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Abstract

In this paper, we consider the submodular function minimization problem with submodular set covering constraints and precedence constraints, and we prove that the algorithm of McCormick, Peis, Verschae, and Wierz for the precedence constrained covering problem can be generalized to our setting.

1 Introduction

Assume that we are given a finite set U. Then a real-valued function g with a domain 2^U is said to be *submodular*, if for every pair of subsets X, Y of U,

$$g(X) + g(Y) \geq g(X \cup Y) + g(X \cap Y).$$

Submodular functions play an important role in many fields, e.g., combinatorial optimization, machine learning, and game theory. One of the most fundamental problems related to submodular functions is the submodular function minimization problem. In this problem, we are given a submodular function g with a domain 2^U. Then the goal of the submodular function minimization problem is to find a subset X of U minimizing $g(X)$ among all subsets of U, i.e., to find a minimizer of the function g. It is known [6, 7, 9, 22] that this problem can be solved in polynomial time if we are given a value oracle for the function g.

Furthermore, constrained variants of the submodular function minimization problem have been extensively studied in various fields [5, 8, 10, 11, 12, 13, 14, 16, 17, 23, 25]. For example, Iwata and Nagano [10] considered the submodular function minimization problem with vertex covering constraints, set covering constraints, and edge covering constraints, and gave approximability and inapproximability results. Goel, Karande, Tripathi, and Wang [5] considered the vertex cover problem, the shortest path problem, the perfect matching problem, and the minimum spanning tree problem with a monotone submodular cost function. Svitkina and Fleischer [23] also considered several optimization problems with a submodular cost function. Jegelka and Bilmes [14] considered the submodular function minimization problem with cut constraints. Zhang and Vorobeychik [25] considered the submodular function minimization problem with routing constraints. Hochbaum [8] considered the submodular minimization problem with linear constraints having at most two variables per inequality. Kamiyama [15] considered the the submodular function minimization problem with covering type linear constraints. Koufogiannakis and Young [17] considered the monotone submodular function minimization problem with general covering constraints. Furthermore, Iyer and Bilmes [11] and Kamiyama [16] considered the submodular function minimization problem

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with submodular set covering constraints (see Section 2 for the formal definition of submodular set covering constraints).

In this paper, we consider the submodular function minimization problem with submodular set covering constraints and precedence constraints. This problem is inspired by the precedence constrained covering problem proposed by McCormick, Peis, Verschae, and Wierz [19]. This problem is an integer program with covering type inequalities and precedence constraints over variables. The submodular set cover problem with a linear cost function was introduced by Wolsey [24]. A greedy algorithm [24] and a primal-dual algorithm [2] were proposed for this problem. Applications of the submodular set cover problem include the capacitated supply-demand problem [4] and the bounded degree deletion problem [3]. See also [11] for its applications. The submodular function minimization problem with submodular set covering constraints was considered by Iyer and Bilmes [11] and Kamiyama [16]. In this paper, we prove that the algorithm of McCormick, Peis, Verschae, and Wierz [19] for the precedence constrained covering problem can be generalized to our setting by using the technique of Kamiyama [16] that is based on the result of Iwata and Nagano [10].

2 Preliminaries

Throughout this paper, we denote by $\mathbb{R}$ and $\mathbb{R}_+$ the sets of real numbers and non-negative real numbers, respectively. Assume that we are given a finite set U. Then for each subset X of U and each vector x in $\mathbb{R}^U$, we define $x(X) := \sum_{i \in X} x(i)$. Furthermore, a function $g\colon 2^U \to \mathbb{R}$ is said to be *monotone*, if $g(X) \leq g(Y)$ for every pair of subsets X, Y of U such that $X \subseteq Y$.

In this paper, we consider the submodular function minimization problem with submodular set covering constraints and precedence constraints defined as follows. We are given a non-empty finite set N and monotone submodular functions $\rho, \mu\colon 2^N \to \mathbb{R}_+$ such that $\rho(\emptyset) = 0$ and $\mu(\emptyset) = 0$. Furthermore, we are given a partial order $\preceq$ over N (i.e., a reflexive, antisymmetric, and transitive order over N). Define $\mathcal{L}$ as the family of subsets X of N satisfying the condition that if $j \in X$, then $i \in X$ for every pair of elements i, j in N such that $i \preceq j$. Then the submodular function minimization problem with submodular set covering constraints and precedence constraints is defined as follows.

$$\begin{array}{ll} \text{Minimize} & \rho(X) \\ \text{subject to} & \mu(X) = \mu(N) \\ & X \in \mathcal{L}. \end{array} \tag{1}$$

We assume without loss of generality that $\mu(N) > 0$. Otherwise, $\mu(X) = 0$ for every subset X of N, and thus $\emptyset$ is an optimal solution of the problem (1).

