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Abstract

The Pareto stability is one of solution concepts in two-sided matching markets with ties. It is known that there always exists a Pareto stable matching in the many-to-many setting. In this paper, we consider the following generalization of the Pareto stable matching problem in the many-to-many setting. Each agent v of one side has a matroid defined on the set of edges incident to v, and the set of agents assigned to v must be an independent set of this matroid. By extending the algorithm of Kamiyama for the many-to-many setting, we prove that there always exists a Pareto stable matching in this setting, and a Pareto stable matching can be found in polynomial time.

1 Introduction

The stable matching problem introduced by Gale and Shapley [12] is one of the most successful two-sided matching market models. In this problem, each agent has a preference list over agents of the other side. In several situations, it is natural to assume that these preference lists contain ties (see, e.g., [1, 6]). It is known that the introduction of ties drastically change the properties of stable matchings (see, e.g., [19] and [27, Chapter 3] for a survey of stable matchings with ties). For the stable matching problem with ties, several solution concepts were introduced (see, e.g., [15, 17, 18, 32]). In this paper, we consider the concept of Pareto stability that is one of solution concepts in two-sided matching markets with ties (see, e.g., [32] for properties of Pareto stable matchings). A matching M is said to be Pareto stable, if M is Pareto efficient and stable. It is known that a Pareto stable matching always exists in the one-to-one setting [6], the many-to-one setting [5], and the many-to-many setting [3, 21]. Furthermore, we can find a Pareto stable matching in polynomial time [3, 5, 6, 21]. It should be noted that Chen and Ghosh [4] considered Pareto stable matchings in the setting where every pair of agents can transact any number of units.

In this paper, we consider the following generalization of the Pareto stable matching problem in many-to-many setting. In our setting, each agent v of one side has a matroid defined on the set of edges incident to v, and the set of agents assigned to v must be an independent set of this matroid. Recently, matroid generalizations of matching problems have been extensively studied (see, e.g., [7, 8, 11, 20, 22, 23, 29, 33]). By extending the algorithm of Kamiyama [21] for the many-to-many setting, we prove that there always exists a Pareto stable matching in this setting, and a Pareto stable matching can be found in polynomial time.

The main technical difficulty in the extension of the algorithm of [21] in the many-to-many setting to the matroid setting is a proof of a key lemma that plays an important role when we prove the correctness of our algorithm. In the many-to-many setting, this lemma can be relatively easily proved because an auxiliary graph used for proving this lemma does not drastically change. However, in the matroid setting, this auxiliary graph may drastically change. For coping with this difficulty, we use the idea of the algorithm of [14] for the independent assignment problem using potential functions.

2 Preliminaries

Let $\mathbb{R}$ and $\mathbb{Z}_+$ be the sets of real numbers and non-negative integers, respectively.

A pair $\mathbf{M} = (U, \mathcal{I})$ of a finite set U and a family $\mathcal{I}$ of subsets of U is called a *matroid*, if it satisfies the following conditions.

(I0) $\emptyset \in \mathcal{I}$.

(I1) If $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$.

(I2) If $I, J \in \mathcal{I}$ and $|I| < |J|$, then there exists an element u in $J \setminus I$ such that $I \cup \{u\} \in \mathcal{I}$.

A subset of U belonging to $\mathcal{I}$ is called an *independent set of* $\mathbf{M}$.

Assume that we are given a matroid $\mathbf{M} = (U, \mathcal{I})$. A subset C of U is called a *circuit* of $\mathbf{M}$, if C is not an independent set of $\mathbf{M}$, but every proper subset of C is an independent set of $\mathbf{M}$. The following property of circuits is known.

Theorem 1 (See, e.g., [30, Lemma 1.1.3])**.** *Assume that we are given a matroid* $\mathbf{M} = (U, \mathcal{I})$ *and distinct circuits* C_1, C_2 *of* $\mathbf{M}$ *such that* $C_1 \cap C_2 \neq \emptyset$. *Then, for every element* u *in* $C_1 \cap C_2$, *there exists a circuit* C *of* $\mathbf{M}$ *such that* $C \subseteq (C_1 \cup C_2) \setminus \{u\}$.

Assume that we are given an independent set I of $\mathbf{M}$. Then, we denote by $\mathbf{sp}_{\mathbf{M}}(I)$ the set of elements u in $U \setminus I$ such that $I \cup \{u\} \notin \mathcal{I}$. It is not difficult to see that for every element u in $\mathbf{sp}_{\mathbf{M}}(I)$, $I \cup \{u\}$ contains a circuit of $\mathbf{M}$ as a subset, and (I1) implies that u belongs to this circuit. Furthermore, Theorem 1 implies that such a circuit is uniquely determined. We call this circuit the *fundamental circuit of* u *with respect to* I *and* $\mathbf{M}$, and we denote by $\mathsf{C}_{\mathbf{M}}(u, I)$ this circuit. It is known [30, p.20, Exercise 5] that for every element u in $\mathbf{sp}_{\mathbf{M}}(I)$, $\mathsf{C}_{\mathbf{M}}(u, I)$ is the set of elements w in $I \cup \{u\}$ such that $(I \cup \{u\}) \setminus \{w\} \in \mathcal{I}$. Furthermore, we define $\mathbf{cl}_{\mathbf{M}}(I) := \mathbf{sp}_{\mathbf{M}}(I) \cup I$.

2.1 Problem formulation

In this paper, we are given a finite simple (not necessarily complete) bipartite graph $G = (V, E)$. We assume that V is partitioned into subsets P, Q, and every edge in E connects a vertex in P and a vertex in Q. If there exists an edge in E connecting a vertex u in P and a vertex w in Q, then we denote by $[u, w]$ this edge. For each vertex v in P (resp., Q) and each subset F of E, we denote by $F(v)$ the set of edges $[u, w]$ in F such that $u = v$ (resp., $w = v$). Without loss of generality, we assume that $E(v) \neq \emptyset$ for any vertex v in V. For each vertex v in V, we are given a transitive and complete binary relation $\succsim_v$ on $E(v)$. Furthermore, we are given a capacity function $c \colon P \to \mathbb{Z}_+ \setminus \{0\}$ such that $c(v) \leq |E|$ for every vertex v in P. Lastly, for each vertex v in Q, we are given a matroid $\mathbf{M}_v = (E(v), \mathcal{I}_v)$. Without loss of generality, we assume that for every vertex v in Q and every edge e in $E(v)$, we have $\{e\} \in \mathcal{I}_v$.

