}(x)\) is \(\mathfrak{F}(\mathcal{C}^d \otimes \mathcal{F})\)-measurable;
(iii) for every \(\gamma \in \Omega\), the following map is continuous,
\[
[0, \infty) \times \mathbb{R}^\nu \ni (t, x) \mapsto \partial_k D_{j,t}(x, \gamma) \in \mathcal{C}(\mathcal{C}^d \otimes \mathcal{F})
\]
(iv) for all \(t > 0\) and \(x \in \mathbb{R}^\nu\), the partial derivative \(\partial_k D_{j,t}(x)\) can be computed by formally applying the Leibniz rule under the integral sign in (7.1).

The previous proposition clarifies pointwise (on \(\Omega\)) differentiability properties of the operator-valued processes \(D_j(x)\). To apply these results in the study of the semi-group we again have to complement them by BDG type bounds. Employing Thm. 5.1 and some bounds used in its proof (like (5.1)) as well as Prop. 7.1(iv), we can, for instance, prove that
\[
\sup_{\|\psi\|_1} \mathbb{E} \left[ \sup_{x \in \mathbb{R}^\nu} \left\| \frac{\partial_k^2 D_{j,s}(x + h e) - \partial_k^2 D_{j,s}(x)}{h} - \partial_k^2 D_{j,s}(x) \right\|^p \right] \xrightarrow{h \to 0} 0,
\]
for all \(p \in \mathbb{N}\) and canonical unit vectors \(e\) in \(\mathbb{R}^\nu\). From this we infer the Leibniz rule
\[
\partial_k^p \mathbb{E} \left[ D_{j,t}(x)^* \Psi(B_x^*) \right] = \sum_{\beta_1 + \beta_2 = \beta} \frac{\beta!}{\beta_1! \beta_2!} \mathbb{E} \left[ \left\{ \partial_k^{\beta_1} D_{j,t}(x) \right\}^* \partial_k^{\beta_2} \Psi(B_x^*) \right],
\]
for all \(\beta \in \mathbb{N}_0^\nu\) and \(\Psi \in C_b^{(\beta)}(\mathbb{R}^\nu, \mathcal{C}^d \otimes \mathcal{F})\).

At this point all essential work has been done and we can follow a well-known proof [2] to arrive at our main theorem:

**Theorem 7.2.** Let \(p \in (1, \infty]\), \(\Psi \in L^p(\mathbb{R}^\nu, \mathcal{C}^d \otimes \mathcal{F})\), and \(t > 0\). Then \(T_t \Psi \in C^p(\mathbb{R}^\nu, \mathcal{C}^d \otimes \mathcal{F})\) and its \(j\)-th partial derivative is given by
\[
(\partial_j T_t \Psi)(x) = \frac{1}{t} \mathbb{E} \left[ D_{j,t}(x)^* \Psi(B_x^*) \right] + \frac{1}{t} \mathbb{E} \left[ B_j \left( W_t(x)^* \Psi(B_x^*) \right) \right], \quad x \in \mathbb{R}^\nu.
\]

Let us sketch the argument of [2] to show how our SDE and heat equation enter. Let \(\psi \in \mathcal{C}^d \otimes \mathcal{F}\), let \(\Phi_0\) satisfy all assumptions appearing in Thm. 6.1, and define \(\Phi\) as in its statement. Then Itô’s formula, (3.1), and Thm. 6.1 \(\mathbb{P}\)-a.s. entail
\[
\langle W_t(x) \psi | \Phi(0, B_x^*) \rangle = \sum_{\ell=1}^{\nu} \int_0^t \left\langle i \varphi(G_{j,t} B_x^*) W_s(x) \psi | \Phi(t-s, B_x^*) \right\rangle dB_{\ell,s} + \langle \psi | \Phi(t, x) \rangle + \sum_{\ell=1}^{\nu} \int_0^t \left\langle W_s(x) \psi | \partial_{x_j} \Phi(t-s, B_x^*) \right\rangle dB_{\ell,s}.
\]

Multiplying this with \(B_{j,t}\), computing expectations, taking (3.4) into account, and applying the semi-group property \(T_t = T_s T_{t-s}\), we deduce that
\[
\mathbb{E} \left[ \langle W_t(x) \psi | \Phi_0(B_x^*) \rangle B_{j,t} \right] - \mathbb{E} \left[ \int_0^t \langle i \varphi(G_{j,t} B_x^*) W_s(x) \psi | \Phi(t-s, B_x^*) \rangle ds \right]
= \mathbb{E} \left[ \int_0^t \langle W_s(x) \psi | \partial_{x_j} \Phi(t-s, B_x^*) \rangle ds \right] - \mathbb{E} \left[ \int_0^t \langle \partial_{x_j} W_s(x) \psi | \Phi(t-s, B_x^*) \rangle ds \right].
\]

\[
= t \langle \psi | \partial_{x_j} (T_t \Phi_0)(x) \rangle
\]
Corollary 7.3. Let $p \in [1, \infty]$, $\Psi \in L^p(\mathbb{R}^\nu, \mathcal{C}^d \otimes \mathcal{F})$, and $t > 0$. Then $T_t \Psi : \mathbb{R}^\nu \to \mathcal{C}^d \otimes \mathcal{F}$ is smooth with bounded partial derivatives of any order.

To prove this corollary we invoke the semi-group property $T_t = T_{t/2}T_{t/2}$ and (7.3) to write

$$(\partial_{x_j} T_t \Psi)(x) = 2 \frac{t}{\nu} \mathbb{E}[D_{j,t/2}(x)^*(T_{t/2} \Psi)(B_{t/2}^j)] + 2 \frac{t}{\nu} \mathbb{E}[B_{j,t/2}W_{t/2}(x)^*(T_{t/2} \Psi)(B_{t/2}^j)],$$

for any $\Psi \in L^p(\mathbb{R}^\nu, \mathcal{C}^d \otimes \mathcal{F})$ with $p \in [1, \infty]$. Notice that $T_{t/2}$ maps $L^p(\mathbb{R}^\nu, \mathcal{C}^d \otimes \mathcal{F})$ continuously into every $L^q(\mathbb{R}^\nu, \mathcal{C}^d \otimes \mathcal{F})$ with $q \in [p, \infty]$. A bootstrap argument employing (6.1) and (7.2) then reveals that $T_t \Phi \in C^\infty_{b}(\mathbb{R}^\nu, \mathcal{C}^d \otimes \mathcal{F})$.

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On the relationship between NLS Dynamics, Gross-Pitaevskii Hierarchy and Liouville’s equation.

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Hamiltonian systems

- Classical Mechanics: a dynamical system with infinite degrees of freedom is described by pairs of momentum-coordinate canonical variables \((p_1, q_1, \ldots, p_n, q_n, \ldots)\). The equation of motion is derived from a classical Hamiltonian:

\[
\mathcal{H}(p, q) = \mathcal{H}(p_1, q_1, \ldots, p_n, q_n, \ldots)
\]

\[
\dot{q}_j = \frac{\delta \mathcal{H}}{\delta p_j}, \quad \dot{p}_j = -\frac{\delta \mathcal{H}}{\delta q_j}, \quad j = 1, \ldots
\] (1)

- Quantum Mechanics: A quantum mechanical system with infinite degrees of freedom is formally described by a Hamiltonian

\[
\mathcal{H}(\hat{p}, \hat{q}) = \mathcal{H}(\hat{p}_1, \hat{q}_1, \ldots, \hat{p}_n, \hat{q}_n, \ldots)
\]

where the pairs \((\hat{p}_j, \hat{q}_j)\) are conjugate canonical variables satisfying the commutation relations

\[
[\hat{q}_j, \hat{p}_k] = i\hbar \delta_{j,k}, \quad [\hat{q}_j, \hat{q}_k] = [\hat{p}_j, \hat{p}_k] = 0.
\]

The equation of motion in this case is given by the Schrödinger equation

\[
i\hbar \partial_t \psi = \mathcal{H}(\hat{p}, \hat{q}) \psi
\] (2)
Correlation Functions

- The Quantum-Classical transition can be understood using correlation functions

\[
\lim_{\hbar \to 0} \langle \psi_\hbar, \hat{p}_j \cdots \hat{q}_k \cdots \psi_\hbar \rangle = \int p_j \cdots q_k \cdots d\mu(p, q).
\]

where \( \mu \) is a probability measure on the classical phase-space usually called semi-classical measures or Wigner measures.

- For instance, consider coherent states

\[
\psi_\hbar = C_\hbar(p^0, q^0)
\]

localized on a given point \((p^0, q^0) = (p^0_1, q^0_1, \cdots)\) of the phase-space:

\[
\lim_{\hbar \to 0} \langle C_\hbar(p^0, q^0), \hat{p}_j \cdots \hat{q}_k \cdots C_\hbar(p^0, q^0) \rangle = p^0_j \cdots q^0_k \cdots .
\]

This means that the semi-classical measure \( \mu \) in this case is equal to the Dirac measure \( \delta_{(p^0, q^0)} \) on the point \((p^0, q^0) = (p^0_1, q^0_1, \cdots)\).

- Conclusion: Quantum correlation functions converge towards classical correlation functions when \( \hbar \to 0 \).

Classical Limit Theorem

Let \((p_1(t), q_1(t), \cdots, p_n(t), q_n(t), \cdots)\) be a solution of the classical Hamiltonian system (1) and let \(\psi_\hbar(t)\) be a solution of the Schrödinger equation (2).

**Theorem (The Classical Limit Theorem)**

*If at time \( t = 0 \) there exist a semi-classical (probability) measure \( \mu_0 \) such that*

\[
\lim_{\hbar \to 0} \langle \psi_\hbar(0), \hat{p}_j \cdots \hat{q}_k \cdots \psi_\hbar(0) \rangle = \int p_j \cdots q_k \cdots d\mu_0(p, q).
\]

*Then at any time \( t \in \mathbb{R} \),*

\[
\lim_{\hbar \to 0} \langle \psi_\hbar(t), \hat{p}_j \cdots \hat{q}_k \cdots \psi_\hbar(t) \rangle = \int p_j(t) \cdots q_k(t) \cdots d\mu_0(p, q)
= \int p_j \cdots q_k \cdots d\mu_t(p, q),
\]

*where \( \mu_t = (\Phi_t)_* \mu_0 \) is the image measure of \( \mu_0 \) by the Hamiltonian flow map \( \Phi_t(p_1, q_1, \cdots) = (p_1(t), q_1(t), \cdots) \).*
The Liouville equation

The Liouville equation is a fundamental equation of statistical mechanics which describes the time evolution of phase-space distribution functions \( \varphi(p, q, t) \),

\[
\frac{\partial \varphi}{\partial t} + \{ \varphi, \mathcal{H} \} = 0 ,
\]

with the Poisson bracket defined as follows,

\[
\{ \varphi, \mathcal{H} \} = \sum_{j=1}^{n} \left[ \frac{\partial \mathcal{H}}{\partial p_j} \frac{\partial \varphi}{\partial q_j} - \frac{\partial \mathcal{H}}{\partial q_j} \frac{\partial \varphi}{\partial p_j} \right].
\]

Liouville’s theorem says that "The distribution function is constant along any trajectory in phase space", i.e.,

\[
\frac{d}{dt} \varphi(p(t), q(t), t) = 0.
\]

where \((p(t), q(t))\) are solutions of the Hamiltonian equations.

A general strategy to prove the CLT

- To establish rigorously the classical limit theorem (CLT), we show that the semi-classical measures \( \mu_t \) satisfy an (infinite dimension) Liouville's equation.
- Then we prove that the Liouville’s equation for Hamiltonian systems with infinite degrees of freedom admits a unique solution \( \mu_t = (\Phi_t)_{*} \mu_0 \) under wide general assumptions.
- Application of the above strategy to the following problems:
  - Mean field limit \( N \to \infty \).
  - Classical limit \( \hbar \to 0 \).

for systems of physical relevance:

- Many-Body theory (N-body Schrödinger operators)
- Relativistic Quantum field theory \((\varphi^4_2, P(\varphi)_2 \text{ models})\)
- Non-relativistic Quantum field theory (Nelson model)
- Quantum electrodynamics (Pauli-Fierz models)
 caracteristic Method

- The characteristics method says that if the classical Hamiltonian $H$ is sufficiently smooth and generates a unique Hamiltonian flow $\Phi_t$ on the phase-space, then the density function $\varrho(p, q, t)$ is uniquely determined by its initial value $\varrho(p, q, 0)$, i.e.,

$$\varrho(p, q, t) = \varrho(\Phi_t^{-1}(p, q), 0).$$

- The characteristic method relates in finite dimension the individual solutions of the classical Hamiltonian system (1) (ODE) and the statistical (probability measure) solutions of the Liouville equation.

- One expects that this uniqueness principle can be extend to non-smooth vector fields or to dynamical systems with infinite degrees of freedom.

- While the non-smooth case has been carefully studied, the extension to dynamical systems with infinite degrees of freedom is less considered and the investigations are not oriented towards the study of classical PDEs.

Results

- Uniqueness theorem for Liouville’s equation is established for Hamiltonian systems with infinite degrees of freedom in [AN11] using arguments from optimal transport theory due to Ambrosio-Gigli-Savaré [AGS08].

- The previous result is improved in [AL] and it can be applied to the following nonlinear equations:
  - Hartree
  - NLS
  - Gross–Pitaevskii
  - Klein-Gordon
  - Schrödinger-Klein-Gordon.

- The Classical Limit Theorem is rigorously established using the strategy with semiclassical measures on the following examples:
  - Many-body theory $\Rightarrow$ Hartree equation (with Coulomb potential): [AN11].
  - Non-relativistic QFT $\Rightarrow$ Schrödinger-Klein-Gordon (without UV cutoff): [AF14].
Uniqueness for Liouville’s equation

**Theorem (Am.-Liard, see [AL])**

Let \( \mathcal{Z}_1 \subset \mathcal{Z} \subset \mathcal{Z}_1' \) be a rigged separable Hilbert space and \( v \) a vector field such that \( v : \mathbb{R} \times \mathcal{Z}_1 \to \mathcal{Z}_1' \) is continuous and bounded on bounded sets. Assume that the cauchy problem

\[
\partial_t z_t = v(t, z_t) \quad , \quad z_{t=0} = z_0
\]

is globally well-posed on \( \mathcal{Z}_1 \) with a continuous flow map \( \phi_t \) satisfying \( z_t = \phi_t(z_0) \). Let \( (\mu_t)_{t \in \mathbb{R}} \) be a weakly narrowly continuous curve of Borel probability measures on \( \mathcal{Z}_1' \) satisfying the Liouville equation

\[
\partial_t \mu_t + \nabla^T(v_t \mu_t) = 0,
\]

in the sense that for all \( \varphi \in C_0^\infty(\mathbb{R} \times \mathcal{Z}_1; \mathbb{R}) \),

\[
\int_{\mathbb{R} \times \mathcal{Z}_1} (\partial_t \varphi(z, t) + \text{Re}\langle v(t, z), \nabla \varphi(z) \rangle_{\mathcal{Z}_1'}) \, d\mu_t(z) \, dt = 0.
\]

Assume additionally:

(i) For all \( T > 0 \), there exists a ball \( B \) of \( \mathcal{Z}_1 \) such that \( \mu_t(B) = 1 \) for all \( t \in [-T, T] \).
(ii) For all \( T > 0 \), \( \|v(t, .)\|_{L^1(\mathcal{Z}_1, \mu_t)} \in L^1([-T, T], dt) \).

Then the measure \( \mu_t \) satisfies

\[ \forall t \in \mathbb{R}, \quad \mu_t = (\phi_t)_* \mu_0. \]

GP Hierarchy

The Gross-Pitaevskii (GP) hierarchy is a well studied system of equations satisfied by sequences of reduced density matrices. As consequence of the techniques developed in [AN11], we have the following result.

**Theorem (Am.-Liard-Rouffort)**

There is a one to one correspondence between the solutions of GP hierarchy and solutions of the Liouville’s equation.

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Radial Density Function
Associated with the \((\alpha, q)\)-Fock Space*

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Abstract

In Bożejko-Ejsmont-Hasebe [BEH15] the Fock space associated with the Coxeter group of type B over a complex Hilbert space \(F_{\alpha,q}(H)\), has been constructed. This Fock space is called the Fock space of type B or \((\alpha,q)\)-Fock space, for short and considered as a deformation of the (algebraic) full Fock space with two parameters \(\alpha\) and \(q\). (The case with \(\alpha = 0\) is equivalent to the \(q\)-deformation by Bożejko-Speicher [BS91] and Bożejko-Kümmerer-Speicher [BKS97].) Their starting point is to replace the Coxeter group of type A, that is, symmetric group \(S_n\) for the \(q\)-Fock space by the Coxeter group of type B, \(\Sigma_n := Z_n^2 \times S_n\) in (A.1) of the Appendix A. This replacement provides us a more generalsymmetry operatoron \(H \otimes n\) than that of [BS91] to definethe \((\alpha,q)\)-inner product \(\langle \cdot, \cdot \rangle_{\alpha,q}\) in (A.3). One can define annihilation \(B_{\alpha,q}^{-}(f)\) and creation \(B_{\alpha,q}^{+}(f)\) operators acting on \(F_{\alpha,q}(H)\) and the \((\alpha,q)\)-Gaussian process (the Gaussian process of type B) \(G_{\alpha,q}(f)\) for \(f \in H\) as the sum of them.

It is one of their main interests to find a probability distribution \(\mu_{\alpha,q,f}\) on \(R\) of \(G_{\alpha,q}(f)\), \(\|f\|_H = 1\), with respect to the vacuum state.

Motivated by [BEH15], the author with M. Bożejko and T. Hasebe [ABH16] examined the radial Bargmann representation of the probability measure \(\mu_{\alpha,q,f}\) on \(R\). The radial density function is explicitly obtained (Theorem 2.6) and expressed under certain conditions by the Rogers-Szegő polynomials. In addition, our results can cover not only known results under \(\alpha = 0\) as the \(q\)-Bargmann representation [LM95] for \(0 \leq q < 1\), ([Barg61][AKK03] for \(\alpha = 0\), \(q = 1\) and [Bi97] for \(\alpha = 0\), \(q = 0\)), but also new examples as those for symmetric free Meixner (Kesten) and \(q^2\)-Gaussian distributions on \(R\) and \(t\)-deformed cases of these [KW14][AKW16].