For each subset X of N, we denote by χ_X the vector in $\{0,1\}^N$ satisfying the condition that $\chi_X(i) = 1$ for every element i in X, and $\chi_X(i) = 0$ for every element i in $N \setminus X$. For each pair of elements i, j in N such that $i \preceq j$ and $i \neq j$, we write $i \prec j$. For each subset S of N, we define the function $\mu_S\colon 2^{N \setminus S} \to \mathbb{R}$ by $\mu_S(X) := \mu(X \cup S) - \mu(S)$.

The *Lovász extension* $\widehat{\rho}\colon \mathbb{R}_+^N \to \mathbb{R}_+$ of ρ is defined as follows [18]. Assume that we are given a vector x in $\mathbb{R}_+^N$. Furthermore, we assume that for non-negative real numbers $x_1, x_2, \ldots, x_s$ such that $x_1 > x_2 > \cdots > x_s$, $\{x_1, x_2, \ldots, x_s\} = \{x(i) \mid i \in N\}$ holds. For each integer p in $\{1, 2, \ldots, s\}$, we define N_p as the set of elements i in N such that $x(i) \geq x_p$. Define $\widehat{\rho}(x)$ by

$$\widehat{\rho}(x) := \sum_{p=1}^{s} (x_p - x_{p+1}) \rho(N_p),$$

where we define $x_{s+1} := 0$. It is not difficult to see that $\rho(X) = \widehat{\rho}(\chi_X)$ holds for every subset X of N. Define $\mathrm{P}(\rho)$ as the set of vectors x in $\mathbb{R}^N$ such that $x(X) \leq \rho(X)$ for every subset X of N.

Theorem 1 (Edmonds [1])**.** *For every vector x in $\mathbb{R}_+^N$,*

$$\widehat{\rho}(x) = \max_{z \in \mathrm{P}(\rho)} \sum_{i \in N} x(i) z(i). \tag{2}$$

By considering the dual problem of (2), Theorem 1 implies that the following theorem.

Theorem 2 (See, e.g., [10])**.** *For every vector x in $\mathbb{R}_+^N$, $\widehat{\rho}(x)$ is equal to the optimal objective value of the following problem.*

$$\begin{aligned} \text{Minimize} \quad & \sum_{X \subseteq N} \rho(X)\xi(X) \\ \text{subject to} \quad & \sum_{X \subseteq N \colon i \in X} \xi(X) = x(i) \quad (i \in N) \\ & \xi \in \mathbb{R}_+^{2^N}. \end{aligned}$$

The following theorem plays an important role in our algorithm.

Theorem 3 (Wolsey [24])**.** *Assume that we are given a subset X of N. Then $\mu(X) = \mu(N)$ holds if and only if for every subset S of N,*

$$\sum_{i \in N \setminus S} \mu_S(\{i\}) \cdot \chi_X(i) \geq \mu_S(N \setminus S).$$

3 Algorithm

In this section, we propose a polynomial-time approximation algorithm for the submodular function minimization problem with submodular set covering constraints and precedence constraints. This algorithm is a natural generalization of the algorithm of McCormick, Peis, Verschae, and Wierz [19] for the precedence constrained covering problem

Theorem 3 implies that the problem (1) is equivalent to the following problem.

$$\begin{aligned} \text{Minimize} \quad & \rho(X) \\ \text{subject to} \quad & \sum_{i \in N \setminus S} \mu_S(\{i\}) \cdot \chi_X(i) \geq \mu_S(N \setminus S) \quad (S \subseteq N) \\ & X \in \mathcal{L}. \end{aligned} \tag{3}$$

Then the problem (3) is equivalent to the following problem.

$$\begin{aligned} \text{Minimize} \quad & \widehat{\rho}(x) \\ \text{subject to} \quad & \sum_{i \in N \setminus S} \mu_S(\{i\}) \cdot x(i) \geq \mu_S(N \setminus S) \quad (S \subseteq N) \\ & x_i \geq x_j \quad (i, j \in N \text{ such that } i \prec j) \\ & x \in \{0,1\}^N. \end{aligned} \tag{4}$$