For each vertex v in V and each pair of edges e, f in $E(v)$, we write $e \succ_v f$ (resp., $e \sim_v f$), if $e \succsim_v f$ and $f \not\succsim_v e$ (resp., $e \succsim_v f$ and $f \succsim_v e$). For every vertex v in V and every pair of edges e, f in $E(v)$, if $e \succ_v f$, then v prefers e to f.

A subset M of E is called a *matching in* G, if the following conditions are satisfied.

(M1) For every vertex v in P, $|M(v)| \leq c(v)$.

(M2) For every vertex v in Q, $M(v)$ is an independent set of $\mathbf{M}_v$.

It should be noted that the condition (M2) can be rewritten as follows. Define $\mathcal{I}^{\oplus}$ as the family of subsets F of E such that $F(v)$ is an independent set of $\mathbf{M}_v$ for every vertex v in Q. Furthermore, we define $\mathbf{M}^{\oplus} := (E, \mathcal{I}^{\oplus})$. Then, it is not difficult to see that $\mathbf{M}^{\oplus}$ is a matroid and the condition (M2) is equivalent to the condition that M is an independent set of $\mathbf{M}^{\oplus}$.

Assume that we are given matchings M, N in G and a vertex v in V. Furthermore, we assume that $M(v) = \{e_1, e_2, \ldots, e_k\}$, $N(v) = \{f_1, f_2, \ldots, f_h\}$,

$$e_1 \succsim_v e_2 \succsim_v \cdots \succsim_v e_k, \quad \text{and} \quad f_1 \succsim_v f_2 \succsim_v \cdots \succsim_v f_h.$$

We say that M *dominates* N *on* v, if (i) $k \geq h$, and (ii) $e_i \succsim_v f_i$ for every integer i in $\{1, 2, \ldots, h\}$. We say that M *strictly dominates* N *on* v, if M dominates N on v and at least one of the following conditions (i) and (ii) is satisfied. (i) $k > h$. (ii) There exists an integer i in $\{1, 2, \ldots, h\}$ such that $e_i \succ_v f_i$.

For each pair of matchings M, N in G, we write $M \gg N$, if M dominates N on every vertex in V and M strictly dominates N on some vertex in V. Furthermore, a matching M in G is said to be *Pareto efficient*, if there does not exist a matching N in G such that $N \gg M$.

Assume that we are given a matching M in G. Define $\mathbf{dom}_P(M)$ as the set of edges $e = [u, w]$ in $E \setminus M$ such that $|M(u)| = c(u)$ and $f \succsim_u e$ for every edge f in $M(u)$. We define $\mathbf{dom}_Q(M)$ as the set of edges $e = [u, w]$ in $E \setminus M$ such that $e \in \mathbf{sp}_{\mathbf{M}_w}(M(w))$ and $f \succsim_w e$ for every edge f in $\mathsf{C}_{\mathbf{M}_w}(e, M(w))$. Then, M is said to be *stable*, if

$$\mathbf{dom}_P(M) \cup \mathbf{dom}_Q(M) = E \setminus M.$$

A matching M in G is said to be *Pareto stable*, if M is Pareto efficient and stable. Then, the goal of this paper is to prove the following theorem (see Section 3 for its proof).

Theorem 2. *There always exists a Pareto stable matching in* G*.*

Since our proof of Theorem 2 is constructive, a Pareto stable matching in G can be found in polynomial time (we assume that for every vertex v in Q and every subset F of $E(v)$, we can decide whether $F \in \mathcal{I}_v$ in time bounded by a polynomial of the input size of G).

It is not difficult to see that our model is a generalization of the Pareto stable matching problem and its variants [5, 6, 3, 21]. Furthermore, matroid constraints can represent the following *laminar capacity constraints* (see, e.g., [13]). In this setting, for each vertex v in Q, we are given a laminar family $\mathcal{C}_v$ of subsets of $E(v)$, i.e., $C_1 \cap C_2 = \emptyset$, or $C_1 \subseteq C_2$, or $C_2 \subseteq C_1$ for every distinct members C_1, C_2 in $\mathcal{C}_v$. For each vertex v in Q, we are given a capacity function $\hat{c}_v \colon \mathcal{C}_v \to \mathbb{Z}_+ \setminus \{0\}$. For each vertex v in Q, we define $\mathcal{I}_v$ as the family of subsets F of $E(v)$ such that for every member C in $\mathcal{C}_v$, $|F \cap C| \leq \hat{c}_v(C)$. It is not difficult to see that $\mathbf{M}_v = (E(v), \mathcal{I}_v)$ is a matroid for every vertex v in Q.

2.2 Key lemma

In this subsection, we assume that we are given a matroid $\mathbf{N} = (S, \mathcal{J})$ such that $\{u\} \in \mathcal{J}$ for every element u in S, a partition $S_1, S_2, \ldots, S_d$ of S such that $S_i \neq \emptyset$ for any integer i in $\{1, 2, \ldots, d\}$, a negative integer $\xi(u)$ for each element u in S, and capacity functions $q_1, q_2 \colon \{1, 2, \ldots, d\} \to \mathbb{Z}_+$ such that $q_1(i) \leq q_2(i)$ for every integer i in $\{1, 2, \ldots, d\}$. For each integer t in $\{1, 2\}$, we define $\mathcal{O}_t$ as the family of subsets I of S satisfying the following conditions.

- For every integer i in $\{1, 2, \ldots, d\}$, we have $|I \cap S_i| \leq q_t(i)$.
- I is an independent set of $\mathbf{N}$.