1 Preliminaries

Let us first recall standard notations from \(q\)-calculus, which can be found in [GR04][KLS10] for example. The \(q\)-shifted factorials are defined by

\[
(a; q)_0 := 1, \quad (a; q)_k := \prod_{\ell=1}^{k} (1 - aq^{\ell-1}), \quad k = 1, 2, \ldots, \infty,
\]

and the product of \(q\)-shifted factorials is defined by

\[
(a_1, a_2; q)_k := (a_1; q)_k (a_2; q)_k, \quad k = 1, 2, \ldots, \infty.
\]

Remark 1.1. The \(q\)-shifted factorials are a natural extension of the Pochhammer symbol \((\cdot)_n\) because one can see that \(\lim_{q \to 1} [k]_q = k\) implies

\[
\lim_{q \to 1} \frac{(q^k; q)_n}{(1 - q)^n} = (k)_n, \quad (1.1)
\]

where \((k)_0 := 1, \quad (k)_n := k(k+1) \cdots (k+n-1), \quad n \geq 1.\]

*This is a summary paper of [ABH16].
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Let \( \{P_n^{(\alpha, q)}(x)\} \) for \( \alpha, q \in (-1, 1) \) denote the \( q \)-Meixner-Pollaczek polynomials defined by the recurrence relation,

\[
\begin{align*}
P_0^{(\alpha, q)}(x) &= 1, \quad P_1^{(\alpha, q)}(x) = x, \\
x P_n^{(\alpha, q)}(x) &= P_{n+1}^{(\alpha, q)}(x) + (1 + \alpha q^n)[n]_q P_{n-1}^{(\alpha, q)}(x), \quad n \geq 1.
\end{align*}
\] (1.2)

where \([n]_q := 1 + q + q^2 + \cdots + q^{n-1}\) for \( n \geq 1 \). Let us omit the explicit form of the density function of the orthogonality measure \( \nu_{\alpha, q} \) for such polynomials. It can be found in [KLS10, 14.9.2] and [BEH15, page 1781]

**Example 1.2.** (1) If \( \alpha = 0 \), then \( q \)-Meixner-Pollaczek polynomials get back to the \( q \)-Hermite polynomials \( \{H_n^{(q)}(x)\} \) whose orthogonality measure is the standard \( q \)-Gaussian measure supported on \((-2/\sqrt{1-q}, 2/\sqrt{1-q})\). The explicit form of the density can be found in [BKS97][BS91]. Note that one can get the standard Gaussian law as \( q \to 1 \), the Bernoulli law as \( q \to -1 \), and the standard Wigner’s semi-circle law if \( q = 0 \).

(2) The measure \( \nu_{\alpha, 0} \) is the symmetric free Meixner law \([An03][BB06][SY01]\).

(3) The measure \( \nu_{q, q} \) is the \( q \)-Gaussian law scaled by \( \sqrt{1+q} \).

(4) If \( \alpha = -q^{2\beta} \) as suggested in Remark 1.1, then the measure \( \nu_{-q^{2\beta}, q} \) under a certain scaling converges to the classical symmetric Meixner law as \( q \uparrow 1 \),

\[
\frac{2^{2\beta}}{2\pi \Gamma(2\beta)} |\Gamma(\beta + ix)|^2 dx, \quad x \in \mathbb{R}.
\] (1.3)

See also [KLS10, 14.9.15].

We shall explain a fundamental idea to calculate the distribution of \( G_{\alpha, q}(f) \) in [BEH15] and show how the theory of orthogonal polynomials of one variable is related with. By direct computations, the following equality under \( \|f\|_H = 1 \),

\[
G_{\alpha, q}(f)f^{\otimes n} = (B_{\alpha, q}^+(f) + B_{\alpha, q}^-(f))(f^{\otimes n}) = f^{\otimes(n+1)} + (1 + \alpha(f, \bar{f})_H q^{n-1})[n]_q f^{\otimes(n-1)},
\]

can be obtained where \( \bar{f} \) denotes a self-adjoint involution of \( f \in H \) in (A.2). Since a linear map, \( \Phi : \text{Span}\{f^{\otimes n} \mid f \in H, \ n \geq 0\} \to L^2(\mathbb{R}, \mu_{\alpha, q,f}) \) given by \( \Phi(f^{\otimes n}) = P_n^{(\alpha(f, \bar{f})_H q, q)}(x) \), is an isometry, the above equality corresponds to the three terms recursion relation satisfied by \( P_n^{(\alpha(f, \bar{f})_H q, q)}(x) \) through \( \Phi \). Then, it is proved that \( \mu_{\alpha, q,f} = \nu_{\alpha(f, \bar{f})_H q, q} \) (see [BEH15],[ABH16]) in the sense of

\[
\langle \Omega, G_{\alpha, q}(f)^n \Omega \rangle_{\alpha, q} = \int x^n \mu_{\alpha, q,f}(dx)
\] (1.4)

where \( \Omega \) denotes the vacuum vector.

Therefore, in order to get the Bargmann representation of \( \nu_{\alpha(f, \bar{f})_H q, q} \), it is enough to consider that of \( \nu_{\alpha, q} \) in the sense of Definition 1.3 given below. It means as a result to solve the following moment problem to realize the inner product by the integral:

**Problem 1.** Find a probability measure \( \gamma_\mu \) satisfying the equality,

\[
\int_{\mathbb{C}} z^m \overline{z^n} \gamma_\mu(d^2 z) = \delta_{m,n}(-\alpha; q)_n [n]_q!, \quad n \in \mathbb{N} \cup \{0\}
\] (1.5)

for all \( m, n \in \mathbb{N} \cup \{0\} \). Here \([n]_q!\) denotes the \( q \)-factorials defined by

\[
[0]_q! := 1, \quad [n]_q! := \prod_{\ell=1}^{n} [\ell]_q = (q; q)_n/(1-q)_n^n, \quad n \geq 1.
\]

**Definition 1.3.** A measure \( \gamma_\mu \) satisfying the equality (1.5) is called a Bargmann representation (measure on \( \mathbb{C} \)) of a measure \( \mu \) on \( \mathbb{R} \).
It was proved in [Sz07] (see also [AKW16],[KW14]) that if a measure $\mu$ admits any Bargmann representation, then it also admits a radial (rotation invariant) Bargmann representation

$$\gamma_\mu(d^2z) = \frac{1}{2\pi}\lambda_{[0,2\pi)}(d\theta)\rho_\mu(dr), \quad z = re^{i\theta}, \quad r \geq 0, \quad \theta \in [0,2\pi),$$

where $\lambda_{[0,2\pi)}$ is the Lebesgue measure on $[0,2\pi)$. It says that the angular part takes care of orthogonality of (1.5). Therefore, Problem 1 can be transformed into the following Problem 2:

**Problem 2.** Find a positive radial measure $\rho_\mu$ satisfying

$$\int_0^\infty r^{2k}\rho_\mu(dr) = (-\alpha; q)_k[k]_q!, \quad k \in \mathbb{N} \cup \{0\}.$$  

Due to Carleman criterion for the moment problem, the inequality for $\alpha, q \in (-1,1)$,

$$\left|(-\alpha; q)_k[k]_q!\right| \leq \left(\frac{4}{1 - |q|}\right)^k, \quad k \in \mathbb{N} \cup \{0\}. \quad (1.6)$$

implies that a radial measure $\rho_{\alpha,q}$ is determined uniquely by the sequence $\{(-\alpha; q)_k[k]_q!\}$.

Our main purpose is to consider Problem 2 in Section 2. Furthermore, commutation relations satisfied by one-mode creation operator $a^+$ and annihilation operator $a^-$ acting on one-mode interacting Bargmann-Fock space $\mathcal{B}$ associated with $\omega_n = (1 + \alpha q^{n-1})[n]_q!$ will be presented in Section 3.

## 2 Construction of $(\alpha, q)$-radial measures

Let

$$\left[\begin{array}{c} n \\ \ell \end{array}\right]_q := \frac{[n]_q!}{[\ell]_q![n-\ell]_q!} = \frac{(q;q)_n}{(q;q)_\ell(q;q)_{n-\ell}}$$

be the $q$-binomial coefficients and $h_n(z | q)$ be the Rogers-Szeg"{o} polynomials [GR04][S05] defined by

$$h_n(z | q) = \sum_{\ell=0}^n \left[\begin{array}{c} n \\ \ell \end{array}\right]_q z^\ell.$$

We shall omit proofs of all results in this section. Consult our paper [ABH16] in detail.

**Lemma 2.1.** Suppose that $\alpha \in (-1,1)$ and $q \in [0,1)$. Let

$$\rho_{\alpha,q} := (-\alpha; q)\infty \sum_{n=0}^\infty (-\alpha)^n_n (q; q)_n \delta_{q^{n/2}},$$

which is a signed measure. Then we have

$$\int_0^\infty r^{2k}\rho_{\alpha,q}(dr) = (-\alpha; q)_k, \quad k \in \mathbb{N} \cup \{0\}.$$  

In particular, if taking $\alpha = -q$, then one can see $\rho_{\alpha,q} = D_{\{1-q\}}^{1/2}(\rho_{-q,q})$, namely,

$$\int_0^\infty r^{2k}D_{\{1-q\}}^{1/2}(\rho_{-q,q})(dr) = \frac{(q;q)_k}{(1-q)^k} = [k]_q!,$$

where $D_t(\lambda)$ is the push-forward of $\lambda$ by the map $x \mapsto tx$ for a measure $\lambda$ on $\mathbb{R}$.

**Proposition 2.2.** Suppose that $\alpha \in (-1,1)$ and $q \in [0,1)$. Let

$$\rho_{\alpha,q} := \begin{cases} (-\alpha; q)\infty \sum_{n=0}^\infty \frac{q^n}{(q; q)_n} h_n(-\alpha q^{-1} | q)\delta_{(1-q)^{-1/2}q^{n/2}}, & q > 0, \\ -\alpha \delta_0 + (1 + \alpha)\delta_1, & q = 0, \end{cases} \quad (2.1)$$

which is a signed measure in general. Then we have

$$\int_0^\infty r^{2k}\rho_{\alpha,q}(dr) = \frac{(-\alpha; q)_k}{(1-q)^k} = (-\alpha; q)_k[k]_q!, \quad k \in \mathbb{N} \cup \{0\}. \quad (2.2)$$
It is known [LM95] that a radial measure $\rho_{\alpha,q} := \rho_{\alpha,\nu_{\alpha,q}}$ does not exist for $q < 0$. In addition, due to the term

$$\delta_{(1-q)^{-1/2}q^{n/2}} \text{ in } \rho_{\nu_{\alpha,q}},$$

it seems impossible for $q \in (-1, 0)$ to define $\rho_{\nu_{\alpha,q}}$. However, if $-1 < \alpha = q < 0$, then $\nu_{q,q}$ coincides with a scaled $q^2$-Gaussian measure, and hence we have

**Proposition 2.3.** Suppose $-1 < \alpha = q < 0$. We define

$$\rho_{\nu_{q,q}} := D_{(1+q)^{1/2}}(\rho_{\nu_{q,q}}) = (q^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n}(q^2; q^2)_n}{(q^2; q^2)_n} \delta_{(1-q)^{-1/2}(-q)^n}.$$  

Then one can see

$$\int_0^{\infty} r^{2k} \rho_{\nu_{q,q}}(dr) = (1 + q)^k [k]_q! = (-q; q)_k [k]_q!.$$  

We need some properties of the Rogers-Szegő polynomials to know when the measure $\rho_{\nu_{\alpha,q}}$ becomes positive.

**Lemma 2.4** ([MGH90]). Suppose that $q \in (-1, 1)$.

1. If $n \geq 0$ is odd, then $h_n(x | q) \geq 0$ if and only if $x \geq -1$. Moreover, the point $x = -1$ is the unique zero of $h_n(x | q)$ on $\mathbb{R}$.
2. If $n \geq 0$ is even, then $h_n(x | q) > 0$ for all $x \in \mathbb{R}$.

We need the following lemma in proof of Theorem 2.6 for the non-existence part of a radial Bargmann measure.

**Lemma 2.5.** Let $\mu$ be a signed measure on $\mathbb{R}$ with compact support and let $\nu$ be a nonnegative measure on $\mathbb{R}$. If $\mu$ and $\nu$ have the same finite moments of all orders, then $\mu = \nu$.

In summary, the complete answer to the unique existence of a radial Bargmann representation of $\nu_{\alpha,q}$ is stated as follows:

**Theorem 2.6.** Suppose that $\alpha, q \in (-1, 1)$. The probability measure $\nu_{\alpha,q}$ has a radial Bargmann representation if and only if either (i) $q \geq 0$ and $\alpha < q$ or (ii) $\alpha = q \neq 0$.

In fact, the radial measure is given uniquely by

$$\rho_{\nu_{\alpha,q}} = \begin{cases} -\alpha \delta_0 + (1 + \alpha)\delta_1 & (\alpha \leq q = 0), \\ (-\alpha, q; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n} h_n(-\alpha q^{-1} | q) \delta_{(1-q)^{-1/2}q^{n/2}} & (q > 0, \alpha < q), \\ (q^2, q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2; q^2)_n} \delta_{(1-q)^{-1/2}q^n} & (\alpha = q \neq 0). \end{cases}$$

**Example 2.7.**

1. The radial measure $\rho_{\nu_{\alpha,q}}$ for $q \in [0,1)$ is of the $q$-Bargmann [LM95]. In particular, $\rho_{\nu_{0,0}}$ is of the free-Bargmann [Bi97].
2. $\lim_{q \uparrow 1} \rho_{\nu_{\alpha,q}}$ is of the classical Bargmann [Barg61][AKK03].
3. $\rho_{\nu_{q,q}}$ for $q \in (-1,1)$ is of the $q^2$-Bargmann.
4. Consider $\alpha = -q^{2\beta}, \beta > 0$. This choice of $\alpha$ is suggested by (1.1) in Remark 1.1. The following limit,

$$\lim_{q \uparrow 1} \frac{(1 - q^{2\beta + n - 1})[n]_q}{4(1 - q)} = \frac{1}{4}(n + 2\beta - 1)n,$$

gives the Jacobi sequence of the symmetric Meixner distribution in (1.3), so that $\rho_{\nu_{-q^{2\beta},q}}$ under suitable scaling converges weakly as $q \uparrow 1$ to the radial measure with the density,

$$\frac{2\pi r}{\Gamma(2\beta)} \int_0^{\infty} h(r, t/4)e^{-t^{2\beta-1}}dt$$
where
\[ h(r, t) = \frac{1}{\pi t} \exp \left( -\frac{r^2}{t} \right), \quad r \in \mathbb{R}, \quad t > 0. \]

This is an integral representation of the radial density for the Bessel kernel measure, which can be also represented by the modified Bessel function [As05][As09].

(5) \( \rho_{\alpha,n} \) for \( \alpha \in (-1, 0] \) is the radial measure for the symmetric free Meixner distribution. This case is closely related with the \( t \)-deformed Bargmann representation of the \( t \)-deformed probability measure on \( \mathbb{R} \) in the sense of Bożejko-Wysoczański [BW98, BW01]. See [ABH16], [AKW16] and [KW14],

3 One-mode interacting Fock space and commutation relations

3.1 One-mode interacting Fock space

Let \( \{\omega_n\}_{n=0}^\infty \) with \( \omega_0 := 1 \) be an infinite sequence of positive real numbers and \( \{\alpha_n\}_{n=0}^\infty \) be of real numbers. A one-mode interacting Bargmann-Fock space \( \mathcal{B} \) is defined as \( \bigoplus_{n=0}^\infty \mathbb{C} \Phi_n \) equipped with \( \Phi_n := z^n/\lfloor \omega_n \rfloor! \), \( \lfloor \omega_n \rfloor! := \prod_{k=0}^n \omega_k \), the inner product \( \langle \Phi_m, \Phi_n \rangle_{\mathcal{B}} = \delta_{m,n} \) for all \( m, n \in \mathbb{N} \cup \{0\} \), operators of creation \( a^+ \), annihilation \( a^- \), and conservation \( a^{\circ} \) defined by

\[
\begin{align*}
\alpha^+ \Phi_n &:= \sqrt{\omega_n+1} \Phi_{n+1}, \quad n \geq 0, \\
a^- \Phi_0 &= 0, \quad a^- \Phi_n := \sqrt{\omega_n} \Phi_{n-1}, \quad n \geq 1, \\
a^{\circ} \Phi_n &= \alpha_n \Phi_n, \quad n \geq 0.
\end{align*}
\]

(3.1)

Consider a sequence of monic polynomials \( \{P_n(x)\} \) recurrently by

\[
\begin{cases}
P_0(x) = 1, \quad P_1(x) = x - \alpha_0, \\
xP_n(x) = P_{n+1}(x) + \omega_nP_{n-1} + \alpha_nP_n(x), \quad n \geq 1.
\end{cases}
\]

(3.2)

Due to Favard theorem (See [Chi78][HO07], for example.), there exists a probability measure \( \mu \) on \( \mathbb{R} \) with finite moments of all orders such that \( \{P_n(x)\} \) is the orthogonal polynomials with \( \langle P_m(x), P_n(x) \rangle_{L^2(\mathbb{R}, \mu)} = \delta_{m,n}[\omega_n]! \) for all \( m, n \in \mathbb{N} \cup \{0\} \). It is easy to see that a linear map

\[ U : \mathcal{B} = \bigoplus_{n=0}^\infty \mathbb{C} \Phi_n \rightarrow L^2(\mathbb{R}, \mu) \]

defined by \( U \left( \sqrt{\omega_n} \Phi_n \right) = P_n(x) \) is an isometry and in addition \( a^+ + a^- + a^{\circ} = U^* X U \) is satisfied due to (3.1) and (3.2), where \( X \) represents the multiplication operator by \( x \) in \( L^2(\mathbb{R}, \mu) \). This intertwining relation provides a notion of the quantum decomposition of a classical random variable \( X \) and

\[
\langle \Phi_0, (a^+ + a^- + a^{\circ})^n \Phi_0 \rangle_{\mathcal{B}} = \int x^n \mu(dx).
\]

(3.3)

3.2 \( \alpha \)-deformed Jackson derivative and commutation relations

Definition 3.1. Suppose that \( \alpha, q \in (-1, 1) \) and \( f \) is analytic on \( \mathbb{C} \).