For each member S in $\mathcal{L}$, we define $\mathsf{Min}(S)$ as the set of elements i in $N \setminus S$ such that there does not exist an element j in $N \setminus S$ such that $j \prec i$. Furthermore, for each member S in $\mathcal{L}$, we define $\mathsf{Min}^+(S) := \mathsf{Min}(S) \cup S$. For each member S in $\mathcal{L}$ and each element i in $N \setminus \mathsf{Min}^+(S)$, we define $D_i(S)$ the set of elements j in $\mathsf{Min}(S)$ such that $j \prec i$. For each member S in $\mathcal{L}$ and each element i in $\mathsf{Min}(S)$, we define

$$\pi_S(i) := \min \Big\{ \mu_S(N \setminus S),\ \mu_S(\{i\}) + \sum_{j \in N \setminus \mathsf{Min}^+(S) \colon i \prec j} \frac{\mu_S(\{j\})}{|D_j(S)|} \Big\}.$$

Then we consider the following problem.

$$\begin{array}{ll} \text{Minimize} & \widehat{\rho}(x) \\ \text{subject to} & \displaystyle\sum_{i \in \mathsf{Min}(S)} \pi_S(i) \cdot x(i) \geq \mu_S(N \setminus S) \quad (S \in \mathcal{L}) \\ & x \in \{0,1\}^N. \end{array} \tag{5}$$

The following lemma is almost the same as [19, Lemma 3].

Lemma 4. *Every feasible solution of the problem* (4) *is a feasible solution of the problem* (5)*.*

Proof. Let x be a feasible solution of the problem (4). Assume that we are given a member S in $\mathcal{L}$. Define X as the set of elements i in N such that $x(i) = 1$. We first consider the case where there exists an element i^* in $\mathsf{Min}(S) \cap X$ such that $\pi_S(i^*) = \mu_S(N \setminus S)$. In this case,

$$\sum_{i \in \mathsf{Min}(S)} \pi_S(i) \cdot x(i) = \sum_{i \in \mathsf{Min}(S) \cap X} \pi_S(i) \geq \pi_S(i^*) = \mu_S(N \setminus S).$$

Next, we consider the case where $\pi_S(i) < \mu_S(N \setminus S)$ for every element i in $\mathsf{Min}(S) \cap X$. Then the second constraint of the problem (4) implies that for every element j in $(N \setminus \mathsf{Min}^+(S)) \cap X$, we have $D_j(S) \subseteq \mathsf{Min}(S) \cap X$. Thus, since $\mu_S(\{j\}) \geq 0$ follows from the monotonicity of μ for every element j in $N \setminus S$, we have

$$\begin{aligned} \sum_{i \in \mathsf{Min}(S) \cap X} \sum_{j \in N \setminus \mathsf{Min}^+(S) \colon i \prec j} \frac{\mu_S(\{j\})}{|D_j(S)|} &\geq \sum_{i \in \mathsf{Min}(S) \cap X} \sum_{j \in (N \setminus \mathsf{Min}^+(S)) \cap X \colon i \prec j} \frac{\mu_S(\{j\})}{|D_j(S)|} \\ &= \sum_{j \in (N \setminus \mathsf{Min}^+(S)) \cap X} \sum_{i \in D_j(S)} \frac{\mu_S(\{j\})}{|D_j(S)|} = \sum_{j \in (N \setminus \mathsf{Min}^+(S)) \cap X} \mu_S(\{j\}) = \sum_{j \in N \setminus \mathsf{Min}^+(S)} \mu_S(\{j\}) \cdot x(j). \end{aligned}$$

This implies that

$$\begin{aligned} \sum_{i \in \mathsf{Min}(S)} \pi_S(i) \cdot x(i) &= \sum_{i \in \mathsf{Min}(S) \cap X} \pi_S(i) \\ &= \sum_{i \in \mathsf{Min}(S) \cap X} \mu_S(\{i\}) + \sum_{i \in \mathsf{Min}(S) \cap X} \sum_{j \in N \setminus \mathsf{Min}^+(S) \colon i \prec j} \frac{\mu_S(\{j\})}{|D_j(S)|} \\ &\geq \sum_{i \in \mathsf{Min}(S)} \mu_S(\{i\}) \cdot x(i) + \sum_{j \in N \setminus \mathsf{Min}^+(S)} \mu_S(\{j\}) \cdot x(j) \\ &= \sum_{i \in N \setminus S} \mu_S(\{i\}) \cdot x(i) \geq \mu_S(N \setminus S). \end{aligned}$$

Notice that the last equality follows from the first constraint of the problem (4). This completes the proof. □

We are now ready to introduce a relaxation problem of our problem. The following technique is based on the technique proposed by Iwata and Nagano [10]. Lemma 4 implies that the following problem (6) is a relaxation problem of the problem (4).