For each subset I of S, we define $\mathsf{c}(I) := \sum_{u \in I} \xi(u)$. For each integer t in $\{1, 2\}$, we define $\mathcal{O}_t^*$ as the family of members I in $\mathcal{O}_t$ such that $\mathsf{c}(I) \leq \mathsf{c}(J)$ for every member J in $\mathcal{O}_t$. The following lemma plays an important role in our algorithm. We will give a proof of this lemma in Section 4

Lemma 3. *Assume that we are given a member* I_1 *in* $\mathcal{O}_1^*$*. Then, there exists a member* I_2 *in* $\mathcal{O}_2^*$ *satisfying the condition that* $I_2 \cap S_i$ *is a subset of* $I_1 \cap S_i$ *for every integer* i *in* $\{1, 2, \ldots, d\}$ *such that* $|I_1 \cap S_i| < q_1(i)$*.*

If we can decide whether $I \in \mathcal{J}$ in time bounded by a polynomial in $|S|$ for every subset I of S, we can find a member I_2 satisfying the conditions in Lemma 3 in polynomial time as follows. Define $\overline{S}$ as the subset of S satisfying

$$\overline{S} \cap S_i = \begin{cases} I_1 \cap S_i & \text{if } |I_1 \cap S_i| < q_1(i) \\ S_i & \text{otherwise} \end{cases}$$

for every integer i in $\{1, 2, \ldots, d\}$. Furthermore, we define $\overline{\mathcal{J}} := \{I \subseteq \overline{S} \mid I \in \mathcal{J}\}$. It is not difficult to see that the pair $\overline{\mathbf{N}} = (\overline{S}, \overline{\mathcal{J}})$ is a matroid. Define $\overline{\mathcal{O}}$ as the family of subsets I of $\overline{S}$ satisfying the following conditions.

- For every integer i in $\{1, 2, \ldots, d\}$, we have $|I \cap S_i| \leq q_2(i)$.
- I is an independent set of $\overline{\mathbf{N}}$.

Furthermore, we define $\overline{\mathcal{O}}^*$ as the family of members I in $\overline{\mathcal{O}}$ such that $\mathsf{c}(I) \leq \mathsf{c}(J)$ for every member J in $\overline{\mathcal{O}}$. Then, the problem of finding I_2 satisfying the conditions in Lemma 3 is equivalent to the problem of finding a member in $\overline{\mathcal{O}}^*$. It is known that this problem can be solved in polynomial time by using, e.g., the algorithms of [9, 10, 14].

3 Main Result

For each vertex v in V, we can partition $E(v)$ into non-empty subsets $E_{v,1}, E_{v,2}, \ldots, E_{v,\delta_v}$ satisfying the following conditions.

- For every integer i in $\{1, 2, \ldots, \delta_v\}$ and every pair of edges e, f in $E_{v,i}$, we have $e \sim_v f$.
- For every pair of integers i, j in $\{1, 2, \ldots, \delta_v\}$ such that $i < j$ and every pair of edges e in $E_{v,i}$ and f in $E_{v,j}$, we have $e \succ_v f$.

Define $\Delta_P := \max_{v \in P} \delta_v$ and $\Delta_Q := \max_{v \in Q} \delta_v$. Define $\mathbb{P} := \{(v, i) \mid v \in P, i \in \{1, 2, \ldots, \delta_v\}\}$. For each pair (v, i) in $\mathbb{P}$, we define $\Sigma_{v,i} := \bigcup_{j=1}^{i} E_{v,j}$. For each vertex v in V, we define $\Sigma_{v,0} := \emptyset$.

For each edge $e = [u, w]$ in E, we define $\mathsf{r}(e)$ (resp., $\overline{\mathsf{r}}(e)$) as the integer i in $\{1, 2, \ldots, \delta_u\}$ (resp, $\{1, 2, \ldots, \delta_w\}$) such that $e \in E_{u,i}$ (resp., $e \in E_{w,i}$). For each edge $e = [u, w]$ in E, we define

$$\omega(e) := -(|E| + 1)^{\Delta_P - \mathsf{r}(e)} - (|E| + 1)^{\Delta_P + \Delta_Q - \overline{\mathsf{r}}(e)}.$$

This is a standard technique in the study of the rank-maximal matching problem (see, e.g., [16, 28]). For each subset F of E, we define $\omega(F) := \sum_{e \in F} \omega(e)$.

Although the following lemma is well known, we give a proof for completeness.

Lemma 4. *For every pair of matchings M, N in G, if $M \gg N$, then $\omega(M) < \omega(N)$.*

Proof. Since $M \gg N$, it is not difficult to see that there exists an injective mapping $\sigma \colon N \to M$ such that $\mathsf{r}(e) \geq \mathsf{r}(\sigma(e))$ for every edge e in N. Furthermore, for the same reason, there exists an injective mapping $\overline{\sigma} \colon N \to M$ such that $\overline{\mathsf{r}}(e) \geq \overline{\mathsf{r}}(\overline{\sigma}(e))$ for every edge e in N. Thus, we have

$$\begin{aligned} \omega(N) &= -\sum_{e \in N} (|E| + 1)^{\Delta_P - \mathsf{r}(e)} - \sum_{e \in N} (|E| + 1)^{\Delta_P + \Delta_Q - \overline{\mathsf{r}}(e)} \\ &\geq -\sum_{e \in N} (|E| + 1)^{\Delta_P - \mathsf{r}(\sigma(e))} - \sum_{e \in N} (|E| + 1)^{\Delta_P + \Delta_Q - \overline{\mathsf{r}}(\overline{\sigma}(e))} \\ &\geq -\sum_{e \in M} (|E| + 1)^{\Delta_P - \mathsf{r}(e)} - \sum_{e \in M} (|E| + 1)^{\Delta_P + \Delta_Q - \overline{\mathsf{r}}(e)} = \omega(M). \end{aligned} \tag{1}$$

Since $M \gg N$, at least one of the following conditions (i) and (ii) holds. (i) There exists an edge e in N such that at least one of $\mathsf{r}(e) > \mathsf{r}(\sigma(e))$ and $\overline{\mathsf{r}}(e) > \overline{\mathsf{r}}(\overline{\sigma}(e))$ holds. (ii) There exists an edge in $M \setminus N$. If the condition (i) holds, then the first inequality in (1) holds strictly. If the condition (ii) holds, then the second inequality in (1) holds strictly. This completes the proof. □

Assume that we are given a function $\overline{c}\colon \mathbb{P} \to \mathbb{Z}_+$. Define $\mathcal{F}(\overline{c})$ as the family of subsets F of E satisfying the following conditions.

- For every pair (v,i) in $\mathbb{P}$, we have $|F \cap E_{v,i}| \le \overline{c}(v,i)$.
- F is an independent set of $\mathbf{M}^{\oplus}$.

Define $\mathcal{F}^*(\overline{c})$ as the family of members F in $\mathcal{F}(\overline{c})$ such that $\omega(F) \le \omega(F')$ for every member F' in $\mathcal{F}(\overline{c})$. Our algorithm is described as follows.

Algorithm 1

Step 1. Define $X^0 := E$. Set $r := 1$.

Step 2. For each vertex v in P, we define θ_v^r as follows.