(1) Let \( Z \) be the multiplication operator defined by

\[ (Zf)(z) := zf(z). \]

(2) Let \( D_q \) be the Jackson derivative given by

\[
(D_qf)(z) = \begin{cases}
\frac{f(z) - f(qz)}{(1 - q)z}, & z \neq 0, \\
f'(0), & z = 0.
\end{cases}
\]

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The α-deformed Jackson derivative is given as

\[
D_{\alpha,q} := \begin{cases} 
D_q + \alpha q^2 N D_1/q, & q \neq 0, \\
D_0 + \alpha \frac{d}{dz} |_0, & q = 0,
\end{cases}
\]

where \( N \) is the number operator.

For \( q \neq 0 \), we can also write

\[
D_{\alpha,q} = D_q + \frac{\alpha}{q^2} D_1/q q^{2N},
\]

which is equivalently expressed as

\[
(D_{\alpha,q} f)(z) = (D_q f)(z) + \alpha (D_{1/q} f)(q^2 z), \quad q \neq 0.
\]

It can be seen that the α-deformed Jackson derivative is an analogue of the operator in [BEH15, Theorem 2.5].

One can realize one-mode analogue of \((\alpha, q)\)-operators on an appropriate domain of the one-mode interacting Bargmann-Fock space \( \mathcal{B} \) by taking \( \omega_n = (1 + \alpha q^{n-1})[n]_q \) and \( \alpha_n = 0 \) as \( a^+ := Z, a^- := D_{\alpha,q} \). In fact, it is easy to check that \( a^- \Phi_n = \sqrt{\omega_{n+1}} \Phi_{n+1} \) and \( a^+ \Phi_n = \sqrt{\omega_n} \Phi_{n-1} \) hold and the \( q \)-commutation relation, one-mode analogue of (A.4), \( [a^-, a^+] := I + \alpha q^{2N} \), is satisfied. By direct computations, one can check

**Theorem 3.2.** Suppose \( \alpha \in (-1, 1) \) and \( q \in (-1, 1) \). Then we have

1. \( [a^-, a^+] = M_{\alpha,q} \), \( [a^-, M_{\alpha,q}] = (1 - q^2) a^- \), \( [M_{\alpha,q}, a^+] = (1 - q^2) a^+ \).
2. \( M_{\alpha,q} = (1 + \alpha) I - \alpha (1 - q^2) ZD_q^2 \).
3. \( [a^-, a^+] = (1 + q) I \) if \( \alpha = q \).

**Example 3.3.** (1) \( \alpha = 0 \) implies \( [a^-, a^+] = I \). Hence \( M_{0,q} = I \) commutes with both \( a^+ \) and \( a^- \),

\[
[a^-, M_{0,q}] = [M_{0,q}, a^+] = 0.
\]

(2) If \( \alpha = -q^{2\beta} \) for \( \beta > 0 \), consider the scaled operators given by

\[
A^\pm := \lim_{q \uparrow 1} \frac{a^\pm}{\sqrt{1 - q^2}}
\]

and obtained as

\[
\lim_{q \uparrow 1} \frac{M_{-q^{2\beta} q}}{1 - q^2} = \lim_{q \downarrow 1} \frac{I - q^{2\beta} q^{2N}}{1 - q^2} = N + \beta.
\]

Then one can get \( [A^-, A^+] = N + \beta, [A^-, N] = A^- \) and \( [N, A^+] = A^+ \). These are the commutation relations for the classical Meixner-Pollaczek polynomials with respect to the symmetric Meixner distribution in (1.3). See [As08].

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A Appendix

Let $\Sigma_n$ be the set of bijections $\sigma$ of the $2n$ points $\{\pm 1, \pm 2, \cdots, \pm n\}$ with $\sigma(-k) = -\sigma(k)$. Equipped with the composition operation as a product, $\Sigma_n$ becomes what is called a Coxeter group of type B. It is generated by $\pi_0 := (1, -1)$ and $\pi_i := (i, i + 1)$, $1 \leq i \leq n - 1$, which satisfy the generalized braid relations

\[
\begin{align*}
\pi_0^2 &= e, & 0 \leq i \leq n - 1, \\
(\pi_0 \pi_1)^4 &= (\pi_i \pi_{i+1})^3 = e, & 1 \leq i \leq n - 1, \\
(\pi_i \pi_j)^2 &= e, & |i - j| \geq 2, 0 \leq i, j \leq n - 1. 
\end{align*}
\]

(A.1)

An element $\sigma \in \Sigma_n$ expresses an irreducible form,

\[
\sigma = \pi_{i_1} \cdots \pi_{i_k}, \quad 0 \leq i_1, \ldots, i_k \leq n - 1,
\]

and in this case

\[
\ell_1(\sigma) := \text{the number of } \pi_0 \text{ in } \sigma, \\
\ell_2(\sigma) := \text{the number of } \pi_i, \quad 1 \leq i \leq n - 1, \text{ in } \sigma
\]

are well defined. Let $H$ be a separable Hilbert space. For a given self-adjoint involution $f \mapsto \overline{f}$ for $f \in H$, an action of $\Sigma_n$ on $H^\otimes n$ is defined by

\[
\begin{align*}
\pi_0(f_1 \otimes \cdots \otimes f_n) &= \overline{f_1} \otimes f_2 \otimes \cdots \otimes f_n, & n \geq 1, \\
\pi_i(f_1 \otimes \cdots \otimes f_n) &= f_1 \otimes \cdots \otimes f_{i-1} \otimes f_i \otimes f_{i+1} \otimes f_{i+2} \otimes \cdots \otimes f_n, & n \geq 2, 1 \leq i \leq n - 1. 
\end{align*}
\]

(A.2)

The $(\alpha, q)$-inner product on the full Fock space $F(H)$ is defined by

\[
\langle f_1 \otimes \cdots \otimes f_m, g_1 \otimes \cdots \otimes g_n \rangle_{\alpha,q} := \delta_{m,n} \sum_{\sigma \in \Sigma_n} \alpha^{\ell_1(\sigma)} q^{\ell_2(\sigma)} \prod_{j=1}^{n} \langle f_j, g_{\sigma(j)} \rangle_H, \quad \alpha, q \in (-1, 1)
\]

(A.3)

with conventions $0^0 = 1$ and $g_{-k} = \overline{g_k}$, $k = 1, 2, \ldots, n$. For example, if one may define the involution as $\overline{f} := -f$, then $g_{-k} = -g_k$. Equipped with this inner product the full Fock space $F(H)$ is denoted by $\mathcal{F}_{\alpha,q}(H)$ to emphasize on the dependence of the inner product on $\alpha, q$.

The $(\alpha, q)$-creation operator $B^+_{\alpha,q}(f)$ is the usual left creation operator on the full Fock space, and the $(\alpha, q)$-annihilation operator $B^-_{\alpha,q}(f)$ is its adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_{\alpha,q}$. They satisfy the commutation relation

\[
B^-_{\alpha,q}(f) B^+_{\alpha,q}(g) - q B^+_{\alpha,q}(g) B^-_{\alpha,q}(f) = \langle f, g \rangle_H I + \alpha \langle f, g \rangle_H q^{2n}, \quad f, g \in H.
\]

(A.4)

The readers can consult [BEH15] for details.

References


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On a mathematical treatment of measurement correlations

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2016/06/08

This talk is based on papers

[00’16] K. Okamura and M. Ozawa,
Measurement theory in local quantum physics,

and on


Axioms of Algebraic Quantum Theory

Axiom 1.
A physical system in a specified experimental situation is described by a
$W^*$-probability space (a pair of a $W^*$-algebra and a normal state on it)
$(\mathcal{M}, \rho)$.

In this talk, we assume that $W^*$-algebras are von Neumann algebras on
separable Hilbert spaces.

Axiom 2 (Born statistical formula).
When an observable $A$ of $\mathcal{M}$ is precisely measured in a normal state $\rho$,
the probability $\Pr\{A \in \Delta | \rho\}$ that the spectrum of $A$ belonging to $\Delta$
emerge is given by

$$\Pr\{A \in \Delta | \rho\} = \rho(B^A(\Delta)).$$ (1)
Historical Remarks

- Under the repeatability hypothesis, a standard assumption in 1930s, von Neumann constructed a model of quantum measurement called a von Neumann model and derived the so-called von Neumann-Luders projection postulate for nondegenerate discrete observables \( A = \int a \, dE^A(a) = \sum_j a_j E(a_j) \) in his famous book [vN’32]: the state change caused by the measurement of \( A \) with the measured output \( \Delta \in B(\mathbb{R}) \) is given by

\[
\rho \mapsto \frac{\sum_{a \in \Delta} E^A(a) \rho E^A(a)}{\rho(E^A(\Delta))} = \left( \frac{\sum_{a \in \Delta} E^A(a) \rho E^A(a)}{\rho(E^A(\Delta))} \right) E^A(\Delta).
\]

**Postulate 1 (Repeatability hypothesis [Ozawa’15]).**

If an observable \( A \) is measured twice in succession in the object system, then we get the same value each time.


- Nakamura and Umegaki pointed out in [NU’62] that the map

\[
\rho \mapsto \sum_{a \in \mathbb{R}} E^A(a) \rho E^A(a)
\]

is nothing but (the dual map of) a conditional expectation onto (the predual of) the von Neumann algebra generated by \( A \), and conjectured that the same argument holds for continuous observables.
- Arveson [W. Arveson, Amer. J. Math. 89, 578 (1967)] proved that such conditional expectations do not exist for the continuous case.
- Following those investigations, Davies and Lewis [DL’70] abandoned the repeatability hypothesis (Postulate 1) and introduced the notion of instrument which describes general state changes caused by the measurement.


### Completely Positive Instrument

\( M \) : a \( \sigma \)-finite von Neumann algebra on a Hilbert space \( \mathcal{H} \) with predual (the set of ultraweakly continuous linear maps on \( M \)) \( M_* \).

\( P(M_*) \) : the set of positive linear maps on \( M_* \).

\((S, \mathcal{F})\) : a measurable space.

**Definition 1 (Instrument [DL’70]):**

1. A map \( \mathcal{I} : \mathcal{F} \to P(M_*) \) is called an instrument for \((M, S)\) if it satisfies the following two conditions:
   1. For all \( \rho \in M_* \), \( M \in M \) and mutually disjoint sequence \( \{\Delta_j\} \subset \mathcal{F} \),
      \[
      \langle \mathcal{I}(\bigcup_j \Delta_j) \rho, M \rangle = \sum_j \langle \mathcal{I}(\Delta_j) \rho, M \rangle; \tag{2}
      \]
   2. \( \|\mathcal{I}(S)\rho\| = \|\rho\| \) for all \( \rho \in M_* \).

The dual map \( \mathcal{I} : M \times \mathcal{F} \to M \) of an instrument \( \mathcal{I} \) for \((M, S)\) is defined by

\[
\langle \rho, \mathcal{I}(M, \Delta) \rangle = \langle \mathcal{I}(\Delta) \rho, M \rangle \tag{3}
\]

for all \( \rho \in M_* \), \( M \in M \) and \( \Delta \in \mathcal{F} \).
The dual map of an instrument \( \mathcal{I} \) for \( (\mathcal{M}, S) \) is characterized by the following conditions:

(i) For every \( \Delta \in \mathcal{F} \), the map \( M \mapsto \mathcal{I}(M, \Delta) \) is positive and linear;

(ii) For all \( \rho \in \mathcal{M}_+ \), \( M \in \mathcal{M} \) and mutually disjoint sequence \( \{\Delta_j\} \subset \mathcal{F} \),

\[
\langle \rho, \mathcal{I}(M, \cup_j \Delta_j) \rangle = \sum_j \langle \rho, \mathcal{I}(M, \Delta_j) \rangle ;
\]

(iii) \( \mathcal{I}(1, S) = 1 \).

That is to say, a map \( \mathcal{I} : \mathcal{M} \times \mathcal{F} \to \mathcal{M} \) satisfying the above conditions is the dual map of an instrument for \( (\mathcal{M}, S) \).

**Definition 2 (Completely positive (CP) instrument [Ozawa’84]).**

An instrument \( \mathcal{I} \) for \( (\mathcal{M}, S) \) is said to be completely positive if, for all \( \Delta \in \mathcal{F} \), the map \( M \mapsto \mathcal{I}(M, \Delta) \) is completely positive.

We denote by \( \text{CPInst}(\mathcal{M}, S) \) the set of CP instruments for \( (\mathcal{M}, S) \).


---

**Davies-Lewis proposal**

For every apparatus \( A(x) \) measuring \( S \), where \( x \) is the output variable of \( A(x) \) taking values in a measurable space \( (S, \mathcal{F}) \), there always exists an instrument \( \mathcal{I} \) for \( (\mathcal{M}, S) \) corresponding to \( A(x) \) in the following sense.

For every input state \( \rho \), the probability distribution \( \Pr(x \in \Delta | \rho) \) of \( x \) in \( \rho \) is given by

\[
\Pr(x \in \Delta | \rho) = \frac{|| \mathcal{I}(\Delta) \rho ||}{|| \mathcal{I}(\Delta) \rho ||} = \langle \mathcal{I}(\Delta) \rho, 1 \rangle \quad (5)
\]

for all \( \Delta \in \mathcal{F} \), and the state \( \rho_{(x \in \Delta)} \) after the measurement under the condition that \( \rho \) is the prepared state and the outcome \( x \in \Delta \) is given by

\[
\rho_{(x \in \Delta)} = \frac{\mathcal{I}(\Delta) \rho}{|| \mathcal{I}(\Delta) \rho ||} \quad (6)
\]

if \( \Pr(x \in \Delta | \rho) > 0 \), and \( \rho_{(x \in \Delta)} \) is indefinite if \( \Pr(x \in \Delta | \rho) = 0 \).

---

**Statistical Approach to Quantum Measurement I**

**Notation.**

We denote a measuring apparatus by \( A(x) \), where \( x \) is the output variable of this apparatus and takes values in a measurable space \( (S, \mathcal{F}) \).

**Postulate 2 [Ozawa’04, Ozawa’14].**

In standard experimental situations, we can specify the following two components by using the measuring apparatus \( A(x) \):

1. **The probability measure** \( \Pr(x \in \Delta | \rho) \) of the outcome \( x \) in an arbitrary normal state \( \rho \) on \( \mathcal{M} \);
2. **The state** \( \rho_{(x \in \Delta)} \) just after the measurement under the condition that \( \rho \in \mathcal{M}_+ \) is the state before the measurement and the outcome \( x \) contained in \( \Delta \) emerges, where \( \rho_{(x \in \Delta)} \) is defined for all \( \Delta \in \mathcal{F} \) with \( \Pr(x \in \Delta | \rho) > 0 \) and normal, and represents an indefinite state otherwise.


Statistical Approach to Quantum Measurement II

Consider the successive measurement carried out by $A(x)$ and an observable $Y$ of $N$ in this order. Then, the joint probability measure $\Pr((Y, x) \in \Delta \| \rho)$ of $x$ and $Y$ on $(\mathbb{R} \times S, B(\mathbb{R}) \times \mathcal{F})$ is uniquely determined by the formula

$$\Pr((Y, x) \in \Delta_2 \times \Delta_1 \| \rho) = \Pr(Y \in \Delta_2 \| \rho_{\{x \in \Delta_1\}}) \Pr(x \in \Delta_1 \| \rho)$$ (7)

for all $\Delta_1 \in \mathcal{F}$ and $\Delta_2 \in B(\mathbb{R})$.

**Postulate 3 [Ozawa '04, Ozawa '14]**

For any successive measurements carried out by $A(x)$ and an observable $Y$ in this order, the joint probability measure $\Pr((Y, x) \in \Delta \| \rho)$ is an affine function of $N_{*1}$.

**Theorem 3 [Ozawa '04, Ozawa '14]**

Under the Postulate 2, the followings are equivalent:

1. The Davies-Lewis proposal.
2. The Postulate 3.

Further Measurement Theory

**Question.**

Let $A$ be an observable of $N$ to be measured.

Von Neumann [vN'32] discussed a model of measurement consisting of

1. a Hilbert space $L^2(\mathbb{R})$,
2. a unit vector $\xi \in L^2(\mathbb{R})$,
3. a meter observable $Q = \int q \, dE^Q(q)$, and
4. a unitary $U = e^{iA \| P}$ on $\mathcal{H} \otimes L^2(\mathbb{R})$, $(0 \neq \gamma \in \mathbb{R}, [Q, P] = i1)$, which defines a CP instrument $I_{A,VN}$ for $(N, R)$ by

$$I_{A,VN}(M, \Delta) = \text{Tr}_{L^2(\mathbb{R})}[U^*(M \otimes E^Q(\Delta))U(1 \otimes |\xi\rangle \langle \xi|)]$$ (8)

for all $\Delta \in B(\mathbb{R})$ and $M \in N$.

Do all CP instruments permit modeling of this kind?

Measuring Process

**Definition 3 (Measuring process).**

A measuring process $M$ for $(N, S)$ is a 4-tuple $M = (K, \sigma, E, U)$ consisting of

1. a Hilbert space $K$,
2. a normal state $\sigma$ on $B(K)$,
3. a spectral measure $E : \mathcal{F} \to B(K)$, and
4. a unitary operator $U$ on $\mathcal{H} \otimes K$

satisfying the following condition:

$$\{I_M(M, \Delta) \mid M \in M, \Delta \in \mathcal{F}\}$$ is contained in $N$,

where $I_M$ is a CP instrument for $(B(\mathcal{H}), S)$ defined by

$$I_M(X, \Delta) = (id \otimes \sigma)(U^*X \otimes E(\Delta)U)$$ (9)

for all $X \in B(\mathcal{H})$ and $\Delta \in \mathcal{F}$.

This is a generalization of von Neumann model of measurement [vN'32].
A Complete Characterization of Quantum Mechanical Measurements

By Stinespring representation theorem and the uniqueness theorem of irreducible normal representation of $B(ℋ)$, the following theorem holds:

\textbf{Theorem 2 [Ozawa '84].}

Let $ℋ$ be a Hilbert space and $(S, ℱ)$ a measurable space. Then there is a one-to-one correspondence between statistical equivalence classes of measuring processes $ℳ = (K, σ, E, U)$ for $(B(ℋ), S)$ and CP instruments $ℐ$ for $(B(ℋ), S)$, which is given by the relation

$$ℐ(M, Δ) = (1 ⊗ σ) [U^* (M ⊗ E(Δ)) U]$$

(10)

for all $Δ ∈ ℱ$ and $M ∈ B(ℋ)$.