$$\begin{array}{ll} \text{Minimize} & \widehat{\rho}(x) \\ \text{subject to} & \displaystyle\sum_{i \in \mathsf{Min}(S)} \pi_S(i) \cdot x(i) \geq \mu_S(N \setminus S) \quad (S \in \mathcal{L}) \\ & x \in \mathbb{R}_+^N. \end{array} \tag{6}$$

Then Theorem 2 implies that the optimal objective value of the problem (6) is equivalent to that of the following problem.

$$\begin{array}{ll} \text{Minimize} & \sum_{X \subseteq N} \rho(X) \cdot \xi(X) \\ \text{subject to} & \sum_{i \in \mathsf{Min}(S)} \pi_S(i) \cdot x(i) \geq \mu_S(N \setminus S) \quad (S \in \mathcal{L}) \\ & \sum_{X \subseteq N \colon i \in X} \xi(X) = x(i) \quad (i \in N) \\ & (x, \xi) \in \mathbb{R}^N \times \mathbb{R}_+^{2^N}. \end{array} \tag{7}$$

Here we neglect the redundant non-negativity constraint of x. The dual problem of the problem (7) is the following problem.

$$\begin{array}{ll} \text{Maximize} & \sum_{S \in \mathcal{L}} \mu_S(N \setminus S) \cdot y(S) \\ \text{subject to} & \sum_{S \in \mathcal{L} \colon i \in \mathsf{Min}(S)} \pi_S(i) \cdot y(S) = z(i) \quad (i \in N) \\ & (y, z) \in \mathbb{R}_+^{\mathcal{L}} \times \mathrm{P}(\rho). \end{array} \tag{8}$$

Assume that we are given a vector z in $\mathrm{P}(\rho)$. Define the function $\rho - z$ with a domain 2^N by $(\rho - z)(X) := \rho(X) - z(X)$. Then $\rho - z$ is submodular, and $\min_{X \subseteq N}(\rho - z)(X) = (\rho - z)(\emptyset) = 0$. Furthermore, it is not difficult to see that for every pair of minimizers X, Y of $\rho - z$, $X \cup Y$ is a minimizer of $\rho - z$. Thus, for every subset S of N, there exists the unique maximal subset X of S such that $\rho(X) = z(X)$.

For each member S in $\mathcal{L}$, we define the vector d_S in $\mathbb{R}_+^N$ by

$$d_S(i) := \begin{cases} \pi_S(i) & \text{if } i \in \mathsf{Min}(S) \\ 0 & \text{if } i \in N \setminus \mathsf{Min}(S). \end{cases}$$

We are now ready to propose our algorithm.

Algorithm 1

Step 1: Define y_1, z_1 as the zero vectors in $\mathbb{R}^{\mathcal{L}}$ and $\mathbb{R}^N$, respectively. Define $S_1 := \emptyset$. Set $t := 1$.

Step 2: If $\mu(S_t) = \mu(N)$, then output S_t and halt.

Step 3: Do the following steps **(3-a)** to **(3-e)**.

(3-a) Define the real number α_t by

$$\alpha_t := \min_{X \subseteq N \colon d_{S_t}(X) \neq 0} \frac{\rho(X) - z_t(X)}{d_{S_t}(X)}.$$

(3-b) Define the vector y_{t+1} in $\mathbb{R}^{\mathcal{L}}$ by

$$y_{t+1}(S) := \begin{cases} y_t(S) + \alpha_t & \text{if } S = S_t \\ y_t(S) & \text{otherwise.} \end{cases}$$

(3-c) Define $z_{t+1} := z_t + \alpha_t \cdot d_{S_t}$.

(3-d) Define S_{t+1} as the maximal subset of $\mathsf{Min}^+(S_t)$ such that $\rho(S_{t+1}) = z_{t+1}(S_{t+1})$.

(3-e) Set $t := t + 1$, and go back to **Step 2**.

End of Algorithm

4 Analysis

We first prove that **Algorithm 1** is well-defined.