- If $|X^{r-1}(v)| < c(v)$, then we define $\theta_v^r := \delta_v + 1$.
- If $|X^{r-1}(v)| \ge c(v)$, then we define θ_v^r as the integer i in $\{1,2,\ldots,\delta_v\}$ such that

$$|X^{r-1} \cap \Sigma_{v,i-1}| < c(v) \quad \text{and} \quad |X^{r-1} \cap \Sigma_{v,i}| \ge c(v).$$

Step 3. Define a function $\overline{c}^r \colon \mathbb{P} \to \mathbb{Z}_+$ as follows.

$$\overline{c}^r(v,i) := \begin{cases} |E|+1 & \text{if } i < \theta_v^r \\ c(v) - |X^{r-1} \cap \Sigma_{v,i-1}| & \text{if } i = \theta_v^r \\ 0 & \text{if } i > \theta_v^r. \end{cases}$$

Step 4. Find a member F^r in $\mathcal{F}^*(\overline{c}^r)$ such that $F^r \cap E_{v,i} \subseteq X^{r-1} \cap E_{v,i}$ for every pair (v,i) in $\mathbb{P}$.

Step 5. If the following conditions hold, then output F^r and halt.

- For every vertex v in P and every integer i in $\{1,2,\ldots,\theta_v^r - 1\}$, $F^r \cap E_{v,i} = X^{r-1} \cap E_{v,i}$.
- For every vertex v in P such that $\theta_v^r \le \delta_v$, we have $|F^r \cap E_{v,\theta_v^r}| = \overline{c}^r(v,\theta_v^r)$.

Otherwise, we define X^r as the subset of E satisfying the following conditions.

- For every vertex v in P and every integer i in $\{1,2,\ldots,\theta_v^r - 1\}$, $X^r \cap E_{v,i} = F^r \cap E_{v,i}$.
- For every vertex v in P such that $\theta_v^r \le \delta_v$, we have

$$X^r \cap E_{v,\theta_v^r} = \begin{cases} F^r \cap E_{v,\theta_v^r} & \text{if } |F^r \cap E_{v,\theta_v^r}| < \overline{c}^r(v,\theta_v^r) \\ E_{v,\theta_v^r} & \text{if } |F^r \cap E_{v,\theta_v^r}| = \overline{c}^r(v,\theta_v^r). \end{cases}$$

- For every vertex v in P and every integer i in $\{\theta_v^r + 1, \theta_v^r + 2, \ldots, \delta_v\}$, $X^r \cap E_{v,i} = E_{v,i}$.

Update $r := r+1$, and go back to **Step 2**.

End of Algorithm

We first prove that **Algorithm 1** is well-defined. Since $X^0 \cap E_{v,i} = E_{v,i}$ for every pair (v,i) in $\mathbb{P}$, there exists a member F^1 in $\mathcal{F}^*(\overline{c}^1)$ such that $F^1 \cap E_{v,i} \subseteq X^0 \cap E_{v,i}$ for every pair (v,i) in $\mathbb{P}$. This implies that the first iteration of **Algorithm 1** is well-defined. Assume that the kth iteration of **Algorithm 1** is well-defined for some positive integer k, i.e., there exists a member F^k in $\mathcal{F}^*(\overline{c}^k)$ such that $F^k \cap E_{v,i} \subseteq X^{k-1} \cap E_{v,i}$ for every pair (v,i) in $\mathbb{P}$. Then, we prove that if **Algorithm 1** does not halt in the kth iteration, then the $(k+1)$st iteration of **Algorithm 1** is well-defined.

Lemma 5. *For every vertex v in P,*

$$|X^k \cap \Sigma_{v,\theta_v^k - 1}| \le |X^{k-1} \cap \Sigma_{v,\theta_v^k - 1}| < c(v).$$

Proof. The definition of X^k implies that $X^k \cap E_{v,i} = F^k \cap E_{v,i} \subseteq X^{k-1} \cap E_{v,i}$ for every vertex v in P and every integer i in $\{1, 2, \ldots, \theta_v^k - 1\}$. This implies the first inequality. The strict inequality follows from the definition of θ_v^k. This completes the proof. □

Lemma 6. *For every vertex v in P, we have $\theta_v^k \leq \theta_v^{k+1}$.*

Proof. Since $|X^k \cap \Sigma_{v,\theta_v^{k+1}}| \geq c(v)$, Lemma 5 implies that $\theta_v^k \leq \theta_v^{k+1}$. □

Lemma 7. *For every pair (v, i) in $\mathbb{P}$, we have $\overline{c}^k(v, i) \leq \overline{c}^{k+1}(v, i)$.*

Proof. Let v be a vertex in P. Lemma 6 implies that $\theta_v^k \leq \theta_v^{k+1}$. We first consider the case where $\theta_v^k < \theta_v^{k+1}$. In this case, $\overline{c}^{k+1}(v, i) = |E| + 1$ for every integer i in $\{1, 2, \ldots, \theta_v^k\}$. Since $c(v) \leq |E|$, this completes the proof. If $\theta_v^k = \theta_v^{k+1}$, then Lemma 5 implies that $\overline{c}^k(v, \theta_v^k) \leq \overline{c}^{k+1}(v, \theta_v^k)$. This completes the proof. □

The following lemma implies that the $(k + 1)$st iteration of **Algorithm 1** is well-defined.

Lemma 8. *There exists a member F^{k+1} in $\mathcal{F}^*(\overline{c}^{k+1})$ such that $F^{k+1} \cap E_{v,i} \subseteq X^k \cap E_{v,i}$ for every pair (v, i) in $\mathbb{P}$.*

Proof. For every pair (v, i) in $\mathbb{P}$, if $X^k \cap E_{v,i} \neq E_{v,i}$, then $X^k \cap E_{v,i} = F^k \cap E_{v,i}$ and $|F^k \cap E_{v,i}| < \overline{c}^k(v, i)$. Thus, this lemma follows from Lemmas 3 and 7 by setting $S := E$, $q_1 := \overline{c}^k$, $q_2 := \overline{c}^{k+1}$, $\xi := \omega$, $\{1, 2, \ldots, d\} := \mathbb{P}$, and $\mathbf{N} := \mathbf{M}^{\oplus}$. □

Next we prove that the number of iterations of **Algorithm 1** is at most $|E| + 2$. Let k be a positive integer. Then, we assume that **Algorithm 1** does not halt in the $(k + 1)$st iteration.