Two measuring processes $ℳ_1 = (K_1, σ_1, E_1, U_1)$ and $ℳ_2 = (K_2, σ_2, E_2, U_2)$ for $(ℳ, S)$ are said to be statistically equivalent if $ℐ_{ℳ_1}(M, Δ) = ℐ_{ℳ_2}(M, Δ)$ for all $M ∈ ℳ$ and $Δ ∈ ℱ$.

---

The Normal Extension Property (NEP)

\textbf{Proposition 1.}

For every CP instrument $ℐ$ for $(ℳ, S)$, there uniquely exists a unital (binormal) CP map $Ψ_ℐ : ℳ ⊗_{\text{min}} L^∞(S, ℱ) → ℳ$ such that

$$Ψ_ℐ(M ⊗ |x_Δ|) = ℐ(M, Δ)$$

(11)

for all $M ∈ ℳ$ and $Δ ∈ ℱ$.

- By Arveson’s extension theorem, there exists a unital CP map $Ψ_ℐ : ℳ ⊗ L^∞(S, ℱ) → B(ℋ)$ such that $Ψ_ℐ|_{ℳ ⊗_{\text{min}} L^∞(S, ℱ)} = Ψ_{ℐ}$, which is not always nonnormal.

\textbf{Definition 4 (Normal extension property [O’O’16]).}

A CP instrument $ℐ$ for $(ℳ, S)$ has normal extension property (NEP) if there exists a unital normal CP map $Ψ_ℐ : ℳ ⊗ L^∞(S, ℱ) → ℳ$ such that $Ψ_ℐ|_{ℳ ⊗_{\text{min}} L^∞(S, ℱ)} = Ψ_{ℐ}$.

\textbf{Theorem 3 (A generalization of Theorem 2 [O’O’16]).}

Let $ℳ$ be a σ-finite von Neumann algebra on a Hilbert space $ℋ$ and $(S, ℱ)$ a measurable space. Then there is a one-to-one correspondence between statistical equivalence classes of measuring processes $ℳ = (K, σ, E, U)$ for $(ℳ, S)$ and CP instruments $ℐ$ for $(ℳ, S)$ with the normal extension property (NEP), which is given by the relation

$$ℐ(M, Δ) = (1 ⊗ σ) [U^* (M ⊗ E(Δ)) U]$$

(12)

for all $Δ ∈ ℱ$ and $M ∈ ℳ$.

Key Facts:

- The structure theorem of normal representations of v.N. algs.
- The existence of CP instrument $ℐ$ for $(B(ℋ), S)$ such that $ℐ(M, Δ) = ℐ(M, Δ)$ for all $Δ ∈ ℱ$ and $M ∈ ℳ$.
Examples of CP Instruments without NEP

Example 1.

\[ m : \text{Lebesgue measure on } [0, 1]. \]

A CP instrument \( I_m \) for \( (L^\infty([0, 1]), [0, 1]) \) is defined by

\[ I_m(f, \Delta) = [x_\Delta] f \tag{13} \]

for all \( \Delta \in B([0, 1]) \) and \( f \in L^\infty([0, 1], m) \).

Example 2.

Let \( \mathcal{N} \) be an AFD factor of type II_1 on a separable Hilbert space \( \mathcal{H} \), \( A \) a self-adjoint element of \( \mathcal{N} \) with continuous spectrum, and \( \mathcal{E} \) a conditional expectation from \( \mathcal{N} \) into \( \{A\}' \cap \mathcal{N} \). We define a CP instrument \( I_A \) for \( (\mathcal{N}, \mathcal{B}) \) by

\[ I_A(\mathcal{N}, \Delta) = \mathcal{E}(\mathcal{N}) E^A(\Delta) \tag{14} \]

for all \( \mathcal{N} \in \mathcal{N} \) and \( \Delta \in B(\mathcal{B}) \).

Theorem 4.

If \( \mathcal{M} \) is atomic, every CP instrument \( I \) for \( (\mathcal{M}, S) \) has the NEP.

Theorem 5.

Suppose that \( \mathcal{M} \) is injective. For any CP instrument \( I \) for \( (\mathcal{M}, S) \), there exists a net \( \{I_a\}_{a \in A} \) of CP instruments with the NEP satisfying the following two conditions:

1. For every \( \varepsilon > 0 \), \( n \in \mathbb{N} \), \( \rho_1, \ldots, \rho_n \in \mathcal{M}_{\text{sa}}, M_1, \ldots, M_n \in \mathcal{M}, \Delta_1, \ldots, \Delta_n \in \mathcal{F} \), there exists \( a \in A \) such that

\[ |I(\Delta_i)\rho_i - I_a(\Delta_i)\rho_i| < \varepsilon \tag{15} \]

for all \( i = 1, \ldots, n \).

2. For every \( a \in A \) and \( \Delta \in \mathcal{F} \), it holds that \( I(1, \Delta) = I_a(1, \Delta) \).

On Complete Positivity of Instruments

Postulate 4 (Trivial extendability [Ozawa ‘04, Ozawa ‘14]).

For any apparatus \( A(x) \) measuring a system \( S \) described by \( \mathcal{M} \) and any quantum system \( S' \) described by \( \mathcal{N} \) statistically independent of \( S \) and not interacting with \( A(x) \), there exists an apparatus \( A(x') \) measuring the composite system \( S \oplus S' \) described by \( \mathcal{M} \otimes \mathcal{N} \) with the following statistical properties:

\[ \Pr\{x' \in \Delta || \rho \otimes \sigma\} = \Pr\{x \in \Delta || \rho\}, \tag{16} \]

\[ (\rho \otimes \sigma)_{(x' \in \Delta_1)} = \rho_{(x \in \Delta_1)} \otimes \sigma \tag{17} \]

for all \( \Delta \in \mathcal{F} \), \( \rho \in \mathcal{M}_{\text{sa}} \), and \( \sigma \in \mathcal{N}_{\text{sa}} \).

Since we have \( A(x') \leftrightarrow I' \) and \( A(x) \leftrightarrow I \), it holds by Eq.(16) and Eq.(17) that

\[ I'(\Delta) = I(\Delta) \otimes \text{id}_{\mathcal{N}}. \tag{18} \]

Consider when \( \mathcal{N} \) is \( M_n(\mathbb{C}) \). Since \( I' \) is positive, \( I \) is \( n \)-positive.

If \( n \) can be arbitrarily chosen, then \( I \) is completely positive.
Another Approach

Observation.

If a measuring process $M = (K, \sigma, E, U)$ for $(B(H), S)$ is given, we can consider correlation functions such as

$$\rho \otimes \sigma(\pi_{\Delta_1}(M_1)\pi_{\Delta_2}(M_2)),$$

where $\pi_X(X) = X \otimes 1_K$ and $\pi_{\Delta}(M) = U^*(M \otimes E(\Delta))U$ for all $M \in B(H)$.

It is natural to consider correlation functions in the context of measurement since measuring processes are also quantum dynamical processes.

Plan.

In terms of quantum stochastic processes in the sense of [AFL’82], we reconsider measuring processes.


We modify here its operator-valued extension of the theory of [AFL’82], which is formulated by Belavkin [Belavkin ’85], to define a generalization of instrument.


Notation.

Let $T^{(1)}$ be a set. We define a set $T$ by $T = \bigcup_{n \geq 1} (T^{(1)})^n$. For each $T \in T$, we denote by $|T|$ the natural number $n$ such that $T \in (T^{(1)})^n$.

For each $T = (t_1, t_2, \ldots, t_n) \in T$, $T^\# = (t_n, t_{n-1}, \ldots, t_1)$.

For any $S = (s_1, s_2, \ldots, s_m), T = (t_1, t_2, \ldots, t_n) \in T$, the product $S \times T$ is defined by

$$S \times T = (s_1, s_2, \ldots, s_m, t_1, t_2, \ldots, t_n).$$

For any $n \in \mathbb{N}$ and $M = (M_1, M_2, \ldots, M_n) \in M^n$,

$$M^\# = (M^*_n, M^*_{n-1}, \ldots, M^*_1).$$

For any $m, n \in \mathbb{N}$, $M = (M_1, M_2, \ldots, M_m) \in M^m$ and $N = (N_1, N_2, \ldots, N_n) \in M^n$,

$$M \times N = (M_1, M_2, \ldots, M_m, N_1, N_2, \ldots, N_n) \in M^{m+n}.$$ 

Let $(S, \mathcal{F})$ be a measurable space. We define a set $T_S$ by

$$T_S = \bigcup_{n=1}^\infty (T_S^{(1)})^n$$

and

$$T_S^{(1)} = \{\text{in} \} \cup \mathcal{F}.$$ 

Definition 5 [O’15].

A family $\{W_T\}_{T \in T}$ of maps $W_T : M^{(1)} = M \times \cdots \times M \to M$ is called a system of measurement correlations for $(M, S)$ if it satisfies $T^{(1)} = T_S^{(1)}$ and the following six conditions:

1. For any $T \in T$, $W_T(M_1, \ldots, M_{\overline{T}})$ is separately ultraweakly continuous and linear in each variable $M_1, \ldots, M_{\overline{T}}$.

2. For any $m \in \mathbb{N}$, $T_1, \ldots, T_m \in T$, $M_1 \in M^{(1)}$, $\ldots$, $M_m \in M^{(1)}$ and $\xi_1, \ldots, \xi_m \in \mathcal{H}$,

$$\sum_{i,j=1}^m <\xi_i | W_{T_1} \times T_2 \times \cdots \times T_m (M_i \times M_j) | \xi_j> \geq 0.$$
(3) For any $T = (t_1, t_2, \ldots, t_{|T|}) \in T$, $M = (M_1, \ldots, M_{|T|}) \in \mathcal{M}^{|T|}$ and $M \in \mathcal{M}$,

$$MW_T(M) = W_{<T}(M, M_1, \ldots, M_{|T|}),$$  \hspace{1cm} (26)

$$W_T(M)M = W_{<T}(M_1, \ldots, M_{|T|}, M),$$  \hspace{1cm} (27)

where $<T = (\text{in}, t_1, t_2, \ldots, t_{|T|-1})$ and $T^\text{in} = (t_1, t_2, \ldots, t_{|T|-1}, \text{in})$.

(4) Let $T = (t_1, t_2, \ldots, t_{|T|}) \in T$. If $t_k = t_{k+1} = \text{in}$ or $t_k, t_{k+1} \in F$ for some $1 \leq k \leq |T| - 1$,

$$W_T(M_1, \ldots, M_k, M_{k+1}, \ldots, M_{|T|}) = W_{T^\text{in}}(M_1, \ldots, M_k, M_{k+1}, \ldots, M_{|T|}),$$  \hspace{1cm} (28)

for all $(M_1, M_2, \ldots, M_{|T|}) \in \mathcal{M}^{|T|}$, where $T^\text{in} = (t_1, t_2, \ldots, t_{k-1}, t_k \wedge t_{k+1}, t_{k+2}, \ldots, t_{|T|})$ and

$$t_k \wedge t_{k+1} = \begin{cases} 
\text{in}, & \text{if } t_k = t_{k+1} = \text{in} \\
\Delta_k \cap \Delta_{k+1}, & \text{if } t_k = \Delta_k, t_{k+1} = \Delta_{k+1}
\end{cases}$$

(5) For any $T = (t_1, t_2, \ldots, t_{|T|}) \in T$ with $t_k = \text{in}$ or $\Delta_k$, and $(M_1, \ldots, M_{|T|}) \in \mathcal{M}^{|T|}$ with $M_k = 1$,

$$W_T(M_1, \ldots, M_{k-1}, 1, M_{k+1}, \ldots, M_{|T|}) = W_{\tilde{T}}(M_1, \ldots, M_{k-1}, M_{k+1}, \ldots, M_{|T|}),$$  \hspace{1cm} (29)

where $\tilde{T} = (t_1, t_2, \ldots, t_{k-1}, t_k, t_{k+1}, t_{k+2}, \ldots, t_{|T|})$. In addition,

$$W_\text{in}(1) = W_S(1) = 1.$$  \hspace{1cm} (30)

(6) For any $n \in \mathbb{N}$, $1 \leq k \leq n$, $t_1, t_2, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n \in T$, a mutually disjoint sequence $(\{t_{k,j}\}) \in F$, and $M \in \mathcal{M}^n$,

$$W_{(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n)}(M) = \sum J W_{(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n)}(M),$$

where $\cup_j t_{k,j} = \cup_j \Delta_j$. \(\Rightarrow W_\Delta\text{ is a CP instrument for } (\mathcal{M}, S)\)

**Measuring Process**

**Definition 3’ (Measuring process (\text{O’15})).**

A measuring process $\mathcal{M}$ for $(\mathcal{M}, S)$ is a 4-tuple $\mathcal{M} = (K, \sigma, E, U)$ which consists of a Hilbert space $K$, a normal state $\sigma$ on $B(K)$, a spectral measure $E : F \to B(K)$, and a unitary operator $U$ on $\mathcal{H} \otimes K$ and defines a system of measurement correlations $\{W^M_T\}_{T \in T_S}$ for $(\mathcal{M}, S)$ as follows:

We define two representations of $\mathcal{M}$ on $\mathcal{H} \otimes K$ by

$$\pi_m(M) = M \otimes 1_K, \quad \pi_\Delta(M) = U^*(M \otimes E(\Delta))U,$$  \hspace{1cm} (32)

for all $M \in \mathcal{M}$. We use the following notation.

$$\pi_T(M) := \pi_m(M_1) \pi_m(M_2) \cdots \pi_m(M_{|T|}).$$  \hspace{1cm} (33)

For each $T \in T_S$, $W^M_T : \mathcal{M}^{|T|} \to \mathcal{M}$ is defined by

$$W^M_T(M) = (id \otimes \sigma)(\pi_T(M)),$$  \hspace{1cm} (34)

for all $M \in \mathcal{M}^{|T|}$.

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Definition 6 [O’15].
Two measuring processes \( \mathcal{M}_1 = (\mathcal{K}_1, \sigma_1, E_1, U_1) \) and 
\( \mathcal{M}_2 = (\mathcal{K}_2, \sigma_2, E_2, U_2) \) for \( (\mathcal{M}, S) \) are said to be \( n \)-equivalent if 
\( W^\mathcal{M}_1 = W^\mathcal{M}_2 \) for all \( T \in \mathcal{T} \) such that \( |T| \leq n \). Two measuring processes 
\( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) for \( (\mathcal{M}, S) \) are said to be completely equivalent if they are 
\( n \)-equivalent for all \( n \in \mathcal{N} \).

The 2-equivalence is the same as the statistical equivalence.

Theorem 6 [O’15].
Let \( \mathcal{H} \) be a Hilbert space and \( (S, \mathcal{F}) \) a measurable space. Then there is a 
one-to-one correspondence between complete equivalence classes of 
measuring processes \( \mathcal{M} = (\mathcal{K}, \sigma, E, U) \) for \( (\mathcal{B}(\mathcal{H}), S) \) 
and systems of measurement correlations \( (W_T)_{T \in \mathcal{T}} \) for \( (\mathcal{B}(\mathcal{H}), S) \), 
which is given by the relation 
\[ W_T(M) = W_T^\mathcal{M}(M) \] (35) 
for all \( T \in \mathcal{T} \) and \( M \in \mathcal{M}(\mathcal{T}) \).

Thank you for your attention!
How is the ground state of the (generalized) quantum Rabi model dressed with a real photon?

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joint work with I. Sasaki & J. S. Møller

at Kyushu Univ. on 8 June, 2016

1. Generalized Quantum Rabi Model

- Preparata claims that there are cases where the photon cannot be emitted from the matter when the matter-radiation coupling is very strong, and the ground state is dressed with the photon which should primarily be emitted outside of the matter.

- He also says that the ground state switch from the perturbative ground state to the non-pertubative ground state.

- Preparata found this phenomena stimulated by and based on the Hepp-Lieb quantum phase transition. We thus call this phenomenon the Hepp-Lieb-Preparata quantum phase transition today.
2. Mathematical Models

- Generalized Quantum Rabi Hamiltonian
  \[ H(\omega_a, \omega_c, g) := \frac{\hbar}{2} (\omega_a \sigma_z - \varepsilon \sigma_x) + \hbar \omega_c \left( a ^ \dagger a + \frac{1}{2} \right) + \hbar g \sigma_x (a + a ^ \dagger). \]

- In the case \( \varepsilon = 0 \), we have the quantum Rabi Hamiltonian:
  \[ H_{\text{Rabi}}(\omega_a, \omega_c, g) := \frac{\hbar \omega_a}{2} \sigma_z + \hbar \omega_c \left( a ^ \dagger a + \frac{1}{2} \right) + \hbar g \sigma_x (a + a ^ \dagger). \]

- Our Hamiltonian with \( A^2 \)-term
  \[ H_{A^2} := H(\omega_a, \omega_c, g) + \hbar C_g (a + a ^ \dagger)^2, \]
  where \( C_g \) is a function of the coupling strength \( g \) given as
  \[ C_g = C_g \quad \text{or} \quad C_g = C_g^2 \]
  with a constant \( C \).

3. Bare-Photon Number Expectation

- We denote by \( |E_\nu \rangle, \nu = 0, 1, 2, \cdots \), the eigenstates of the total Hamiltonian \( H_{A^2} \), and by \( E_\nu \) the corresponding eigenenergies with \( E_0 \leq E_1 \leq E_2 \leq \cdots \).

- The ground-state expectation of photon defined by
  \[ N_0^{\text{bare}} := \langle E_0 | a ^ \dagger a | E_0 \rangle \]
  is for the bare photons, and includes the number of virtual photons as well as that of real photons (see p.76 & p.77 of Ref.[HT]).

- Let \( \Delta \Phi \) be the fluctuation including that of the virtual-photon field
  \[ \Phi = (a + a ^ \dagger)/ \sqrt{2 \omega_c} \] at the ground state \( |E_0 \rangle \), i.e.,
  \[ \Delta \Phi = \sqrt{\langle E_0 | (\Phi - \langle E_0 | \Phi | E_0 \rangle) | E_0 \rangle}. \]
3. Bare-Photon Number Expectation

Estimate of $N^{\text{bare}}_0$

Suppose $C_g = C_g^\ell$, $\ell = 1, 2$.