Lemma 5. *Assume that we are given a member S in $\mathcal{L}$ such that $\mu(S) < \mu(N)$. Then there exists an element i in $\mathsf{Min}(S)$ such that $\pi_S(i) \neq 0$.*

Proof. Assume that $\pi_S(i) = 0$ for every element i in $\mathsf{Min}(S)$. Since $\mu_S(N \setminus S) = \mu(N) - \mu(S) > 0$, we have

$$\mu_S(\{i\}) + \sum_{j \in N \setminus \mathsf{Min}^+(S)\colon i \prec j} \frac{\mu_S(\{j\})}{|D_j(S)|} = 0$$

for every element i in $\mathsf{Min}(S)$. This implies that $\mu_S(\{i\}) = \mu_S(\{j\}) = 0$ for every element i in $\mathsf{Min}(S)$ and every element j in $N \setminus \mathsf{Min}^+(S)$ such that $i \prec j$. Thus, since for every element j in $N \setminus \mathsf{Min}^+(S)$, there exists an element i in $\mathsf{Min}(S)$ such that $i \prec j$, we have $\mu_S(\{i\}) = 0$ for every element i in $N \setminus S$. Thus, $\mu(S) = \mu(S \cup \{i\})$ holds for every element i in $N \setminus S$. Furthermore, since μ is a submodular function, we have

$$\sum_{i \in N \setminus S} \mu(S \cup \{i\}) \geq \mu(N) + (|N \setminus S| - 1)\mu(S).$$

Recall that $\mu(S) = \mu(S \cup \{i\})$ for every element i in $N \setminus S$. Thus, this implies that $\mu(S) \geq \mu(N)$. This contradicts the fact that $\mu(S) < \mu(N)$. This completes the proof. □

Lemma 5 implies that for every member S in $\mathcal{L}$ such that $\mu(S) < \mu(N)$, there exists a subset X of N such that $d_S(X) \neq 0$. The following lemma implies that **Algorithm 1** is well-defined.

Lemma 6. *Assume that we are given a vector z in $\mathrm{P}(\rho)$ and a member S of $\mathcal{L}$ such that $\mu(S) < \mu(N)$, $\rho(S) = z(S)$, and $z(i) = 0$ for every element i in $N \setminus \mathsf{Min}^+(S)$. Define*

$$\alpha := \min_{X \subseteq N\colon d_S(X) \neq 0} \frac{\rho(X) - z(X)}{d_S(X)}, \qquad z' := z + \alpha \cdot d_S.$$

Then we have

(1) $z' \in \mathrm{P}(\rho)$.

Furthermore, we define S' as the maximal subset of $\mathsf{Min}^+(S)$ such that $\rho(S') = z'(S')$. Then the following statements hold.

(2) $S \subsetneq S'$ *and* $S' \in \mathcal{L}$.

(3) $z'(i) = 0$ *for every element i in $N \setminus \mathsf{Min}^+(S')$.*

Proof. (1) For every subset X of N such that $d_S(X) = 0$, we have $z'(X) = z(X) \leq \rho(X)$. For every subset X of N such that $d_S(X) \neq 0$,

$$z'(X) = z(X) + \alpha \cdot d_S(X) \leq z(X) + \frac{\rho(X) - z(X)}{d_S(X)} \cdot d_S(X) = \rho(X). \tag{9}$$

This completes the proof.

(2) Since $z(i) = z'(i)$ for every element i in S and $\rho(S) = z(S)$, we have $\rho(S) = z'(S)$. Thus, since $S \subseteq \mathsf{Min}^+(S)$, we have $S \subseteq S'$. Furthermore, since $S' \setminus S \subseteq \mathsf{Min}(S)$, we have $S' \in \mathcal{L}$. What remains is to prove that $S \neq S'$. For proving this, we first prove that there exists a subset Z of $\mathsf{Min}^+(S)$ such that $d_S(Z) \neq 0$ and

$$\alpha = \frac{\rho(Z) - z(Z)}{d_S(Z)}.$$

Let X be a subset of N such that $d_S(X) \neq 0$ and

$$\alpha = \frac{\rho(X) - z(X)}{d_S(X)}.$$

Furthermore, we assume that X minimizes $|X \setminus \mathsf{Min}^+(S)|$ among all subsets of N satisfying these conditions. If $|X \setminus \mathsf{Min}^+(S)| = 0$, then the proof is done. Assume that $|X \setminus \mathsf{Min}^+(S)| \neq 0$. Let j be an element in $X \setminus \mathsf{Min}^+(S)$. Then since $z(j) = 0$, $d_S(j) = 0$, and $\rho(X \setminus \{j\}) \leq \rho(X)$ follows from the monotonicity of ρ, we have

$$d_S(X \setminus \{j\}) = d_S(X) \neq 0, \qquad \alpha = \frac{\rho(X) - z(X)}{d_S(X)} \geq \frac{\rho(X \setminus \{j\}) - z(X \setminus \{j\})}{d_S(X \setminus \{j\})} \geq \alpha.$$

This contradicts the definition of X. This completes the proof of the existence of Z. We are now ready to prove that $S \neq S'$. Notice that $z \in \mathrm{P}(\rho)$ implies that $\alpha \geq 0$. If $\alpha = 0$, then $z = z'$ and $\rho(Z) = z(Z)$. Thus, $\rho(Z) = z'(Z)$. If $\alpha > 0$, then (9) implies that $\rho(Z) = z'(Z)$. Thus, in both cases, we have $\rho(Z) = z'(Z)$. Since $Z \subseteq \mathsf{Min}^+(S)$, the maximality of S' implies that $Z \subseteq S'$. Since $d_S(Z) \neq 0$, we have $Z \not\subseteq S$. Thus, we have $S \neq S'$. This completes the proof.