Lemma 9. *For every vertex v in P, if $|F^k \cap E_{v,\theta_v^k}| < \overline{c}^k(v, \theta_v^k)$, then $\theta_v^k < \theta_v^{k+1}$.*

Proof. It suffices to prove that $|X^k \cap \Sigma_{v,\theta_v^k}| < c(v)$. Lemma 5 implies that

$$\begin{aligned} |X^k \cap \Sigma_{v,\theta_v^k}| &= |X^k \cap \Sigma_{v,\theta_v^k - 1}| + |X^k \cap E_{v,\theta_v^k}| \leq |X^{k-1} \cap \Sigma_{v,\theta_v^k - 1}| + |X^k \cap E_{v,\theta_v^k}| \\ &= |X^{k-1} \cap \Sigma_{v,\theta_v^k - 1}| + |F^k \cap E_{v,\theta_v^k}| < |X^{k-1} \cap \Sigma_{v,\theta_v^k - 1}| + \overline{c}^k(v, \theta_v^k) = c(v). \end{aligned}$$

This completes the proof. □

Lemma 10. *Assume that we are given a vertex v in P such that $\theta_v^{k+1} \leq \delta_v$. For every integer i in $\{\theta_v^{k+1}, \theta_v^{k+1} + 1, \ldots, \delta_v\}$, we have $X^k \cap E_{v,i} = E_{v,i}$.*

Proof. Lemma 6 implies that $\theta_v^k \leq \theta_v^{k+1}$. If $\theta_v^k < \theta_v^{k+1}$, then this lemma immediately follows from the definition of X^k. If $\theta_v^k = \theta_v^{k+1}$, then Lemma 9 implies that $|F^k \cap E_{v,\theta_v^k}| = \overline{c}^k(v, \theta_v^k)$. Thus, the definition of X^k completes the proof. □

Lemma 11. $X^{k+1} \subsetneq X^k$.

Proof. For proving this lemma, it suffices to prove that (i) $X^{k+1} \cap E_{v,i} \subseteq X^k \cap E_{v,i}$ for every pair (v, i) in $\mathbb{P}$, and (ii) $X^{k+1} \cap E_{v,i} \subsetneq X^k \cap E_{v,i}$ for some pair (v, i) in $\mathbb{P}$.

We first prove the statement (i). Let v be a vertex in P. If $\theta_v^{k+1} > \delta_v$, then

$$X^{k+1} \cap E_{v,i} = F^{k+1} \cap E_{v,i} \subseteq X^k \cap E_{v,i} \tag{2}$$

for every integer i in $\{1, 2, \ldots, \delta_v\}$. Next we consider the case where $\theta_v^{k+1} \leq \delta_v$. In this case, for every integer i in $\{1, 2, \ldots, \theta_v^{k+1} - 1\}$, (2) holds. Furthermore, Lemma 10 implies that

$$X^{k+1} \cap E_{v,i} = F^{k+1} \cap E_{v,i} \subseteq E_{v,i} = X^k \cap E_{v,i}$$

for every integer i in $\{\theta_v^{k+1}, \theta_v^{k+1} + 1, \ldots, \delta_v\}$. This completes the proof of the statement (i).

Next we prove the statement (ii). If there exist a vertex v in P and an integer i in $\{1, 2, \ldots, \theta_v^{k+1}\}$ such that $F^{k+1} \cap E_{v,i} \subsetneq X^k \cap E_{v,i}$, then the proof is done. Otherwise, since **Algorithm 1** does not halt in the $(k+1)$st iteration, there exists a vertex v in P such that $\theta_v^{k+1} \leq \delta_v$ and

$$|F^{k+1} \cap E_{v,\theta_v^{k+1}}| < \overline{c}^{k+1}(v, \theta_v^{k+1}) \leq |X^k \cap E_{v,\theta_v^{k+1}}|,$$

where the second inequality follows from

$$c(v) \leq |X^k \cap \Sigma_{v,\theta_v^{k+1}}| = |X^k \cap \Sigma_{v,\theta_v^{k+1}-1}| + |X^k \cap E_{v,\theta_v^{k+1}}|.$$

Thus, $X^{k+1} \cap E_{v,\theta_v^{k+1}} = F^{k+1} \cap E_{v,\theta_v^{k+1}} \subsetneq X^k \cap E_{v,\theta_v^{k+1}}$. This completes the proof. □

Lemma 12. *The number of iterations of* **Algorithm 1** *is at most* $|E| + 2$.

Proof. Since $X^1 \subseteq E$, this lemma immediately follows from Lemma 11. □

Lastly, we prove the correctness. Assume that **Algorithm 1** halts when $r = o$.

Lemma 13. *F^o is a matching in G.*

Proof. Since $F^o \in \mathcal{F}(\overline{c}^o)$, F^o is an independent set of $\mathbf{M}^{\oplus}$. This implies that $F^o(v)$ is an independent set of $\mathbf{M}_v$ for every vertex v in Q. What remains is to prove that $|F^o(v)| \leq c(v)$ for every vertex v in P. Let v be a vertex in P. Then, since $F^o \cap E_{v,i} \subseteq X^{o-1} \cap E_{v,i}$ for every integer i in $\{1, 2, \ldots, \theta_v^o - 1\}$ and $F^o \in \mathcal{F}(\overline{c}^o)$, we have

$$|F^o(v)| = |F^o \cap \Sigma_{v,\theta_v^o-1}| + |F^o \cap (E(v) \setminus \Sigma_{v,\theta_v^o-1})| \leq |X^{o-1} \cap \Sigma_{v,\theta_v^o-1}| + \overline{c}^o(v, \theta_v^o) = c(v).$$

This completes the proof. □

Lemma 14. *For every pair (v, i) in $\mathbb{P}$, if $|F^o \cap E_{v,i}| = \overline{c}^o(v, i)$, then $i \geq \theta_v^o$ and $|F^o(v)| = c(v)$.*

Proof. Since $\overline{c}^o(v, i) > |E|$ for every integer i in $\{1, 2, \ldots, \theta_v^o - 1\}$, $|X^{o-1}(v)| \geq c(v)$ and $i \geq \theta_v^o$. If $|F^o(v)| < c(v)$, then at least one of the following conditions (i) and (ii) holds. (i) $F^o(v) \cap E_{v,i} \subsetneq X^{o-1} \cap E_{v,i}$ for some integer i in $\{1, 2, \ldots, \theta_v^o - 1\}$. (ii) $|F^o \cap E_{v,\theta_v^o}| < \overline{c}^o(v, \theta_v^o)$. This contradicts the fact that **Algorithm 1** halts when $r = o$. This completes the proof. □