(a) $(\Delta \Phi)^2 \leq \left( \frac{2N^{\text{bare}}_0 + 1}{\omega_c} \right)$.

(b) $N^{\text{bare}}_0 \geq \frac{1}{2} \left( \frac{\sqrt{\omega_c}}{2 \sqrt{\omega_c + 4C_g^{\ell+1}}} + \frac{\sqrt{\omega_c + 4C_g^{\ell+1}}}{2 \sqrt{\omega_c}} - 1 \right) - \epsilon(g)$, where

$$0 < \epsilon(g) \rightarrow \begin{cases} \frac{1}{8C_\omega c} & \text{if } \ell = 1, \\ 0 & \text{if } \ell = 2. \end{cases}$$

and therefore,

$$\lim_{g \to \infty} N^{\text{bare}}_0 = \infty.$$

4. Hopfield-Bogoliubov Transformation

- We follow the pair theory [HT] to avoid the increase of the number of virtual photons.

**Hopfield-Bogoliubov Transformation**

For arbitrary $\omega_c, g, C_g$, the Hopfield-Bogoliubov transformation $U$ is unitary such that

$$U^* H_A^2(\omega_a, \omega_c, g, C_g) U = H(\omega_a, \omega_g, \tilde{g}),$$

where

$$\omega_g = \sqrt{\omega_c^2 + 4C_g g \omega_c} \quad \text{and} \quad \tilde{g} = g \sqrt{\frac{\omega_c}{\omega_g}}.$$

We here note that the Hopfield-Bogoliubov transformation is unitary for our model without any restriction.
5. Physical-Photon Number Expectation

- Thanks to the unitarity of the Hopfield-Bogoliubov transformation, we can define the normalized eigenstates of the renormalized Hamiltonian $H(\omega_a, \epsilon, \omega_g, \tilde{g})$ with the $A^2$-term effect by

$$|E^\text{ren}_\nu\rangle := U^n |E_\nu\rangle,$$

and then, the eigenenergy $E^\text{ren}_\nu$ of each eigenstate $|E^\text{ren}_\nu\rangle$ is $E_\nu$.

- We can show that $E_0$ is always less than $E_1$, i.e., $E_0 < E_1$.

- We consider the renormalized ground-state expectation of real photon,

$$N^\text{ren}_0 := \langle E^\text{ren}_0 | a^\dagger a | E^\text{ren}_0 \rangle,$$

and how the ground state of the renormalized quantum Rabi Hamiltonian is dressed with real photons.

*Estimate of $N^\text{ren}_0$*

$$L^\text{ren}(g)^2 \leq N^\text{ren}_0 \leq \frac{\tilde{g}^2}{\omega_g^2} \omega_c^{-1/2} \left( \frac{\omega_c}{g^{4/3}} + \frac{4C_g}{g^{1/3}} \right)^{-3/2},$$

provided that $L^\text{ren}(g) \geq 0$, where

$$L^\text{ren}(g) := \frac{\tilde{g}}{\omega_g} - \left\{ \sqrt{\omega_a^2 + \epsilon^2} \left( 1 - e^{-2\tilde{g}^2/\omega_g^2} \right) / 2\omega_g \right\}^{1/2}.$$

*How to Be Dressed*

(a) $N^\text{ren}_0 = |\langle E^\text{ren}_0 | E^\text{ren}_0 \rangle|^2 \frac{\tilde{g}^2}{\omega_g^2} + \hbar^2 \frac{g^2}{\omega_g^2} \sum_{\nu=1}^{\infty} \frac{|\langle E^\text{ren}_\nu | E^\text{ren}_0 \rangle|^2}{(E_\nu - E_0 + \hbar \omega_g)^2}$.

(b) There is an $\nu_*$ so that $\hbar^2 \frac{g^2}{\omega_g^2} \frac{|\langle E^\text{ren}_{\nu_*} | E^\text{ren}_0 \rangle|^2}{(E_{\nu_*} - E_0 + \hbar \omega_g)^2} \leq N^\text{ren}_0$. 

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Hirokawa (Hiroshima Univ.) Quantum Phase Transition Fukuoka, June 2016 8 / 8
1. Classical SAW

2. SAW on random conductors

3. Keys to the proof of the lower bound

4. Towards critical behavior

5. Closing remark

1.1. Definition

A self-avoiding walk (SAW) is a (nearest-neighbor) SAW if

\( \omega = (\omega_0, \omega_1, \ldots, \omega_{|\omega|}) \subset \mathbb{Z}^d \)

is a SAW if

\[
\forall i \neq j \quad |\omega_i - \omega_j| \geq 1 \quad (= 1 \text{ when } |i - j| = 1).
\]

\[\Omega(x) = \{ \text{SAW } \omega : \omega_0 = x \}, \quad \Omega(x; n) = \{ \omega \in \Omega(x) : |\omega| = n \}.\]

\[
X(h) = \sum_{\omega \in \Omega(x)} e^{-h|\omega|} \quad \text{are translation-invariant (i.e., independent of } x). \]

\[c_n = |\Omega(x; n)| \]
1. Classical SAW

2. SAW on random conductors

3. Keys to the proof of the lower bound

4. Towards critical behavior

5. Closing remark

1.2. Basic facts

**Subadditivity**

\[ \forall m, n \in \mathbb{Z}_+, \quad c_{m+n} \leq c_m c_n. \]

**Proof.**

\[
    c_{m+n} = \sum_{y \in \mathbb{Z}^d} \sum_{\omega \in \Omega(x;m)} \mathbb{1}_{\omega(y)=v} \sum_{\eta \in \Omega(y;n)} \mathbb{1}_{\omega \eta \in \Omega(x;m+n)} \leq |\Omega(y;n)| = c_n
\]

**The connective constant** ([Hammersley & Morton : 1954]

\[ \exists \mu = \lim_{n \to \infty} c_n^{1/n} = \inf_{n \in \mathbb{N}} c_n^{1/n} \quad (= 2d \quad \text{for simple random walk}). \]

**Proof.**

Fix \( k \in \mathbb{N} \). Then \( \forall n = km + l \) for some \( m \in \mathbb{Z}_+, \ l \in \{0, \ldots, k - 1\} \). By subadditivity,

\[
    c_n^{1/n} \leq (c_k \cdots c_k c_l)^{1/n} \leq c_k^{1/k} c_l^{1/n},
\]

which implies \( \limsup_{m \to \infty} c_n^{1/n} \leq c_k^{1/k} \). Then, take the infimum over \( k \in \mathbb{N} \).

---

The critical point for classical SAW

\[
    \chi_h \equiv \sum_{\omega \in \Omega(x)} e^{-h|\omega|} < \infty \quad \iff \quad h > h_0 \equiv \log \mu \quad (= 0 \quad \text{for } d = 1).
\]

**Proof.**

By monotonicity, \( \chi_h < \infty \) if \( h > h_0 \) and \( \chi_h = \infty \) if \( h < h_0 \). Since \( \mu = \inf_{n \in \mathbb{N}} c_n^{1/n} \),

\[
    \chi_h = \sum_{n=0}^{\infty} (e^{-h} c_n^{1/n})^n \geq \sum_{n=0}^{\infty} (e^{-h} \mu)^n = \frac{1}{1 - e^{-(h-h_0)}} \geq \frac{1}{h-h_0} \quad [h > h_0],
\]

which implies \( \chi_{h_0} = \infty \).

**Other interesting results for classical SAW**

- \( d = 2, \ \text{hexagonal} \quad \Rightarrow \quad h_0 = \log \sqrt{2 + \sqrt{2}} \quad \text{[Duminil-Copin & Smirnov : 2010].} \)

- \( d > 4 \quad \Rightarrow \quad \chi_h \sim \frac{3A}{h-h_0} \quad \text{[Hara & Slade : 1992].} \)
2.2. Main result

Theorem [with Chino (2016)]

Let $d \geq 1$ and $\beta \geq 0$. Then $\hat{h}_{\beta,X}^q(x)$ is $\mathbb{P}$-almost surely a constant that does not depend on $x \in \mathbb{Z}^d$. Moreover,

$$\mathbb{P}\left(h_0 - \beta \mathbb{E}[X_b] \leq \hat{h}_{\beta,X}^q(x) \leq h_{\beta}^a\right) = 1.$$ 

Remark 1:

- For i.i.d. $X$ with finite generating function $\lambda_\beta \equiv \mathbb{E}[e^{-\beta X_b}]$,
  $$\mathbb{E}[\hat{X}_{h,\beta,X}(x)] = \sum_{\omega \in \Omega(x)} \prod_{j=1}^{\text{length}} e^{-h_\beta^{j} \mathbb{E}[e^{-\beta X_b_j}]} = X_{h - \log \lambda_\beta}. \quad \therefore \, h_{\beta}^a = h_0 + \log \lambda_\beta.$$

- By Jensen’s inequality, $\log \lambda_\beta \geq -\beta \mathbb{E}[X_b]$. The gap is $O(\beta^2)$.

- For $d = 1$, $\hat{h}_{\beta,X}(x) \overset{a.s.}{=} -\beta \mathbb{E}[X_b]$ (n.b., $h_0 = 0$, since $c_{n \geq 1}$ is always 2) because, by denoting $\Delta_j \equiv X_{(j-1,j)} - \mathbb{E}[X_b]$,
  $$\hat{X}_{h,\beta,X}(o) = 1 + \sum_{n=1}^{\infty} e^{-(h+\beta \mathbb{E}[X_b])n} \left( e^{-\beta \sum_{j=1}^{n} \Delta_j} + e^{-\beta \sum_{j=1}^{n-1} \Delta_j} \right).$$

  $\sim 2e^{\alpha n}$ by ergodicity.
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2.2. Main result

**Theorem [with Chino (2016)]**

Let $d \geq 1$ and $\beta \geq 0$. Then $\hat{h}^q_{\beta, X}(x)$ is $\mathbb{P}$-almost surely a constant that does not depend on $x \in \mathbb{Z}^d$. Moreover,

$$\mathbb{P}\left(h_0 - \beta \mathbb{E}[X_b] \leq \hat{h}^q_{\beta, X}(x) \leq h^a_{\beta}\right) = 1.$$

**Remark 2:**

- For $d \geq 2$, since $c_n \geq \mu^n$, $\hat{h}^q_{\beta, X}(x)$ is expected to be $> h_0 - \beta \mathbb{E}[X_b]$.
- The key to the proof of the above theorem: for $Z \geq 0$,

$$\mathbb{P}(Z \geq \varepsilon \mathbb{E}[Z]) \leq \varepsilon^{-1} \left(1 - \varepsilon\right)^2 \mathbb{E}[Z]^2 / \mathbb{E}[Z^2] \quad \text{[Markov]}. $$

**Markov**

$$\mathbb{E}[\hat{c}_{\beta, X}(x;n) \geq n^2 \mathbb{E}[\hat{c}_{\beta, X}(x;n)]] \leq n^{-2}$$

**Borel-Cantelli**

$$\mathbb{P}\left(\hat{c}_{\beta, X}(x;n) \leq n^2 c_n h^a_{\beta} \text{ eventually} \right) = 1.$$

**Remark 3:**

- For $d = 2$, $\hat{h}^q_{\beta, X}(x) \overset{\text{a.s.}}{\leq} h^a_{\beta}$ may be proven by adapting the fractional-moment method [Lacoin: 2014]:

$$\exists \delta, \theta \in (0, 1) : \mathbb{E}[\hat{c}_{\beta, X}(x;n) \theta] \leq \left(\delta^n \mathbb{E}[\hat{c}_{\beta, X}(x;n)]\right)\theta.$$

Then

**Markov**

$$\mathbb{P}\left(\hat{c}_{\beta, X}(x;n) \theta \geq n^2 \left(\delta^n \mathbb{E}[\hat{c}_{\beta, X}(x;n)]\right)\theta \leq n^{-2}$$

**Borel-Cantelli**

$$\mathbb{P}\left(\hat{c}_{\beta, X}(x;n) \leq n^{2/\theta} \delta^n c_n h^a_{\beta} \text{ eventually} \right) = 1,$$

which implies $\hat{h}^q_{\beta, X}(x) \overset{\text{a.s.}}{\leq} h^a_{\beta} - \frac{1}{\delta}$. 
3.1. The 1st key lemma

**Lemma 1** \( \hat{h}_{\beta,X}^q(x) \) is almost surely a constant function of \( x \in \mathbb{Z}^d \).

**Proof.** Let \( p_\omega \equiv p_{\omega,x} = e^{-\sum_{j=1}^n (h+\beta X_j)} \). For any neighbor \( y \in \mathbb{Z}^d \) of \( x \),

\[ \hat{h}_{\beta,X}(x) = \sum_{\omega \in \Omega(x)} p_\omega (\mathbb{1}_{\{y \in \omega\}} + \mathbb{1}_{\{y \notin \omega\}}). \]

Due to subadditivity and reversibility,

\[
\sum_{\omega \in \Omega(x), y \in \omega} p_\omega = \sum_{\eta \in \Omega(x); \eta_0 = y} p_{\eta} \sum_{\xi \in \Omega(y); \xi_0 = y} p_{\xi} \leq \hat{h}_{\beta,X}(y) \sum_{\eta \in \Omega(x); \eta_0 = y} p_{\eta} \leq \hat{h}_{\beta,X}(y)^2,
\]

\[
\sum_{\omega \in \Omega(x), y \notin \omega} p_\omega \leq p_{(x,y)}^{-1} \sum_{\omega \in \Omega(x), y \notin \omega} p_{(y,x) \cup \omega} \leq p_{(x,y)}^{-1} \hat{h}_{\beta,X}(y).
\]

Since \( X_{(x,y)} \) is integrable and thus almost surely finite, the above implies \( \hat{h}_{\beta,X}(x) < \infty \iff \hat{h}_{\beta,X}(y) < \infty \). Repeat this argument to all connected vertices.

\[ \square \]

---

3.2. The 2nd and 3rd key lemmas

**Lemma 2** \( \hat{h}_{\beta,X}^q(x) \equiv \hat{h}_{\beta,X}^q(x) \) is a degenerate random variable.

**Proof.** Since \( \mathbb{P} \) is ergodic, any translation-invariant event is trivial. By Lemma 1, \( \{ \hat{h}_{\beta,X}^q(x) = h \} \) for every \( h \in \mathbb{R} \) is a translation-invariant event, hence \( \mathbb{P}(\hat{h}_{\beta,X}^q(x) = h) = 0 \) or 1.

**Lemma 3** \( h = h_0 - \beta \mathbb{E}[X_b] - \beta \delta \Rightarrow \mathbb{P}(\hat{X}_{\beta,X}(x) = \infty) = 1. \)

**Proof.** \( \Delta_b = X_b - \mathbb{E}[X_b], \hat{X}_{\beta,X}(x; n) = \{ \omega \in \Omega(x; n) : \frac{1}{n} \sum_{j=1}^n \Delta_b(\omega) < \delta \} \)

\[ \Rightarrow \hat{X}_{\beta,X}(x) = \sum_{\omega \in \Omega(x)} p_\omega = \sum_{n=0}^\infty \frac{1}{\mu^n} \sum_{\omega \in \Omega(x; n)} \mathbb{e}^{\beta n \left( \frac{1}{n} \sum_{j=1}^n \Delta_b(\omega) \right)} \geq \sum_{n=0}^\infty \frac{1}{\mu^n} \mathbb{E}[\hat{X}_{\beta,X}(x; n)] \]

\[ \Rightarrow \mathbb{P}(\hat{X}_{\beta,X}(x) = \infty \limsup_{n \to \infty} \{ |\hat{X}_{\beta,X}(x; n)| \geq \frac{1}{2} c_n \}) = 1. \]

\[ \square \]
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3.2. The 2nd and 3rd key lemmas

**Lemma 3**

\[ h = h_0 - \beta \mathbb{E}[X_b] - \beta \delta \quad \delta > 0 \Rightarrow \mathbb{P}(\hat{X}_{h,\beta,\mathcal{X}}(x) = \infty) = 1. \]

**Proof (cont.).**

\[ \mathbb{P}(\hat{X}_{h,\beta,\mathcal{X}}(x) = \infty) \geq \mathbb{P}(\limsup_{n \to \infty} |\hat{\Omega}_{h,\beta,\mathcal{X}}(x; n)| \geq \frac{1}{2} c_n) \]
\[ = \mathbb{P}(\limsup_{n \to \infty} |\hat{\Omega}_{h,\beta,\mathcal{X}}(x; n)| \geq \frac{1}{2} c_n) \]
\[ \geq \lim_{n \to \infty} \mathbb{P}(|\hat{\Omega}_{h,\beta,\mathcal{X}}(x; n)| \geq \frac{1}{2} c_n). \]

However, by the Paley-Zygmund inequality with \( Z = |\hat{\Omega}_{h,\beta,\mathcal{X}}(x; n)| (\leq c_n) \),

\[ \mathbb{P}(\limsup_{n \to \infty} |\hat{\Omega}_{h,\beta,\mathcal{X}}(x; n)| \geq \frac{1}{2} c_n) \geq (1 - \varepsilon)^2 \frac{\mathbb{E}[|\hat{\Omega}_{h,\beta,\mathcal{X}}(x; n)|]}{c_n^2}. \]

By ergodicity, \( \mathbb{E}[|\hat{\Omega}_{h,\beta,\mathcal{X}}(x; n)|] = \sum_{\omega \in \Omega(x; n)} \mathbb{P}(\frac{1}{n} \sum_{j=1}^{n} A_{b_j}(\omega) < \delta) \geq c_n(1 - o(1)), \)

hence \( \lim_{n \to \infty} \mathbb{P}(\hat{\Omega}_{h,\beta,\mathcal{X}}(x; n) \geq \frac{1}{2} c_n) \geq \frac{1}{4} > 0 \) and \( \mathbb{P}(\hat{X}_{h,\beta,\mathcal{X}}(x) = \infty) = 1. \) \[ \blacksquare \]

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SAW on random conductors

4. Towards critical behavior

\[ \hat{G}_{h,\beta,\mathcal{X}}(x, y) = \sum_{\omega \in \Omega(y): \omega_0 = x} p_\omega, \quad B_1 = \mathbb{E}\left[ \sum_x \hat{G}_{h,\beta,\mathcal{X}}(o, x)^2 \right], \]

the bubble diagram

\[ B_2 = \mathbb{E}\left[ \sum_{x, y} \hat{G}_{h,\beta,\mathcal{X}}(o, x) \hat{G}_{h,\beta,\mathcal{X}}(x, y)^2 \hat{G}_{h,\beta,\mathcal{X}}(y, o) \right]. \]

**Proposition** (for i.i.d. \( X \) with finite generating function \( \lambda_\beta \))

If \( B_1, B_2 < \infty \), then, for any \( L(h) \downarrow 0 \) slowly varying as \( h \downarrow h_\beta \),

\[ \liminf_{h \downarrow h_\beta} \mathbb{P}(\hat{X}_{h,\beta,\mathcal{X}}(x) \geq \frac{L(h)}{h - h_\beta}) \geq 1 - O(\beta^2). \]

**Remark:**

- \( d > 4 \), classical \( \Rightarrow \) \( B_2 \leq B_1^2 < \infty \) [Hara & Slade : 1992].
- **The lace expansion** [ongoing with Helmuth]:

\[ \hat{G}_{h,\beta,\mathcal{X}}(x, y) = \mathbb{1}_{\{x = y\}} + \sum_{z \in \mathbb{Z}^d} \hat{\Pi}_{h,\beta,\mathcal{X}}(x, z) \hat{G}_{h,\beta,\mathcal{X}}(z, y). \]
Proposition (for i.i.d. $X$ with finite generating function $\lambda_\beta$) 

If $B_1, B_2 < \infty$, then, for any $L(h) \downarrow 0$ slowly varying as $h \downarrow h_\beta^a$, 

$$ \liminf_{h \downarrow h_\beta^a} \mathbb{P}\left( \hat{h}_{\beta, X}(x) \geq \frac{L(h)}{h - h_\beta^a} \right) \geq 1 - O(\beta^2). $$

Rough proof. By the Paley-Zygmund inequality with $Z = \hat{h}_{\beta, X}(x)$,

$$ \mathbb{P}\left( \hat{h}_{\beta, X}(x) \geq \varepsilon \mathbb{E}[\hat{h}_{\beta, X}(x)] \right) \geq (1 - \varepsilon)^2 \left( 1 + \frac{\mathbb{E}[\hat{h}_{\beta, X}(x)]^2 - \mathbb{E}[\hat{h}_{\beta, X}(x)]^2}{\mathbb{E}[\hat{h}_{\beta, X}(x)]^2} \right)^{-1}. $$

LHS is obtained by setting $\varepsilon = L(h)$ and using $\mathbb{E}[\hat{h}_{\beta, X}(x)] \geq (h - h_\beta^a)^{-1}$. 