(3) Recall that for every element i in N, $d_S(i) > 0$ only if $i \in \mathsf{Min}(S)$. Since $S \subseteq S' \subseteq \mathsf{Min}^+(S)$, we have $\mathsf{Min}(S) \subseteq \mathsf{Min}^+(S')$. This completes the proof. □

The statement 2 of Lemma 6 implies that the number of iterations of **Algorithm 1** is at most $|N| + 1$. It is known [21] that α_t can be computed in polynomial time. Furthermore, it is known (see, e.g., [20, Note 10.11] and [21, Lemma 1]) that we can find the maximal subset S_{t+1} of $\mathsf{Min}^+(S_t)$ such that $\rho(S_{t+1}) = z_{t+1}(S_{t+1})$ in polynomial time. These imply that **Algorithm 1** is a polynomial-time algorithm.

Next, we evaluate the approximation ratio of **Algorithm 1**. Assume that **Algorithm 1** halts when $t = k$.

Lemma 7. *For every integer t in $\{1, 2, \ldots, k\}$, (y_t, z_t) is a feasible solution of the problem* (8).

Proof. We prove this lemma by induction on t. Since $\rho(X) \geq 0$ for every subset X of N, (y_1, z_1) is a feasible solution of the problem (8). Assume that (y_t, z_t) is a feasible solution of the problem (8) for some integer t in $\{1, 2, \ldots, k-1\}$. The statement 1 of Lemma 6 implies that $z_{t+1} \in \mathrm{P}(\rho)$. Furthermore, since $\alpha_t \geq 0$, $y_{t+1} \in \mathbb{R}_+^{\mathcal{L}}$. For every element i in $N \setminus \mathsf{Min}(S_t)$, $z_{k+1}(i) = z_k(i)$ and

$$\sum_{S \in \mathcal{L} \colon i \in \mathsf{Min}(S)} \pi_S(i) \cdot y_t(S) = \sum_{S \in \mathcal{L} \colon i \in \mathsf{Min}(S)} \pi_S(i) \cdot y_{t+1}(S).$$

For every element i in $\mathsf{Min}(S_t)$, $z_{t+1}(i) - z_t(i) = \alpha_t \cdot \pi_{S_t}(i)$ and

$$\sum_{S \in \mathcal{L} \colon i \in \mathsf{Min}(S)} \pi_S(i) \cdot y_{t+1}(S) - \sum_{S \in \mathcal{L} \colon i \in \mathsf{Min}(S)} \pi_S(i) \cdot y_t(S) = \pi_{S_t}(i) \cdot (y_{t+1}(S_t) - y_t(S_t)) = \alpha_t \cdot \pi_{S_t}(i).$$

This completes the proof. □

Define $\mathcal{A}$ as the family of subsets S of N such that $i \not\preceq j$ for any pair of distinct elements i, j in S. Notice that for every member S in $\mathcal{L}$, we have $\mathsf{Min}(S) \in \mathcal{A}$. Define $\Pi := \max_{S \in \mathcal{A}} |S|$.

Lemma 8. *For every member S in $\mathcal{L}$ such that $\mu(S) = \mu(N)$, we have $\rho(S_k) \leq \Pi \cdot \rho(X)$*

Proof. Let S be a member in $\mathcal{L}$ such that $\mu(S) = \mu(N)$. Lemma 7 implies that

$$\sum_{S \in \mathcal{L}} \mu_S(N \setminus S) \cdot y_k(S) \leq \rho(S).$$

Thus, we have

$$\begin{aligned}\rho(S_k) = z_k(S_k) &= \sum_{i \in S_k} \sum_{S \in \mathcal{L}\colon i \in \mathsf{Min}(S)} \pi_S(i) \cdot y_k(S) = \sum_{S \in \mathcal{L}} \sum_{i \in \mathsf{Min}(S) \cap S_k} \pi_S(i) \cdot y_k(S) \\ &\leq \sum_{S \in \mathcal{L}} \sum_{i \in \mathsf{Min}(S) \cap S_k} \mu_S(N \setminus S) \cdot y_k(S) \leq \sum_{S \in \mathcal{L}} \Pi \cdot \mu_S(N \setminus S) \cdot y_k(S) \leq \Pi \cdot \rho(S).\end{aligned}$$

This completes the proof. □

We are now ready to prove the main result of this paper.