Lemma 15. *F^o is a stable matching in G.*

Proof. Let $e = [u, w]$ be an edge in $E \setminus F^o$. If $|F^o \cap E_{u,\mathsf{r}(e)}| = \overline{c}^o(u, \mathsf{r}(e))$, then Lemma 14 implies that $|F^o(u)| = c(u)$ and $f \succsim_u e$ for every edge f in $F^o(u)$. Assume that $|F^o \cap E_{u,\mathsf{r}(e)}| < \overline{c}^o(u, \mathsf{r}(e))$. If $e \notin \mathbf{sp}_{\mathbf{M}_w}(F^o(w))$, then $F^o \cup \{e\} \in \mathcal{I}^{\oplus}$. This contradicts the fact that $F^o \in \mathcal{F}^*(\overline{c}^o)$.

Assume that $e \in \mathbf{sp}_{\mathbf{M}_w}(F^o(w))$ and there exists an edge f in $\mathsf{C}_{\mathbf{M}_w}(e, F^o(w))$ such that $e \succ_w f$. Then, it is not difficult to see that $(F^o \cup \{e\}) \setminus \{f\} \in \mathcal{F}(\overline{c}^o)$. Furthermore, since $|E| + 1 \geq 2$,

$$\begin{aligned}
&\omega((F^o \cup \{e\}) \setminus \{f\}) - \omega(F^o) = \omega(e) - \omega(f) \\
&= -(|E|+1)^{\Delta_P - \mathsf{r}(e)} - (|E|+1)^{\Delta_P + \Delta_Q - \overline{\mathsf{r}}(e)} + (|E|+1)^{\Delta_P - \mathsf{r}(f)} + (|E|+1)^{\Delta_P + \Delta_Q - \overline{\mathsf{r}}(f)} \\
&< -(|E|+1)^{\Delta_P + \Delta_Q - \overline{\mathsf{r}}(f)+1} + (|E|+1)^{\Delta_P - 1} + (|E|+1)^{\Delta_P + \Delta_Q - \overline{\mathsf{r}}(f)} \\
&\leq -2 \cdot (|E|+1)^{\Delta_P + \Delta_Q - \overline{\mathsf{r}}(f)} + (|E|+1)^{\Delta_P - 1} + (|E|+1)^{\Delta_P + \Delta_Q - \overline{\mathsf{r}}(f)} \\
&\leq -(|E|+1)^{\Delta_P + \Delta_Q - \Delta_Q} + (|E|+1)^{\Delta_P - 1} = (|E|+1)^{\Delta_P - 1}(1 - (|E|+1)) < 0,
\end{aligned}$$

which contradicts the fact that $F^o \in \mathcal{F}^*(\overline{c}^o)$. This completes the proof. □

Lemma 16. *F^o is a Pareto efficient matching in G.*

Proof. Assume there exists a matching M in G such that $M \gg F^o$. Then, Lemma 4 implies that $\omega(M) < \omega(F^o)$. Since $F^o \in \mathcal{F}^*(\overline{c}^o)$, $M \notin \mathcal{F}(\overline{c}^o)$. Since M is a matching in G, $M(v) \in \mathcal{I}_v$ for every vertex v in Q. This implies that M is an independent set of $\mathbf{M}^{\oplus}$. Thus, there exists a pair (v, i) in $\mathbb{P}$ such that $|M \cap E_{v,i}| > \overline{c}^o(v, i)$. The definition of $\overline{c}^o$ implies that $|X^{o-1}(v)| \geq c(v)$ and $i \geq \theta_v^o$. Since **Algorithm 1** halts when $r = o$, we have $|F^o(v)| = c(v)$. If $i = \theta_v^o$, then

$$|F^o \cap \Sigma_{v,\theta_v^o-1}| \geq c(v) - \overline{c}^o(v, \theta_v^o) > |M(v)| - |M \cap E_{v,\theta_v^o}| \geq |M \cap \Sigma_{v,\theta_v^o-1}|,$$

where the first inequality follows from

$$|F^o \cap \Sigma_{v,\theta_v^o-1}| = |F^o(v)| - |F^o \cap E_{v,\theta_v^o}| - |F^o \cap (E(v) \setminus \Sigma_{v,\theta_v^o})| \geq c(v) - \overline{c}^o(v, \theta_v^o) - 0.$$

This contradicts the fact that M dominates F^o on v. Next we consider the case where $i > \theta_v^o$. In this case, since $|M \cap E_{v,i}| > \overline{c}^o(v, i)$, we have $|M \cap \Sigma_{v,\theta_v^o}| < c(v)$. Thus,

$$|F^o \cap \Sigma_{v,\theta_v^o}| = |F^o(v)| = c(v) > |M \cap \Sigma_{v,\theta_v^o}|.$$

This contradicts the fact that M dominates F^o on v. This completes the proof. □

Lemma 17. *The output of* **Algorithm 1** *is a Pareto stable matching in G.*

Proof. This lemma immediately follows from Lemmas 13, 15, and 16. □

We are now ready to prove the main result of this paper.

Proof of Theorem 2. This theorem immediately follows from Lemma 17. □

4 Proof of Lemma 3

For proving Lemma 3, we give an algorithm for constructing I_2 satisfying the conditions in Lemma 3 from I_1. Our algorithm is based on the algorithm proposed by Iri and Tomizawa [14] for the optimal independent assignment problem using potential functions. In what follows, we use well-known results (e.g., Lemmas 18 and 20). For completeness, we give their proofs (see Section 4.1) because our setting is slightly different from original settings. Without loss of generality, we assume that $|I_1 \cap S_i| < q_1(i)$ for some integer i in $\{1, 2, \ldots, d\}$.