On the other hand, by using $p_{\eta, X} = e^{-\sum_{i=1}^d (h + \beta X_{h, i})}$,

$$ \mathbb{E}[\hat{h}_{\beta, X}(x)]^2 - \mathbb{E}[X_{\beta, X}(x)]^2 = \sum_{\omega, \eta \in \Omega(x)} \mathbb{E}\left[ p_{\omega, \eta, X} \hat{h}_{\beta, X}(x) \right]. $$

By the telescopic-sum decomposition of $p_{\eta, X} - p_{\eta, X'}$, the i.i.d. property of $X, X'$ and the self-avoidance constraint on $\omega, \eta$,

$$ \frac{\mathbb{E}[\hat{h}_{\beta, X}(x)]^2 - \mathbb{E}[X_{\beta, X}(x)]^2}{\mathbb{E}[\hat{h}_{\beta, X}(x)]^2} \leq e^{-2h(2dB_1 + e^h \lambda_{\beta}^{-1}B_2)}(\lambda_{\beta}^{-1} - \beta) = O(\beta^2). \quad \blacksquare $$

Ongoing work:

- Is $h_0 - \beta \mathbb{E}[X_b] < \hat{h}_{\beta, X}(x)$ true for $d \geq 2$? ([with Chino])
- Is $\hat{h}_{\beta, X}(x) = h_\beta^a$ true in high dimensions? How high? ([with Chino])

Recent development [Chino (2016)]:

On the degree-$d$ ($\geq 3$) tree with i.i.d. random conductors,

- $\beta < \beta_c \equiv \sup\{\beta \geq 0 : h_\beta^a - (h_\beta^a)^{\beta} > 0\} \Rightarrow \hat{h}_{\beta, X}(x) = h_\beta^a$ (& MF behavior).
- $\beta > \beta_c \Rightarrow \exists \delta \equiv \delta(d, \beta) > 0 : \hat{h}_{\beta, X}(x) \leq h_\beta^a - \delta$.
- MF behavior of $\hat{h}_{\beta, X}(x)$ and $\hat{G}_{h_{\beta, X}(x), \beta, X}(x, y)$ in high dimensions? How high? ([with Helmuth])
Descriptive analysis of self-adjoint operators and the Weyl–von Neumann Theorem

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Abstract

This notes summarizes the first named author’s talk at the conference “Mathematical quantum field theory and related topics” held at IMI Kyushu University in June 2016. We give a survey of our works [AM15-1, AM15-2] on the descriptive set theoretic study of the space $SA(H)$ of self-adjoint operators on a separable infinite-dimensional Hilbert space $H$. In the appendices, we include proofs of several results which were removed from the published version of [AM15-1].

Keywords. Weyl–von Neumann Theorem, Self-adjoint operators, Borel equivalence relation.

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1 Introduction

The spectral theory for self-adjoint operators has been one of the most powerful tools in the study of quantum physics and harmonic analysis. Especially, unbounded self-adjoint operators appear more naturally than bounded ones in quantum theory. At the same time, unbounded operators are rather difficult to deal with, due mainly to the domain issue. In this respect we are interested in understanding the difference between bounded and unbounded self-adjoint operators in some quantitative form. One approach to this problem can be achieved by the theory of Borel equivalence relations [Gao09, Hjo00] and the study of their complexities. Before introducing basic concepts of the theory, let us start from looking at the celebrated Weyl-von Neumann Theorem, a typical example where the unbounded vs bounded issue appears explicitly. Let $H$ be a separable Hilbert space $\cong \ell^2$, and we denote by $\mathcal{B}(H)_{sa}$ (resp. $SA(H)$) the space of all bounded (resp. not necessarily bounded) self-adjoint operators on $H$. The group of unitaries (resp. the space of compact self-adjoint operators) on $H$ is denoted by $U(H)$ (resp. $K(H)_{sa}$). For $A \in SA(H)$, we denote by $\sigma(A)$ (resp. $\sigma_e(A)$) the spectrum (resp. the essential spectrum) of $A$. Let us say that $A, B \in SA(H)$ are Weyl–von Neumann equivalent ($A \overset{WvN}{\sim} B$), if they are unitarily equivalent modulo the compact, i.e., if there exists $u \in U(H)$ and $K \in K(H)_{sa}$ such that $uAu^* + K = B$. Weyl–von Neumann Theorem below shows that we can completely classify bounded self-adjoint operators up to $\overset{WvN}{\sim}$ by the essential spectrum.

Theorem 1.1 (Weyl-von Neumann). Let $A, B \in \mathcal{B}(H)_{sa}$. Then the following conditions are equivalent:

1. $\sigma_e(A) = \sigma_e(B)$.
2. $A \overset{WvN}{\sim} B$. 

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Note that by Weyl’s Theorem, (2) ⇒ (1) is true for all \( A, B \in \text{SA}(H) \). Can we similarly classify all (possibly unbounded) self-adjoint operators up to \( \sim_{WvN} \) by \( \sigma_e(\cdot) \)? Notice that if \( A \sim_{WvN} B \), then their domains \( \text{dom}(A), \text{dom}(B) \) must be unitarily equivalent \( (\exists u \in \mathcal{U}(H) [u \cdot \text{dom}(A) = \text{dom}(B)]) \). There exist operators \( A, B \in \text{SA}(H) \) with \( \sigma_e(A) = \sigma_e(B) \) while \( A \) is not Weyl–von Neumann equivalent to \( B \). The next natural question would be:

**Question 1.2.** Let \( A, B \in \text{SA}(H) \). If \( \sigma_e(A) = \sigma_e(B) \) and \( \text{dom}(A), \text{dom}(B) \) are unitarily equivalent, does it follow that \( A \sim_{WvN} B \)?

It turns out that there exists a continuous family \( \{B_t\}_{t \in [0,1]} \) in \( \text{SA}(H) \) such that \( \text{dom}(B_t) \) is the same for all \( t \) and \( \sigma_e(B_t) = \mathbb{N} \) for all \( t \), but no two of them are Weyl–von Neumann equivalent. Of course, this does not exclude the possibility that there is yet another invariant \( I(\cdot) \) such that for all \( A, B \in \text{SA}(H) \)

\[
\sigma_e(A) = \sigma_e(B), \quad \exists u \ [u \cdot \text{dom}(A) = \text{dom}(B)], I(A) = I(B) \iff A \sim_{WvN} B.
\]

Notice that this as a question does not even make sense if we do not impose any assumption on \( A \mapsto I(A) \): the above equivalence is obviously true if e.g. we set \( I(A) = \sim_{WvN} \) class of \( A \): the problem is that we do not know how to calculate (at least theoretically) all the \( \sim_{WvN} \)-classes. But if we require \( A \mapsto I(A) \) to be “computable” in a weak sense, then we can actually show as a mathematical theorem, that such \( I(\cdot) \) does not exist. To make this statement precise, we need the framework of descriptive set theory. In the next section we recall basics of this theory.

## 2 Classification and Borel reduction

Here we briefly describe basic concepts from descriptive set theory. For details we recommend the reader to consult [Gao09, Hjo00, Kec96]. Recall that what mathematicians call a “classification” is the following procedure:

- Fix a parameter space \( X \). Points \( x \in X \) parametrizes objects we want to classify.
- Introduce an equivalence relation \( E \) (classification scheme) on \( X \) and identify all (or at least many of) the \( E \)-classes.

As a primitive example, consider the following:

**Example 2.1.** Let \( X \) be the set of all points \((p_1, p_2, p_3) \in (\mathbb{R}^2)^3 \) where \( p_i = (p_{ix}, p_{iy}) \in \mathbb{R}^2 \) \((i = 1, 2, 3, )\), such that \( \{p_1, p_2, p_3\} \) is the set of vertices of an triangle in the plane \( \mathbb{R}^2 \). We fix an ordering of \( p_1, p_2, p_3 \), so that for each triangle \( \Delta \) in the plane, there is a unique point \((p_1, p_2, p_3) \in X^3 \) which parametrizes the vertices of \( \Delta \). With an abuse of language we say an element \( \Delta = (p_1, p_2, p_3) \in X \) a triangle in the plane. Let us identify triangles \( \Delta_1, \Delta_2 \in X \) if they are congruent. In other words, we introduce the equivalence \( E \) of congruence on the space \( X \) parametrizing triangles. What is the complete list of \( E \)-classes? Then we know that \( \Delta_1 E \Delta_2 \) if and only if the lengths of 3 sides are equal. So the map \( I: X \ni \Delta = (p_1, p_2, p_3) \mapsto (|p_1 - p_2|, |p_2 - p_3|, |p_3 - p_1|) \in (0, \infty)^3 = Y \) satisfies

\[
\Delta_1 E \Delta_2 \iff I(\Delta_1) \equiv I(\Delta_2)
\]

so \( I \) is a complete invariant for the congruence relation \( E \). But here appears “\( \equiv \)” instead of “=”, because \( \Delta_1 E \Delta_2 \) if and only if *after permuting the lengths*, they are equal. In other words we have introduced an equivalence relation \( F \) on \( Y = (0, \infty)^3 \) by claiming

\[
I_1 = (p_i)_i^{3} F I_2 = (q_i)_i^{3} \iff \sigma \in \mathfrak{S}_3 \ \forall i \in \{1, 2, 3\} \ \sigma \cdot I_1 = I_2,
\]

where \( \sigma \cdot I := (p_{\sigma^{-1}(1)}, p_{\sigma^{-1}(2)}, p_{\sigma^{-1}(3)}) \). Then

\[
\Delta_1 E \Delta_2 \iff I(\Delta_1) F I(\Delta_2), \quad \Delta_1, \Delta_2 \in X.
\]

\[\text{For example, we require that } p_{1x} \leq p_{2x} \leq p_{3x} \text{ and if } p_{1x} = p_{2x}(< p_{3x}), \text{ then } p_{1y} < p_{2y} \text{ and if } (p_{1x} <) p_{2x} = p_{3x}, \text{ then } p_{2y} < p_{3y}.\]

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This way we can classify (parametrize) $E$-classes by $F$-classes, or we can say $E$ is less complex than $F$.

Note that we can introduce suitable topologies on $X,Y$ and then the map $I:X \to Y$ is Borel (even continuous in this case). The above example can be generalized to the following abstract notion of Borel reduction.

Recall that a Polish space is a topological space which is separable and metrizable by a complete metric. On a Polish space $X$, we define $B(X)$ to be the $\sigma$-algebra on $X$ generated by open sets. Elements of $B(X)$ are called Borel sets, and a map $f:X \to Y$ between Polish spaces is called Borel, if $f^{-1}(B) \in B(X)$ for all $B \in B(Y)$. A Polish group is a topological group which is Polish as a topological space. For example, $U(H)$ (resp. $\mathcal{K}(H)_{sa}$) is a Polish group with respect to the strong operator topology (SOT) (resp. the norm topology). In the Appendix, we give a proof that the space $SA(H)$ is a Polish space with respect to the strong resolvent topology (SRT). It is often more flexible to work on standard Borel spaces, but in this notes, we stick to Polish spaces.

**Definition 2.2.** Let $E$ (resp. $F$) be an equivalence relation on a Polish spaces $X$ (resp. $Y$). We say that $E$ is Borel reducible to $F$, in symbols $E \leq_B F$, if there is a Borel map $f:X \to Y$ such that $x_1E x_2 \iff f(x_1)F f(x_2)$ holds for every $x_1, x_2 \in X$. We say $E, F$ are Borel bireducible ($E \sim_B F$) if $E \leq_B F$ and $F \leq_B E$ hold. We also write $E <_B F$ if $E \leq_B F$ but $F \not\leq_B E$.

Also, many equivalence relations appearing in functional analysis are Borel (or analytic, which we do not discuss here).

**Definition 2.3.** An equivalence relation $E$ on a Polish space $X$ is called Borel, if we regard $E \subset X \times X$, it is a Borel set.

Here are basic examples of Borel equivalence relations.

**Example 2.4.**

(i) For a Polish space $X$, the identity relation $\text{id}(X)$ is $\text{id}(X)x \sim x'$. It is easy to show that

$$\text{id}(1) <_B \text{id}(2) <_B \cdots <_B \text{id}(n) <_B \text{id}(n+1) <_B \cdots <_B \text{id}(\mathbb{N}) <_B \text{id}(\mathbb{R}).$$

Any uncountable Polish space has the size of continuum (i.e., continuum hypothesis is true for Polish spaces). Thus there is no Polish $Y$ such that $\text{id}(\mathbb{N}) <_B \text{id}(Y) <_B \text{id}(\mathbb{R})$. But this does not exclude the possibility that there is an equivalence relation $E$ such that $\text{id}(\mathbb{N})<_B E <_B \text{id}(\mathbb{R})$. That such $E$ does not exist is a consequence of Silver’s dichotomy Theorem (see [Gao09]). An equivalence relation $F$ is called smooth if $F \leq_B \text{id}(\mathbb{R})$ holds.

(ii) The tail equivalence relation $E_0$ on the Cantor space $\mathcal{C}=2^\mathbb{N}$ of all $\{0,1\}$-valued sequences on $\mathbb{N}$. This space is compact with respect to the product topology, and $aE_0 b$ if and only if they are eventually equal: $\exists N \in \mathbb{N} \forall n \geq N \ [a_n = b_n]$. This seemingly very specific equivalence relation has a rather surprising property. Harrington–Kechris–Louveau’s dichotomy Theorem says that any Borel $E$ is either smooth or satisfies $E_0 \leq_B E$ (for details about this and many other dichotomies, see [Gao09]).

(iii) The Vitali equivalence relation $E_\nu$ on $\mathbb{R}$ is given by $xE_\nu y \iff x-y \in \mathbb{Q}$. It is known that $E_0 \sim_B E_\nu$.

(iv) Let $\alpha$ be a continuous action of a Polish group $G$ on a Polish space $X$. The orbit equivalence relation of $\alpha$ is an equivalence relation $E^\alpha_G$ (or written $E^X_\alpha$ if the action is clear from the context) on $X$ given by $xE^\alpha_G y \iff \exists g \in G \ [\alpha_g(x) = y]$. Note that the equivalence relations in (i)-(iii) are all orbit equivalence relations: in (i), we take $G = \{1\}$ to act trivially on $X$. For (ii), let $G = \bigoplus_{n=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$ to act on $2^{\mathbb{N}}$ by coordinatewise addition mod 2. In (iii), we let $\mathbb{Q}$ (as a countable discrete group) act on $\mathbb{R}$ by addition. There are many others. For example one may consider the action of $\mathcal{P}(1 \leq p < \infty)$ on $\mathbb{R}^n$ or the action of the self-adjoint Schatten class operators $S^p(H)_{sa}$ on $SA(H)$ by addition. A slightly different example relevant to harmonic analysis is consider a Polish group $G$ and let $\text{Rep}(G,H)$ be the space of all strongly continuous unitary representations of $G$ on $H$. With respect to a certain topology, it is a Polish space. Then let $U(H)$ act on $X = \text{Rep}(G,H)$ by

$$[u \cdot \pi](g) = u\pi(g)u^*, \quad g \in G, \quad \pi \in \text{Rep}(G,H).$$

Then $E^{{\text{Rep}(G,H)}}_{U(H)}$ is nothing but the unitary equivalence between unitary representations of $G$. 121
Let \( S_\infty = \text{Aut}(\mathbb{N}) \) be the space of all (not necessarily finitely supported) bijections of \( \mathbb{N} \). We say that an equivalence relation \( E \) admits a classification by countable structures if \( E \leq_B E^X_{S_\infty} \) for some continuous action \( S_\infty \curvearrowright X \) on a Polish space. All actions of locally compact Polish groups are classifiable by countable structures (in fact by Kechris’ Theorem, any such equivalence relation is \( \leq_B \) to an orbit equivalence relation of a countable group action), whereas \( F^p \curvearrowright \mathbb{R}^n \) or \( S^p(\mathbb{H})_{sa} \curvearrowright \text{SA}(\mathbb{H}) \) do not. The proof is a typical application of Hjorth’s turbulence Theorem [Hjo00] which we discuss below.