Theorem 9. **Algorithm 1** *is a* Π*-approximation algorithm for the submodular function minimization problem with submodular set covering constraints and precedence constraints*

Proof. This theorem immediately follows from the statement 2 of Lemma 6 and Lemma 8. □

5 Special Cases

In this section, we consider the case where $i \not\preceq j$ for any pair of distinct elements i, j in N, i.e., $\mathcal{L} = 2^N$.

Assume that **Algorithm 1** halts when $t = k$. Theorem 10 can be regarded as a generalization of [19, Corollary 2], and matches the result of [16] under the assumption that ρ is monotone.

Theorem 10. *Assume that* $i \not\preceq j$ *for any pair of distinct elements* i, j *in* N*. Then for every member* S *in* $\mathcal{L}$ *such that* $\mu(S) = \mu(N)$*, we have*

$$\rho(S_k) \leq \Big[\max_{S \subseteq N\colon \mu(S) < \mu(N)} \frac{\sum_{i \in N \setminus S} \mu_S(\{i\})}{\mu_S(N \setminus S)} \Big] \cdot \rho(S).$$

Proof. Since $\mu(N) > 0$, then $k > 1$. The assumption in this theorem implies that for every member S in $\mathcal{L}$ and every element i in $\mathsf{Min}(S)$, $\pi_S(i) \leq \mu_S(\{i\})$ holds. Thus, since $y_k(S) = 0$ for every member S in $2^N \setminus \{S_1, \ldots, S_{k-1}\}$,

$$\begin{aligned}\rho(S_k) = z_k(S_k) &= \sum_{i \in S_k} \sum_{S \in \mathcal{L}\colon i \in \mathsf{Min}(S)} \pi_S(i) \cdot y_k(S) = \sum_{S \in \mathcal{L}} \sum_{i \in \mathsf{Min}(S) \cap S_k} \pi_S(i) \cdot y_k(S) \\ &\leq \sum_{S \in \mathcal{L}} \sum_{i \in \mathsf{Min}(S) \cap S_k} \mu_S(\{i\}) \cdot y_k(S) \leq \sum_{t=1}^{k-1} \sum_{i \in \mathsf{Min}(S_t) \cap S_k} \mu_{S_t}(\{i\}) \cdot y_k(S_t)\end{aligned} \tag{10}$$

Since $\mu_S(\{i\}) \geq 0$ for every member S in $\mathcal{L}$ and every element i in $N \setminus S$, (10) implies that

$$\rho(S_k) \leq \sum_{t=1}^{k-1} \sum_{i \in N \setminus S_t} \mu_{S_t}(\{i\}) \cdot y_k(S_t) \tag{11}$$

Furthermore,

$$\rho(S) \geq \sum_{S \in \mathcal{L}} \mu_S(N \setminus S) \cdot y_k(S) = \sum_{t=1}^{k-1} \mu_{S_t}(N \setminus S_t) \cdot y_k(S_t). \tag{12}$$

Define

$$\beta := \max_{t \in \{1,2,\ldots,k-1\}} \frac{\sum_{i \in N \setminus S_t} \mu_{S_t}(\{i\})}{\mu_{S_t}(N \setminus S_t)}.$$

Then (11) and (12) imply that

$$\rho(S_k) \leq \sum_{t=1}^{k-1} \sum_{i \in N \setminus S_t} \mu_{S_t}(\{i\}) \cdot y_k(S_t) \leq \beta \cdot \sum_{t=1}^{k-1} \mu_{S_t}(N \setminus S_t) \cdot y_k(S_t) \leq \beta \cdot \rho(S). \tag{13}$$

Since $\mu(S_t) < \mu(N)$ for every integer t in $\{1, 2, \ldots, k-1\}$, we have

$$\beta \leq \max_{S \subseteq N \colon \mu(S) < \mu(N)} \frac{\sum_{i \in N \setminus S} \mu_S(\{i\})}{\mu_S(N \setminus S)}.$$

This completes the proof. □

References

[1] J. Edmonds. Submodular functions, matroids, and certain polyhedra. In R. Guy, H. Hanani, N. Sauer, and J. Schönheim, editors, *Combinatorial Structures and their Applications*, pages 69–87. Gordon and Breach, 1970.

