Development of the Theory of Integrable Systems

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Research Interests: Theory of Integrable Systems

My research activities are based on the study of integrable systems, which originates from the study of nonlinear waves with a particle-like characteristic called solitons. Solitons are one of the fundamental modes characterizing the nonlinear phenomena, together with chaos and fractals. The existence of solitons seems miraculous both in physics and mathematics. In physics, a solitary wave that stably propagates cannot be derived from linear theory, and thus it is considered to exist on a subtle balance between nonlinearity and dispersion. In mathematics, although fundamental equations to describe solitons are nonlinear partial differential equations, which are generally difficult to analyze, they possess a remarkable property in that they can be exactly solved. Behind these miracles lies the mathematics of infinite-dimensional space with the symmetry of infinite degrees of freedom. A family of functional equations that share this property is called integrable systems. A deep understanding of the underlying mathematics of integrable systems enables various applications. Three such examples follow.

1. Discretization and ultra-discretization: Recently, a method to discretize both independent and dependent variables of the soliton equations preserving the integrability has been developed (ultra-discretization). Namely, one can construct integrable difference equations or cellular automata by a systematic method from given integrable differential equations. A typical interaction of solitons is shown on the left side of Fig. 1. In this figure, a large soliton with a higher velocity comes from the left and passes a smaller, slower soliton. Although their amplitudes, velocities, and shapes do not change through the interaction, the locations of each wave are shifted, confirming that the interaction is nonlinear. An automaton that describes solitons is shown on the right side of Fig. 1. There are rows of boxes and balls. At each moment in time, from the left to the right, each ball is moved to the empty box closest to the right once, and the time is incremented by one when all balls have been moved. This simple model describes solitons, and a sound correspondence to a partial differential equation can be established through ultra-discretization. Discretization and ultra-discretization preserving integrability have been applied to a wide range of mathematical sciences and engineering, including numerical analysis and traffic flow analysis. In addition, the underlying geometric structure of ultra-discrete systems has recently been clarified in terms of tropical geometry.

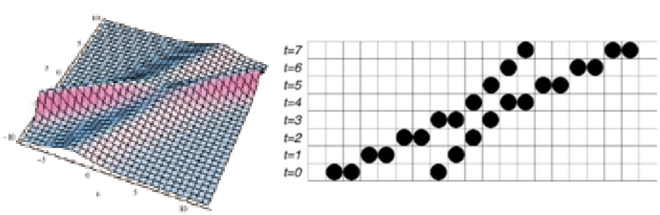

Figure 1

2. Discrete Painlevé equations and elliptic curves: A family of difference equations called discrete Painlevé equations is formulated as an addition theorem on moving pencils of cubic curves in the complex projective plane (Fig. 2). These equations are closely related to soliton equations, and they admit special functions of hypergeometric type, such as the Bessel functions, as particular solutions. Thus, the integrable systems are closely related to pure mathematics, such as algebraic geometry. Additionally, discrete Painlevé equations were recently discovered to be closely related to probability theory and combinatorics.

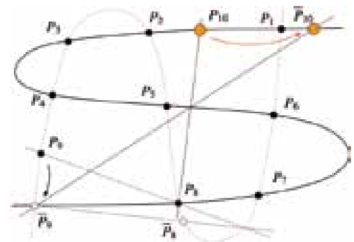

Figure 2

3. Discrete integrable systems and discrete differential geometry: Various integrable systems arise as fundamental equations describing curves and surfaces in space and their dynamics, and integrability corresponds to the consistency of geometric objects. The theory of discrete curves and surfaces consistent with discrete integrable systems has recently been well developed under the name of discrete differential geometry. I am actively trying to expand my research in this area based on the theory of discrete integrable systems. Figure 3 shows a surface constructed from a solution of the soliton equation called the sine-Gordon equation (left) and a discrete surface from that of the discrete sine-Gordon equation (right). The dynamics of a planar discrete curve described by the discrete modified KdV equation, which was constructed in our recent study, is shown in Fig. 4.

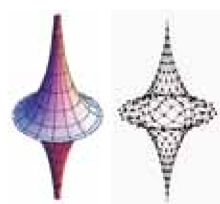

Figure 3

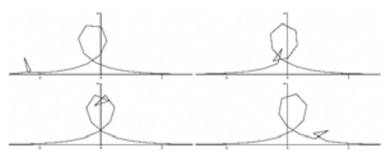

Figure 4

I would like to develop the theory of integrable systems as a methodology for objects that admit exact analysis to contribute to the various activities of Institute of Mathematics for Industry.