**Definition 2.5.** Let \( E \) (resp. \( F \)) be an equivalence relation on a Polish space \( X \) (resp. \( Y \)). A homomorphism from \( E \) to \( F \) is a map \( f : X \to Y \) such that \( xEy \Rightarrow f(x)Ff(y) \) for all \( x, y \in X \). We say that \( E \) is generically \( F \)-ergodic, if for every Baire measurable homomorphism \( f \) from \( E \) to \( F \), there is a comeager set \( A \subset X \) which \( f \) maps into a single \( F \)-class.

Hjorth’s notion of turbulence provides us with a convenient criterion for finding an obstruction of a given equivalence relation to be classifiable by countable structures.

**Definition 2.6.** Let \( G \) be a Polish group and \( X \) a Polish \( G \)-space.

1. Let \( x \in X \). For an open neighborhoods \( U \) of \( x \) in \( X \) and \( V \) of 1 in \( G \), the local \( U \)-orbit of \( x \), denoted \( O(x, U, V) \), is the set of all \( y \in U \) for which there exist \( l \in \mathbb{N} \), \( x = x_0, x_1, \cdots, x_l = y \in U \), and \( g_0, \cdots, g_{l-1} \in V \), such that \( x_{i+1} = g_i \cdot x_i \) for all \( 0 \leq i \leq l-1 \).

2. The action \( \alpha \) is turbulent at \( x \in X \) if the local orbits \( O(x, U, V) \) of \( x \) are somewhere dense (i.e., its closure has nonempty interior) for every open \( U \subset X \) and \( V \subset G \) with \( x \in U \) and \( 1 \in V \).

3. The action \( \alpha \) is said to be generically turbulent if
   
   (a) There is a dense orbit.
   
   (b) Every orbit is meager (Baire first category).
   
   (c) There exists a comeager set \( X_0 \subset X \) such that the action is turbulent at each \( x \in X_0 \).

Next theorem is called Hjorth turbulence Theorem. Proof can be found in [Hjo00].

**Theorem 2.7 (Hjorth).** Let \( G \) be a Polish group and \( X \) a Polish \( G \)-space with every orbit meager and some orbit dense. Then the following statements are equivalent:

(i) For any Borel \( S_\infty \)-space \( Y \), \( E^X_G \) is generically \( E^Y_{S_\infty} \)-ergodic.

(ii) \( X \) is generically turbulent.

To illustrate the technique, in Appendix B, we apply Theorem 2.7 to show that the action of \( \mathbb{K}(\mathbb{H})_{sa} \) on \( \text{SA}(\mathbb{H}) \) by addition is generically turbulent.

### 3 Equivalence relations on the space SA(\( H \))

Below we summarize the results of [AM15-1, AM15-2]. First, we compare the Weyl–von Neumann equivalence relation \( \simW \) and its restriction \( \simW |_{\mathbb{K}(\mathbb{H})_{sa}} \) to bounded self-adjoint operators. We realize \( \simW \) as the orbit equivalence relation \( E^{\text{SA}(\mathbb{H})}_G \), where the action \( \alpha \) of the semidirect product group \( G := \mathbb{K}(\mathbb{H})_{sa} \rtimes \mathcal{U}(\mathbb{H}) \) on \( \text{SA}(\mathbb{H}) \) is given by

\[
(K, u) \cdot A := uAu^* + K, \quad A \in \text{SA}(\mathbb{H}), \ u \in \mathcal{U}(\mathbb{H}), \ K \in \mathbb{K}(\mathbb{H})_{sa}.
\]

One can show that the action is continuous. To show that Theorem 1.1 gives a nice classification in the context of Borel complexity, we need to show that the essential spectrum is a Borel map. But where does it map operators to? The answer is the Effros Borel space.

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Definition 3.1. Let $X$ be a topological space. We denote by $\mathcal{F}(X)$ the set of closed subsets of $X$. The Effros Borel structure of $\mathcal{F}(X)$ is the $\sigma$-algebra on $\mathcal{F}(X)$ generated by the sets

$$\{F \in \mathcal{F}(X); F \cap U \neq \emptyset \}.$$ 

If $X$ is second countable and $\{U_n\}_{n=1}^{\infty}$ is an open basis for $X$, it is sufficient to consider $U$ in this basis.

If $X$ is Polish, then the Effros Borel space of $\mathcal{F}(X)$ is a Polish space with respect to a certain topology [Kec96, Theorem 12.6].

Theorem 3.2 ([AM15-1]). Let $H$ be a separable infinite-dimensional Hilbert space. Then

(i) The essential spectrum map $\sigma_e(\cdot): \sigma(A) \ni A \mapsto \sigma_e(A) \in \mathcal{F}(\mathbb{R})$ is Borel. Therefore $E_G^{\mathbb{R}(H)_{sa}} \leq_B \text{id}_{\mathcal{F}(\mathbb{R})}$, so that $E_G^{\mathbb{R}(H)_{sa}}$ is smooth.

(ii) On the other hand, the full Weyl–von Neumann equivalence $E_G^{\mathcal{S}A(H)}$ does not admit classification by countable structures.

Of course we want to apply Hjorth’s turbulence Theorem 2.7 to prove (ii). However, it turns out that the action $G \curvearrowright \mathcal{S}A(H)$ is not generically turbulent, because there exists a dense $G_δ$ (non-meager) orbit:

Theorem 3.3 ([AM15-1, AM15-2]). The following statements hold.

(i) The set $\mathcal{S}A_{\text{null}}(H) := \{A \in \mathcal{S}A(H); \sigma_e(A) = \mathbb{R}\}$ consists of a single dense $G_δ$ orbit of the $G$-action.

(ii) The set of all $A \in \mathcal{S}A(H)$ which has purely singular continuous spectrum equal to $\mathbb{R}$, is a dense $G_δ$ subset of $\mathcal{S}A(H)$.

It is known that no locally compact Polish group admits a non-transitive action with a dense $G_δ$ orbit. (i) reflects the fact that the $G$-action is rather complicated. (ii) tells us that although any self-adjoint operator can be approximated by diagonal ones, generic self-adjoint operators have rather pathological spectral properties. The proof relies on Simon’s Wonderland Theorem [Sim95]. To circumvent the issue (i), we restrict our attention to the following subspace $\mathcal{E}E\mathcal{S}(H)$, which is not $\mathcal{N}RT$-separable. The space $\mathcal{E}E\mathcal{S}(H)$ can be made Polish by introducing the norm resolvent topology (NRT). Here, $A_n \in \mathcal{S}A(H)$ converges to $A \in \mathcal{S}A(H)$ in NRT, if and only if $\|((A_n - i)^{-1} - (A - i)^{-1})^{-1} n \to \infty 0$. The space $\mathcal{E}E\mathcal{S}(H)$ is Polish with respect to the NRT. Note that $\mathcal{S}A(H)$ is not Polish for NRT, because it is not $\mathcal{N}RT$-separable.

Proposition 3.4 ([AM15-1]). The $G$-action on $\mathcal{E}E\mathcal{S}(H)$ is continuous with respect to the NRT. Moreover, the action is generically turbulent.

Since NRT is stronger than SRT, the natural inclusion $\mathcal{E}E\mathcal{S}(H) \hookrightarrow \mathcal{S}A(H)$ is continuous, which induces $E_G^{\mathcal{E}E\mathcal{S}(H)} \leq_B E_G^{\mathcal{S}A(H)}$. Therefore $E_G^{\mathcal{S}A(H)}$ is unclassifiable by countable structures.

One of the reasons for which $E_G^{\mathbb{R}(H)_{\text{sa}}} \leq_B E_G^{\mathcal{S}A(H)}$ is that it is difficult to tell when two operators have the same (resp. unitarily equivalent) domains. We therefore considered the following two equivalence relations $E_{\text{dom}}$ and $E_{\text{dom, u}}$ on $\mathcal{S}A(H)$ given by

$$AE_{\text{dom}} B \Leftrightarrow \text{dom}(A) = \text{dom}(B), \quad AE_{\text{dom, u}} B \Leftrightarrow \exists u \in \mathcal{U}(H) [u \cdot \text{dom}(A) = \text{dom}(B)].$$

Can we tell which one of these is more complex? The answer is the following. We compare them with the orbit equivalence $E_{\mathbb{R}^\infty}$ of the additive action $\ell^\infty \curvearrowright \mathbb{R}^\mathbb{N}$.

Theorem 3.5 ([AM15-2]). $E_{\text{dom, u}} \leq_B E_{\text{dom}} \sim_B E_{\mathbb{R}^\infty}$ holds.

The key steps in the proof are (a) show that $E_{\text{dom, u}}$ is $\leq_B$ to a $\sigma$-compact equivalence relation, (b) show $E_{\text{dom}} \sim_B E_{\mathbb{R}^\infty}$ by using the theory of operator ranges, and (c) use Rosendal’s Theorem [Ros05] that any $\sigma$-compact equivalence relation is $\leq_B$ to $E_{\mathbb{R}^\infty}$. Finally, we remark that the following question is still open:
Remark 3.8. with respect to SRT. Consequently, 

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (\alpha_i) = \sigma, \quad (\alpha_i)_{n=1}^{\infty} := (x_{\sigma^{-1}(n)} + a_n)_{n=1}^{\infty}, \quad (\alpha_n)_{n=1}^{\infty} \in c_0, \sigma \in S_{\infty}, (x_n)_{n=1}^{\infty} \in \mathbb{R}^N. \]

Is \( E^{SA(H)}_G \leq E^{\mathbb{R}^N}? \)

Appendix A: Polish Space \( SA(H) \)

In this appendix, we show that the space \( SA(H) \) of all self-adjoint operators on \( H \) equipped with strong resolvent topology (SRT) is a Polish space when \( H \) is separable infinite-dimensional. This is probably known to experts, but since we could not find a reference where this was explicitly discussed, we include a full proof. The proof was originally included in the first draft of [AM15-1], but it was removed in the published version. Fix a CONS \( \{\xi_n\}_{n=1}^{\infty} \) for \( H \), and define a metric \( d \) on \( SA(H) \) by

\[
d(A, B) := \sum_{n=1}^{\infty} \frac{1}{2^n} \left| \left| e^{itA} \xi_n - e^{itB} \xi_n \right| \right|, \quad A, B \in SA(H).
\]

**Proposition 3.7.** \( d \) is a complete metric on \( SA(H) \) compatible with SRT, and \( SA(H) \) is separable with respect to SRT. Consequently, \( SA(H) \) is a Polish space.

**Remark 3.8.** Note that the following apparently suitable compatible metric \( d' \) given by

\[
d'(A, B) := \sum_{n=1}^{\infty} \frac{1}{2^n} \left| \left| (A - i)^{-1} \xi_n - (B - i)^{-1} \xi_n \right| \right|
\]

is not complete. Indeed, \( A_n = n1 \in SA(H) \) \((n \in \mathbb{N})\) is \( d' \)-Cauchy which has no SRT-limit.

We need the following classical result. Proof can be found e.g. in [RS81, Theorem VIII.21].

**Lemma 3.9 (Trotter).** Let \( \{A_k\}_{n=1}^{\infty} \subset SA(H) \). Then \( A_k \) converges to \( A \in SA(H) \) in SRT, if and only if for each \( \xi \in H \) and each compact subset \( K \) of \( \mathbb{R} \), \( \sup_{t \in K} \|e^{itA_k} \xi - e^{itA} \xi \| \) tends to 0.

**Proof of Proposition 3.7.** We first show that \( SA(H) \) is separable. Let \( F : SA(H) \to \prod_{n \in \mathbb{N}} H \) be a map defined by \( F(A) := ((A - i)^{-1} \xi_n)_{n=1}^{\infty} \). Then \( F \) is injective. Indeed, if \( F(A_1) = F(A_2) \) for \( A_1, A_2 \in SA(H) \), then as resolvents are bounded, \( (A_1 - i)^{-1} = (A_2 - i)^{-1} \), which implies \( A_1 = A_2 \). It is also easy to see that for a net \( \{A_n\} \) and \( A \) in \( SA(H) \) \((k \in \mathbb{N})\),

\[
A_n \xrightarrow{\text{SRT}} A \iff (A_n - i)^{-1} \xi \to (A - i)^{-1} \xi, \quad \xi \in H
\]

\[
\iff (A_n - i)^{-1} \xi_n \to (A - i)^{-1} \xi_n, \quad n \in \mathbb{N}.
\]

Hence \( F \) is a homeomorphism of \( SA(H) \) onto its range. Therefore as \( \prod_{n \in \mathbb{N}} H \) is Polish, its subspace \( F(SA(H)) \) is separable and metrizable, whence so is \( SA(H) \). It is easy to see that \( d \) is a metric. Set \( I_m := [-m, m] \). By Lemma 3.9, we have

\[
d(A_k, A) \xrightarrow{k \to \infty} 0 \iff \sup_{t \in I_m} \|e^{itA_k} - e^{itA} \xi \| \xrightarrow{k \to \infty} 0, \quad n, m \in \mathbb{N}
\]

\[
\iff \sup_{t \in I_m} \|e^{itA_k} - e^{itA} \xi \| \xrightarrow{k \to \infty} 0, \quad \xi \in H, m \in \mathbb{N}
\]

\[
\iff A_k \xrightarrow{\text{SRT}} A.
\]

Therefore \( d \) is compatible with SRT. Finally, we show that \( d \) is complete. Suppose that \( \{A_k\}_{k=1}^{\infty} \) is a \( d \)-Cauchy sequence in \( SA(H) \). Then for each \( n, m \in \mathbb{N} \), we have

\[
\sup_{t \in I_m} \|e^{itA_k} - e^{itA_l} \xi_n \| \xrightarrow{k, l \to \infty} 0.
\]
Now fix $t \in \mathbb{R}$ and let $\xi \in H$. We show that $\{e^{itA_k}\xi\}_{k=1}^\infty$ is Cauchy in $H$. Given $\varepsilon > 0$, find $\xi_0 \in \text{span}\{\xi_n; n \geq 1\}$ such that $\|\xi - \xi_0\| < \varepsilon/4$. By (2), we see that $\{e^{itA_k}\xi_0\}_{k=1}^\infty$ is Cauchy in $H$. Therefore there exists $k_0$ such that $\|e^{itA_k}\xi_0 - e^{itA_l}\xi_0\| < \varepsilon/2$ for all $k, l \geq k_0$. Then for $k, l \geq k_0$,

$$
\|e^{itA_k}\xi - e^{itA_l}\xi\| \leq \|(e^{itA_k} - e^{itA_l})(\xi - \xi_0)\| + \|(e^{itA_k} - e^{itA_l})\xi_0\| \\
\leq 2\|\xi - \xi_0\| + \varepsilon/2 < \varepsilon.
$$

Therefore $\{e^{itA_k}\xi\}_{k=1}^\infty$ is Cauchy, and let $\varphi(t, \xi) \in H$ be its limit. It is easy to see that for a fixed $t \in \mathbb{R}$, $\xi \mapsto \varphi(t, \xi)$ is linear. Moreover, $\|\varphi(t, \xi)\| = \|\xi\|$ for each $t \in \mathbb{R}$, $\xi \in H$. Therefore for each $t \in \mathbb{R}$, there exists an isometry $u(t) \in \mathcal{B}(H)$ such that $\varphi(t, \xi) = u(t)\xi$ ($\xi \in H$). It is clear that $u(0) = 1$. Moreover, for $s, t \in \mathbb{R}$ and $\xi \in H$, it holds that

$$
\|u(s)u(t)\xi - u(s+t)\xi\| = \lim_{k \to \infty} \|e^{isA_k}u(t)\xi - e^{i(s+t)A_k}\xi\| \\
= \lim_{k \to \infty} \|u(t)\xi - e^{itA_k}\xi\| \\
= \|u(t)\xi - u(t)\xi\| = 0,
$$

which implies that $\{u(t)\}_{t \in \mathbb{R}}$ is a one-parameter unitary group. We show that $t \mapsto u(t)$ is strongly continuous. Since it is a one-parameter unitary group (hence uniformly bounded), it suffices to show that $t \mapsto u(t)\xi_n$ is continuous at $t = 0$ for each $n \in \mathbb{N}$. So let $\varepsilon > 0$ and $n \in \mathbb{N}$ be given. By (2), there exists $k_0 \in \mathbb{N}$ such that for each $t \in [-1, 1]$ and $k, l \geq k_0$,

$$
\|(e^{itA_k} - e^{itA_l})\xi_n\| < \varepsilon/2.
$$

Letting $l \to \infty$ in (3), we obtain that $\|(e^{itA_k} - u(t)\xi_n)\| \leq \varepsilon/2$ for each $t \in [-1, 1]$ and $k \geq k_0$. On the other hand, there exists $(1) > \delta > 0$ such that $\|(e^{itA_k}\xi_n - \xi_n)\| < \varepsilon/2$ for $|t| < \delta$. Therefore for $|t| < \delta$, we obtain

$$
\|u(t)\xi_n - \xi_n\| \leq \|u(t)\xi_n - e^{itA_k}\xi_n\| + \|e^{itA_k}\xi_n - \xi_n\| < \varepsilon.
$$

Therefore $t \mapsto u(t)$ is strongly continuous, and by Stone Theorem [RS81, Theorem VIII.8] let $A \in \text{SA}(H)$ be such that $u(t) = e^{itA}$ ($t \in \mathbb{R}$) holds. We show that $d(A_k, A) \to 0$. To this purpose it suffices to show that $\sup_{t \in I_m} \|(e^{itA_k} - e^{itA})\xi_n\|$ tends to $0$ for each $n, m \in \mathbb{N}$. Let $\varepsilon > 0$. Then there exists $k_0 \in \mathbb{N}$ such that $\sup_{t \in I_m} \|(e^{itA_k} - e^{itA})\xi_n\| < \varepsilon/2$ for all $k, l \geq k_0$. Now for each $t \in I_m$, choose $l = l(t) \geq k_0$ such that $\|(e^{itA_k} - e^{itA_k})\xi_n\| < \varepsilon/2$. It then follows that for all $k \geq k_0$,

$$
\sup_{t \in I_m} \|(e^{itA_k} - e^{itA_k})\xi_n\| \leq \sup_{t \in I_m} \|(e^{itA_k} - e^{itA_k})\xi_n\| + \sup_{t \in I_m} \|(e^{itA_k} - e^{itA_k})\xi_n\| < \varepsilon,
$$

whence the claim is proved. Therefore $d$ is complete.

\[\square\]

**Appendix B: The action $\mathbb{K}(H)_{sa} \rtimes \text{SA}(H)$ is generically turbulent**

Let us give a typical application of turbulence Theorem to show that the action of the group $\mathbb{K}(H)_{sa}$ on $\text{SA}(H)$ by addition is generically turbulent. This result is extracted from the first draft of [AM15-1] which was removed in the published version.