For each element u in S, we define $\mathsf{i}(u)$ as the integer i in $\{1, 2, \ldots, d\}$ such that $u \in S_i$. Define a directed graph $\mathsf{D} = (\mathsf{V}, \mathsf{A})$ as follows. Define the vertex set V by

$$\mathsf{V}^{\mathsf{p}} := \{\mathsf{p}[i] \mid i \in \{1, 2, \ldots, d\}\}, \quad \mathsf{V}^{\mathsf{q}} := \{\mathsf{q}[u] \mid u \in S\}, \quad \mathsf{V}^{\mathsf{s}} := \{\mathsf{s}, \mathsf{t}\}, \quad \mathsf{V} := \mathsf{V}^{\mathsf{p}} \cup \mathsf{V}^{\mathsf{q}} \cup \mathsf{V}^{\mathsf{s}}.$$

The arc set A is the union of A^+, A^-, A^{p}, and A^{q} defined as follows.

$$\mathsf{A}^+ := \{\mathsf{a}[u] = (\mathsf{p}[\mathsf{i}(u)], \mathsf{q}[u]) \mid u \in S\}, \quad \mathsf{A}^- := \{\mathsf{b}[u] = (\mathsf{q}[u], \mathsf{p}[\mathsf{i}(u)]) \mid u \in S\}$$
$$\mathsf{A}^{\mathsf{p}} := \{(\mathsf{s}, \mathsf{p}[i]), (\mathsf{p}[i], \mathsf{t}) \mid i \in \{1, 2, \ldots, d\}\}$$
$$\mathsf{A}^{\mathsf{q}} := \{(\mathsf{q}[u], \mathsf{q}[v]) \mid u, v \in S, u \neq v\} \cup \{(\mathsf{s}, \mathsf{q}[u]), (\mathsf{q}[u], \mathsf{t}) \mid u \in S\},$$

where (v, w) represents an arc from a vertex v to a vertex w.

Define a function $\ell \colon \mathsf{A} \to \mathbb{R}$ as follows. For each arc $\mathsf{a}[u]$ in A^+, we define $\ell(\mathsf{a}[u]) := \xi(u)$. For each arc $\mathsf{b}[u]$ in A^-, we define $\ell(\mathsf{b}[u]) := -\xi(u)$. For each arc a in $\mathsf{A}^{\mathsf{p}} \cup \mathsf{A}^{\mathsf{q}}$, we define $\ell(a) := 0$.

For each integer t in $\{1, 2\}$ and each member I in $\mathcal{O}_t$, we define a subgraph $\mathsf{D}_t(I) = (\mathsf{V}, \mathsf{A}_t(I))$ of D as follows. The arc set $\mathsf{A}_t(I)$ is the union of $\mathsf{A}^+(I)$, $\mathsf{A}^-(I)$, $\mathsf{A}_t^{\mathsf{p}}(I)$, and $\mathsf{A}^{\mathsf{q}}(I)$ defined as follows.

$$\mathsf{A}^+(I) := \{\mathsf{a}[v] \mid v \in S \setminus I\}, \quad \mathsf{A}^-(I) := \{\mathsf{b}[u] \mid u \in I\}$$
$$\mathsf{A}_t^{\mathsf{p}}(I) := \{(\mathsf{s}, \mathsf{p}[i]) \mid i \in \{1, 2, \ldots, d\}, |I \cap S_i| < q_t(i)\}$$
$$\cup \{(\mathsf{p}[i], \mathsf{t}) \mid i \in \{1, 2, \ldots, d\}, |I \cap S_i| > 0\}$$
$$\mathsf{A}^{\mathsf{q}}(I) := \{(\mathsf{q}[v], \mathsf{q}[u]) \mid v \in \mathbf{sp}_{\mathbf{N}}(I), u \in \mathsf{C}_{\mathbf{N}}(v, I)\}$$
$$\cup \{(\mathsf{q}[v], \mathsf{t}) \mid v \in S \setminus I, I \cup \{v\} \in \mathcal{J}\} \cup \{(\mathsf{s}, \mathsf{q}[u]) \mid u \in I\}.$$

Assume that we are given an integer t in $\{1,2\}$ and a member I in $\mathcal{O}_t$. Furthermore, we assume that we are given a sequence $L = (a_1, a_2, \ldots, a_h)$ of arcs in $\mathsf{A}_t(I)$ and $a_p = (\mathsf{v}_p, \mathsf{w}_p)$ for each integer p in $\{1, 2, \ldots, h\}$. Then, L is called a *directed path in* $\mathsf{D}_t(I)$ *from* v *to* w, if $\mathsf{v} = \mathsf{v}_1$, $\mathsf{w} = \mathsf{w}_h$, and $\mathsf{w}_p = \mathsf{v}_{p+1}$ for every integer p in $\{1, 2, \ldots, h-1\}$. Furthermore, L is said to be *simple*, if L satisfies the condition that if there exists a pair of integers m, n in $\{1, 2, \ldots, h\}$ such that $m < n$ and $\mathsf{v}_m = \mathsf{w}_n$, then $m = 1$ and $n = h$. We call L a *directed cycle in* $\mathsf{D}_t(I)$, if L is a simple directed path in $\mathsf{D}_t(I)$ from v to w and $\mathsf{v} = \mathsf{w}$. For each arc a in $\mathsf{A}_t(I)$, if $a = a_p$ for some integer p in $\{1, 2, \ldots, h\}$, we write $a \in L$.

Assume that we are given an integer t in $\{1,2\}$, a member I in $\mathcal{O}_t$, and a function $\nu \colon \mathsf{A} \to \mathbb{R}$. For each directed path $L = (a_1, a_2, \ldots, a_h)$ in $\mathsf{D}_t(I)$, we define $\mathsf{len}(L; \nu) := \sum_{p=1}^{h} \nu(a_p)$ and $\mathsf{nb}(L) := h$. For each simple directed path $L = (a_1, a_2, \ldots, a_h)$ in $\mathsf{D}_t(I)$, we denote by $\mathbf{p}(L)$ (resp., $\mathbf{n}(L)$) the set of elements u in S such that $a_p = \mathsf{a}[u]$ (resp., $a_p = \mathsf{b}[u]$) for some integer p in $\{1, 2, \ldots, h\}$. If there exists a directed cycle in $\mathsf{D}_t(I)$, then we define $\mathcal{C}_t(I; \nu)$ as the set of directed cycles C in $\mathsf{D}_t(I)$ such that $\mathsf{len}(C; \nu) \leq \mathsf{len}(C'; \nu)$ for every directed cycle C' in $\mathsf{D}_t(I)$, and we define $\mathsf{mc}_t(I, \nu) := \mathsf{len}(C; \nu)$ for a directed cycle C in $\mathcal{C}_t(I; \nu)$. Furthermore, if there does not exist a directed cycle in $\mathsf{D}_t(I)$, we define $\mathsf{mc}_t(I, \nu) := \infty$.