[2] T. Fujito. On approximation of the submodular set cover problem. *Operations Research Letters*, 25(4):169–174, 1999.

[3] T. Fujito. Approximating bounded degree deletion via matroid matching. In *Proceedings of the 10th International Conference on Algorithms and Complexity*, volume 10236 of *Lecture Notes in Computer Science*, pages 234–246, 2017.

[4] T. Fujito and T. Yabuta. Submodular integer cover and its application to production planning. In *Proceedingsof the 2nd Workshop on Approximation and Online Algorithms*, volume 3351 of *Lecture Notes in Computer Science*, pages 154–166, 2004.

[5] G. Goel, C. Karande, P. Tripathi, and L. Wang. Approximability of combinatorial problems with multi-agent submodular cost functions. In *Proceedingsof the 50th Annual Symposium on Foundations of Computer Science*, pages 755–764, 2009.

[6] M. Grötschel, L. Lovász, and A. Schrijver. The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica*, 1(2):169–197, 1981.

[7] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization.* Springer-Verlag, 1988.

[8] D. S. Hochbaum. Submodular problems - approximations and algorithms. Technical Report arXiv:1010.1945, 2010.

[9] S. Iwata, L. Fleischer, and S. Fujishige. A combinatorial strongly polynomial algorithm for minimizing submodular functions. *Journal of the ACM*, 48(4):761–777, 2001.

[10] S. Iwata and K. Nagano. Submodular function minimization under covering constraints. In *Proceedingsof the 50th Annual Symposium on Foundations of Computer Science*, pages 671–680, 2009.

[11] R. K. Iyer and J. A. Bilmes. Submodular optimization with submodular cover and submodular knapsack constraints. In *Advances in Neural Information Processing Systems 26*, pages 2436–2444, 2013.

[12] R. K. Iyer, S. Jegelka, and J. A. Bilmes. Curvature and optimal algorithms for learning and minimizing submodular functions. In *Advances in Neural Information Processing Systems 26*, pages 2742–2750, 2013.

[13] R. K. Iyer, S. Jegelka, and J. A. Bilmes. Monotone closure of relaxed constraints in submodular optimization: Connections between minimization and maximization. In *Proceedingsof the 30th Conference on Uncertainty in Artificial Intelligence*, pages 360–369, 2014.

[14] S. Jegelka and J. A. Bilmes. Graph cuts with interacting edge weights: examples, approximations, and algorithms. *Mathematical Programming*, pages 1–42, 2016.

[15] N. Kamiyam. A note on submodular function minimization with covering type linear constraints. Technical Report MI Preprint Series 2016-15, Kyushu University, 2016.

[16] N. Kamiyama. Submodular function minimization under a submodular set covering constraint. In *Proceedingsof the 8th International Conference on Theory and Applications of Models of Computation*, volume 6648 of *Lecture Notes in Computer Science*, pages 133–141, 2011.

[17] C. Koufogiannakis and N. E. Young. Greedy Δ-approximation algorithm for covering with arbitrary constraints and submodular cost. *Algorithmica*, 66(1):113–152, 2013.

[18] L. Lovász. Submodular functions and convexity. In A. Bachem, M. Grötschel, and B. Korte, editors, *Mathematical Programming–The State of the Art*, pages 235–257. Springer-Verlag, 1983.

[19] S. T. McCormick, B. Peis, J. Verschae, and A. Wierz. Primal–dual algorithms for precedence constrained covering problems. *Algorithmica*, pages 1–17, 2016.

[20] K. Murota. *Discrete Convex Analysis*, volume 10 of *SIAM Monographs on Discrete Mathematics and Applications*. Society for Industrial and Applied Mathematics, 2003.

[21] K. Nagano. A faster parametric submodular function minimization algorithm and applications. Technical Report METR 2007-43, The University of Tokyo, 2007.

[22] A. Schrijver. A combinatorial algorithm minimizing submodular functions in strongly polynomial time. *Journal of Combinatorial Theory, Series B*, 80(2):346–355, 2000.

[23] Z. Svitkina and L. Fleischer. Submodular approximation: Sampling-based algorithms and lower bounds. *SIAM Journal on Computing*, 40(6):1715–1737, 2011.

[24] L. A. Wolsey. An analysis of the greedy algorithm for the submodular set covering problem. *Combinatorica*, 2(4):385–393, 1982.

[25] H. Zhang and Y. Vorobeychik. Submodular optimization with routing constraints. In *Proceedingsof the 30th AAAI Conference on Artificial Intelligence*, pages 819–826, 2016.

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