**Theorem 3.10.** The action of $\mathbb{K}(H)_{sa}$ on $\text{SA}(H)$ by addition is continuous. Moreover, the action is generically turbulent.

The action is continuous by [AM15-1, Proposition 3.9]. We will need another Theorem due to Weyl–von Neumann.

**Theorem 3.11.** Let $A \in \text{SA}(H)$ and $\varepsilon > 0$. Then there exists $K \in \mathbb{K}(H)_{sa}$ with $\|K\| < \varepsilon$ such that $A + K$ is diagonal.
The rest of the proof is divided into steps.

**Lemma 3.12.** For every $A \in \mathcal{SA}(H)$, the orbit $[A]_{\mathcal{SA}(H)_s}$ is dense in $\mathcal{SA}(H)$.

**Proof.** By Theorem 3.11, there exists $K_0 \in \mathcal{K}(H)_s$ such that $A + K_0$ is of the form $A + K_0 = \sum_{n=1}^{\infty} a_n e_n$, where $a_n \in \mathbb{R}$ and $(e_n)_{n=1}^{\infty}$ is a sequence of mutually orthogonal rank one projections with sum equal to 1. Then the sequence $A_m := (A + K_0) - \sum_{n=1}^{m} a_n e_n \in [A]_{\mathcal{K}(H)_s}$ ($m \in \mathbb{N}$) satisfies $A_m \to \infty$ (SRT). Therefore for each $K \in \mathcal{K}(H)_s$, $A_m + K \in [A]_{\mathcal{K}(H)_s}$ converges to $K$ in SRT. This shows that $\mathcal{K}(H)_s \subset [A]_{\mathcal{K}(H)_s}$, and since $\mathcal{K}(H)_s$ is dense in $\mathcal{SA}(H)$, so is $[A]_{\mathcal{K}(H)_s}$.

We next show that every orbit $[A]_{\mathcal{K}(H)_s}$ is meager. We first treat the case where $A$ is bounded.

**Lemma 3.13.** Let $B \in \mathcal{B}(H)_s$. Then its orbit $[B]_{\mathcal{B}(H)_s}$ is meager in $\mathcal{SA}(H)$.

**Proof.** Since $B$ is bounded, we have $[B]_{\mathcal{K}(H)_s} \subset [B]_{\mathcal{B}(H)_s}$. We show:

**Claim.** $\mathcal{B}(H)_s$ is a meager $F_\sigma$ subset of $\mathcal{SA}(H)$.

Let $F_n := \{ A \in \mathcal{B}(H)_s : \| A \| \leq n \}$ ($n \in \mathbb{N}$). Then $\mathcal{B}(H)_s = \bigcup_{n=1}^{\infty} F_n$. We show that each $F_n$ is SRT-closed. Let $A_k \in F_n$ and assume that $A_k \to A$ in $\mathcal{SA}(H)$. We show that $A \in F_n$: let $\xi \in H$, and let $A_k = \int_{-n}^{n} \lambda dE_k(\lambda)$ be the spectral resolution of $A_k$ ($k \in \mathbb{N}$). Then for each $k \geq 1$ we have

$$\|(A_k - i)^{-1}\xi\|^2 = \int_{[-n,n]} \frac{1}{|\lambda + i|^2} d\|E_k(\lambda)\xi\|^2 \geq \frac{1}{n^2 + 1} \|\xi\|^2. \quad (4)$$

Therefore

$$\|(A - i)^{-1}\xi\|^2 = \lim_{k \to \infty} \|(A_k - i)^{-1}\xi\|^2 \geq \frac{1}{n^2 + 1} \|\xi\|^2, \quad \xi \in H. \quad (5)$$

If there exists $\lambda \in \sigma(A) \cap \mathbb{R} \setminus [-n,n]$, choose $\varepsilon > 0$ such that $|\lambda| - \varepsilon > n$, and $\xi \in \text{dom}(A)$ such that $\|A\xi - \lambda\xi\| < \varepsilon \|\xi\|$. Then $\|A\xi\| \geq \|\lambda\xi\| - \|A\xi - \lambda\xi\| > |(\lambda - \varepsilon)| \|\xi\|$, so that

$$\|(A - i)\xi\|^2 = \langle A\xi - i\xi, A\xi - i\xi \rangle = \|A\xi\|^2 + \|\xi\|^2 > (|\lambda - \varepsilon|^2 + 1)\|\xi\|^2,$$

which by (5) implies that

$$\|\xi\|^2 = \|(A - i)^{-1}(A - i)\xi\|^2 \geq \frac{1}{n^2 + 1} \{(|\lambda| - \varepsilon)^2 + 1\} \|\xi\|^2 > \|\xi\|^2,$$

a contradiction. Therefore $\sigma(A) \subset [-n,n]$, and $A \in F_n$. Therefore $\mathcal{B}(H)_s$ is $F_\sigma$ in $\mathcal{SA}(H)$.

Next, we show that $F_n$ has empty interior in $\mathcal{SA}(H)$, whence $\mathcal{B}(H)_s$ is meager. Assume by contradiction that there is $A_0 \in \text{Int}(F_n)$. Then by Theorem 3.11, there exists $K_0 \in \mathcal{K}(H)_s$ such that $A_0 + K_0$ is in $\text{Int}(F_n)$ and has the form $\sum_{m=1}^{\infty} \lambda_m e_m$, where $(e_m)_{m=1}^{\infty}$ is a sequence of mutually orthogonal rank one projections with sum equal to 1, and $(\lambda_m)_{m=1}^{\infty} \subset \mathbb{R}$. Let $A_k := \sum_{m=1}^{k} \lambda_m e_m + \sum_{m=k+1}^{\infty} m e_m$. Then for each $\xi \in H$, we have

$$\|(A_k - i)^{-1}\xi - (A_0 + K_0 - i)^{-1}\xi\|^2 = \sum_{m=k+1}^{\infty} \left| \frac{1}{m - i} - \frac{1}{\lambda_m - i} \right|^2 \|e_m\xi\|^2 \leq \sum_{m=k+1}^{\infty} 4\|e_m\xi\|^2 k^{-\infty} 0. \quad (6)$$

Since $A_0 + K_0 \in \text{Int}(F_n)$, this shows that $A_k \in F_n$ for large enough $k$, which is a contradiction because each $A_k$ is unbounded. Thus $\mathcal{B}(H)_s$ is meager in $\mathcal{SA}(H)$, whence so is $[B]_{\mathcal{K}(H)_s}$.

To prove the meagerness of $[A]_{\mathcal{K}(H)_s}$ for an unbounded $A$, we need easy lemmata. We say that a subset $A$ of a Polish space $X$ has the **Baire property** if there exists an open set $U \subset X$ such that the symmetric difference $A \triangle U$ is meager (Baire first category).
Lemma 3.14. In a Polish space $X$, there is no uncountable disjoint family of non-meager subsets of $X$ each of which has the Baire property.

Proof. Assume by contradiction that there exists an uncountable disjoint family of non-meager subsets $(X_i)_{i \in I}$ of $X$ such that each $X_i$ has the Baire property ($i \in I$). Then for each $i \in I$, there exists a nonempty open subset $U_i$ of $X$ such that $U_i \setminus X_i$ is meager in $X_i$ (this is equivalent to that $U_i \setminus X_i$ is meager in $U_i$ with subspace topology, since $U_i$ is open). Since $(U_i)_{i \in I}$ is an uncountable family of nonempty open sets in a second countable space $X$, there exists $i_1, i_2 \in I$ ($i_1 \neq i_2$) such that $V := U_{i_1} \cap U_{i_2} \neq \emptyset$. For $k = 1, 2$, $V \setminus X_{i_k} \subset U_{i_k} \setminus X_{i_k}$ is meager in $X$, whence $V \cap X_{i_k}$ is comeager in $V$. Therefore $(V \cap X_{i_1}) \cap (V \cap X_{i_2})$ is also comeager in $V$. Since $V$ is open hence Baire, this shows in particular that $V \cap X_{i_1} \cap X_{i_2} \neq \emptyset$, which is a contradiction.

□

Lemma 3.15. Let $A \in SA(H)$ be unbounded. Then for each $s, t \in \mathbb{R} \setminus \{0\}$ with $s \neq t$, $[sA]_{\mathbb{K}(H)_{sa}}$ and $[tA]_{\mathbb{K}(H)_{sa}}$ are disjoint and homeomorphic.

Proof. Suppose by contradiction that $[sA]_{\mathbb{K}(H)_{sa}} \cap [tA]_{\mathbb{K}(H)_{sa}} \neq \emptyset$. Then there exist $K_1, K_2 \in \mathbb{K}(H)_{sa}$ such that $sA + K_1 = tA + K_2$. Therefore for $\xi \in \text{dom}(A)$, $\frac{1}{s} - \frac{1}{t} (K_1 - K_2)\xi = \phi \xi$. Since $\text{dom}(A)$ is dense and $K_1 - K_2$ is bounded, this implies that $A$ is also bounded, a contradiction. Therefore $[sA]_{\mathbb{K}(H)_{sa}} \cap [tA]_{\mathbb{K}(H)_{sa}} = \emptyset$. To show the latter claim, it is enough to show that for each $s \neq 0$, $[A]_{\mathbb{K}(H)_{sa}}$ and $[sA]_{\mathbb{K}(H)_{sa}}$ are homeomorphic. Define $\varphi : [A]_{\mathbb{K}(H)_{sa}} \to [sA]_{\mathbb{K}(H)_{sa}}$ by $\varphi(A + K) := sA + sK$ ($K \in \mathbb{K}(H)_{sa}$). It is straightforward to see that this is a well-defined homeomorphism. □

Proposition 3.16. Let $A \in SA(H)$ be unbounded. Then $[A]_{\mathbb{K}(H)_{sa}}$ is meager in $SA(H)$.

Proof. Suppose that $[A]_{\mathbb{K}(H)_{sa}}$ were non-meager. Since any orbit of a continuous action of a Polish group on a Polish is Borel, $[A]_{\mathbb{K}(H)_{sa}}$ is a Borel subset of $SA(H)$ by [Gao09, Proposition 3.1.10]. By Lemma 3.15, $\{[sA]_{\mathbb{K}(H)_{sa}} \mid s \in \mathbb{R} \setminus \{0\}\}$ would be an uncountable family of disjoint Borel subsets of $SA(H)$ any two of which are homeomorphic. Thus, each $[sA]_{\mathbb{K}(H)_{sa}}$ ($s \neq 0$) would be non-meager and has the Baire property. This is a contradiction to Lemma 3.14. Therefore $[A]_{\mathbb{K}(H)_{sa}}$ is meager. □

Proof of Theorem 3.10. We have shown that every orbit is dense (Lemma 3.12) and meager (Lemma 3.13 and Proposition 3.16). Therefore to show the generic turbulence, it suffices to show that there exists at least one orbit on which the action is turbulent (cf.[Gao09, Exercise 10.3.7]). We thus show that every local orbit of $0 \in SA(H)$ is somewhere-dense. Let $U$ be an open neighborhood of $0$ in $SA(H)$, $V$ be an open neighborhood of $0$ in $\mathbb{K}(H)_{sa}$. We may and do assume that $U, V$ are of the following form

$$U = \bigcap_{j=1}^{m} \{ B \in SA(H) : \| (B - i)^{-1} \xi_j - (0 - i)^{-1} \xi_j \| < \varepsilon \},$$

$$V = \{ K \in \mathbb{K}(H)_{sa} : \| K \| < \delta \}$$

for some unit vectors $\xi_1, \cdots, \xi_m \in H$ and $\varepsilon, \delta > 0$. We show that $U \subset \mathcal{O}(0, U, V)$. Let $B \in U$. By Theorem 3.11 and spectral Theorem, there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of finite-rank self-adjoint operators contained in $U$ such that $B_n \xrightarrow{\text{SRT}} B$ in $SA(H)$. Therefore to show that $B \in \mathcal{O}(0, U, V)$, it suffices to prove that $B_n \in \mathcal{O}(0, U, V)$ for each $n \in \mathbb{N}$. Thus we may assume that $B$ is of finite-rank. Let $B = \sum_{k=1}^{n} \lambda_k p_k$ be the spectral decomposition of $B$. Choose $N \in \mathbb{N}$ so that $\frac{1}{N} \| B \| < \delta$. Then for each $1 \leq j \leq m$ and $1 \leq l \leq N$,

$$\left\| \left( \frac{1}{N} B - i \right)^{-1} \xi_j - (0 - i)^{-1} \xi_j \right\|^2 = \sum_{k=1}^{n} \left| \frac{1}{N} \lambda_k - i \right|^2 \| p_k \xi_j \|^2 \leq \| (B - i)^{-1} \xi_j - (0 - i)^{-1} \xi_j \|^2 < \varepsilon^2.$$
Therefore $\frac{1}{N}B \in U$ for each $0 \leq l \leq N$. Since $\frac{1}{N}B \in V$, this shows that $B = \frac{1}{N}B + \cdots + \frac{1}{N}B \in \mathcal{O}(0, U, V)$. Therefore $\mathcal{O}(0, U, V)$ is somewhere-dense, and the action is turbulent at 0. This finishes the proof.

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1. Introduction

- I would like to use electromagnetically induced transparency (EIT) in a solid medium for the memory of a single qubit.

- Ichimura’s TOSHIBA group succeeded in demonstrating the EIT in $\text{Y}_2\text{SiO}_5$ crystal doped with $\text{Pr}^{3+}$ ions, denoted by $\text{Pr}^{3+}:\text{Y}_2\text{SiO}_5$ [Phys. Rev. A 58, 4116 (1998)].

- Zhu and other collaborators studied the superconducting qubit coupled with the nitrogen-vacancy (NV) centers in diamond, and observed a dark state for such a superconducting qubit in the experiment [Nature Comm. 5, 3424 (2014)].


- I found a duality between the dark state and the quasi-dark state, which enables to swap them [arXiv:1503.04386].
1. Introduction

I took interest in Hashizume’s research on developing a laser system with the soft-X-ray range in a compact laboratory. Thus, our target laser has the soft-X-ray range. That is, 100 eV (i.e., 10 nm wavelength).

[But]

Ichimura’s TOSHIBA team succeeded in demonstrating the EIT in Pr$^{3+}$:Y$_2$SiO$_5$ [Phys. Rev. A 58, 4116 (1998)].

They dope the crystal, Y$_2$SiO$_5$, with the rare-earth ion, Pr$^{3+}$, and use the 3-level quantum states of Pr$^{3+}$ for the EIT.

The distribution of the Pr$^{3+}$ ions depends on each sample because their doping method is not the ion implantation. Thus, the distribution changes with each change in sample. They handle an ensemble of the Pr$^{3+}$ ions.

I am interested in the EIT for a one qubit of a Pr$^{3+}$ ion. To achieve that, we need a laser with very short wave length.
2. A Physical Problem

- Let $\omega_c$ be the (angular) frequency of the 1-mode light of a laser, and $\lambda_c$ be the wave length: for the speed $c$ of light
  \[
  \omega_c = \frac{2\pi c}{\lambda_c}.
  \]

- The laser is the ensemble of many copies of the photon with $\omega_c$ by the stimulated emission using the difference between the two energy levels of the electron in a atom. The light emission follows QED. Thus, $\omega_c$ is determined by the sort of the atom, and thus, we cannot control the value of $\omega_c$ freely.

- The laser of which wave length $\lambda_c$ is a dozen of nanometer can be realized by the so-called free electron laser using synchrotron radiation.

- Setting up the system of such a free electron laser requires a huge facilities such as of the X-ray free electron laser.

- The laser of which wave length $\lambda_c$ is a dozen of nanometer can be realized by the so-called free electron laser using synchrotron radiation.

- Setting up the system of such a free electron laser requires a huge facilities such as of the X-ray free electron laser.

- We would like a laser system producing the laser with a dozen of nanometer wave length in a laboratory to handle a single qubit.

- How can we describe the synchrotron radiation in QED?
2. A Physical Problem

To solve the problem

- Physicists irradiate a non-linear optical medium with laser(s), and use several non-linear optical effects.
- The sum-frequency generation (SFG) is among them.

![Diagram showing lasers and polarization density](image)

3. Overview of SFG in Non-Linear Optics

SFG

- When the electric field $E$ is small, the polarization density $P$ of a non-linear optical medium is given as

$$P = \varepsilon_0 \chi^{(1)} E,$$

where $\varepsilon_0$ is the electric permittivity of free space, and $\chi^{(1)}$ the electric susceptibility.

- When the strength of electric field $E$ gets larger in the non-linear optical medium, the strength of polarization also becomes large, and we cannot ignore the non-linear terms:

$$P = \varepsilon_0 \left( \chi^{(1)} E + \chi^{(2)} E^2 + \cdots \right) =: P_L + P_{NL}^{(2)} + \cdots.$$

- For simplicity, we assume

$$E = E_1 \sin \omega_1 t + E_2 \sin \omega_2 t.$$
4. Mathematical Problems

My Problem

The above explanation of SFG is according to classical physics. I would like to know its quantum version, in particular, using quantum field theory.

The Hamiltonian may be written as

$$ H_{SFG} = \hbar \omega_1 a_1^\dagger a_1 + \hbar \omega_2 a_2^\dagger a_2 + H_{\text{plasmon}} + H_{\text{int}}. $$

Then,

Problem 1) How can we define $H_{\text{plasmon}}$ and $H_{\text{int}}$?
Problem 2) Can we derive the laser by SFG from $H_{SFG}$ in quantum field theory?
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