Assume that we are given an integer t in $\{1,2\}$, a member I in $\mathcal{O}_t$, and a function $\nu \colon \mathsf{A} \to \mathbb{R}$ such that $\mathsf{mc}_t(I, \nu) \geq 0$. Furthermore, we are given a vertex v in V such that there exists a directed path in $\mathsf{D}_t(I)$ from v to t. Then, since $\mathsf{mc}_t(I, \nu) \geq 0$, it is not difficult to see that there exists a simple directed path L in $\mathsf{D}_t(I)$ from v to t such that $\mathsf{len}(L; \nu) \leq \mathsf{len}(L'; \nu)$ for every directed path L' in $\mathsf{D}_t(I)$ from v to t. Define $\mathcal{L}_t(\mathsf{v}; I, \nu)$ as the set of simple directed paths L in $\mathsf{D}_t(I)$ from v to t such that $\mathsf{len}(L; \nu) \leq \mathsf{len}(L'; \nu)$ for every directed path L' in $\mathsf{D}_t(I)$ from v to t. Furthermore, we define $\mathsf{mp}_t(\mathsf{v}; I, \nu) := \mathsf{len}(L; \nu)$ for a directed path L in $\mathcal{L}_t(\mathsf{v}; I, \nu)$. For each vertex w in V such that there does not exist a simple directed path L in $\mathsf{D}_t(I)$ from w to t, we define $\mathsf{mp}_t(\mathsf{w}; I, \nu) = \infty$.

Assume that we are given an integer t in $\{1,2\}$ and a member I in $\mathcal{O}_t$. A function $\pi \colon \mathsf{V} \to \mathbb{R}$ is called a *compatible potential function* for $\mathsf{D}_t(I)$, if $\ell(a) \geq \pi(\mathsf{v}) - \pi(\mathsf{w})$ holds for every arc $a = (\mathsf{v}, \mathsf{w})$ in $\mathsf{A}_t(I)$.

The following lemma is known [10, 25]. For completeness, we give a proof in Section 4.1.

Lemma 18. *Assume that we are given an integer t in $\{1,2\}$ and a member I in $\mathcal{O}_t$. Then, $I \in \mathcal{O}_t^*$ if and only if $\mathsf{mc}_t(I, \ell) \geq 0$ and $\mathsf{mp}_t(\mathsf{s}; I, \ell) \geq 0$.*

Since $I_1 \in \mathcal{O}_1^*$, Lemma 18 implies that $\mathsf{mc}_1(I_1, \ell) \geq 0$. Furthermore, it is not difficult to see that for every vertex v in V, there exists a directed path in $\mathsf{D}_1(I_1)$ from v to t (recall that we assume that there exists an integer i in $\{1, 2, \ldots, d\}$ such that $|I_1 \cap S_i| < q_1(i)$). This implies that we can define a function $\pi_1 \colon \mathsf{V} \to \mathbb{R}$ by $\pi_1(\mathsf{v}) := \mathsf{mp}_1(\mathsf{v}; I_1, \ell)$. Since $\mathsf{mp}_1(\mathsf{v}; I_1, \ell) \leq \ell(a) + \mathsf{mp}_1(\mathsf{w}; I_1, \ell)$ for every arc $a = (\mathsf{v}, \mathsf{w})$ in $\mathsf{A}_1(I_1)$, π_1 is a compatible potential function for $\mathsf{D}_1(I_1)$.

Lemma 19. *For every integer i in $\{1, 2, \ldots, d\}$ such that $|I_1 \cap S_i| < q_1(i)$, we have $\pi_1(\mathsf{p}[i]) \geq 0$.*

Proof. Let i be an integer in $\{1, 2, \ldots, d\}$ such that $|I_1 \cap S_i| < q_1(i)$. The definition of $\mathsf{A}_1^{\mathsf{p}}$ implies that $(\mathsf{s}, \mathsf{p}[i]) \in \mathsf{A}_1(I_1)$. Thus, if $\pi_1(\mathsf{p}[i]) < 0$, then since $\mathsf{mp}_1(\mathsf{s}; I_1, \ell) \leq \ell((\mathsf{s}, \mathsf{p}[i])) + \mathsf{mp}_1(\mathsf{p}[i]; I_1, \ell) = 0 + \pi_1(\mathsf{p}[i])$, Lemma 18 implies that $I_1 \notin \mathcal{O}_1^*$. This contradicts that fact that $I_1 \in \mathcal{O}_1^*$. □

Define $\pi^1 \colon \mathsf{V} \to \mathbb{R}$ as follows.

$$\pi^1(\mathsf{v}) := \begin{cases} \pi_1(\mathsf{v}) & \text{if } \mathsf{v} \neq \mathsf{s} \\ \min\{\pi_1(\mathsf{s}), \min\{\pi_1(\mathsf{p}[i]) \mid \mathsf{p}[i] \in \mathsf{V}^{\mathsf{p}}, |I_1 \cap S_i| = q_1(i) < q_2(i)\}\} & \text{if } \mathsf{v} = \mathsf{s}. \end{cases}$$

Since $q_1(i) \leq q_2(i)$ for every integer i in $\{1, 2, \ldots, d\}$, $I_1 \in \mathcal{O}_2$. Since $\mathsf{A}_2(I_1) \setminus \mathsf{A}_1(I_1) \subseteq \mathsf{A}_2^{\mathsf{p}}$, π^1 is a compatible potential function for $\mathsf{D}_2(I_1)$.

We can construct a desired member in $\mathcal{O}_2^*$ by using the following **Algorithm 2**.

Algorithm 2

Step 1. Define $I^1 := I_1$. Set $r := 1$.

Step 2. Define $\ell^r \colon \mathsf{A} \to \mathbb{R}$ by $\ell^r(a) := \ell(a) - \pi^r(\mathsf{v}) + \pi^r(\mathsf{w})$ for each arc $a = (\mathsf{v}, \mathsf{w})$ in A.

Step 3. If $\mathsf{mp}_2(\mathsf{s}; I^r, \ell^r) = \infty$, then ouput I^r and halt. Otherwise, compute a directed path L^r in $\mathcal{L}_2(\mathsf{s}; I^r, \ell^r)$ such that $\mathsf{nb}(L^r) \le \mathsf{nb}(L)$ for every directed path L in $\mathcal{L}_2(\mathsf{s}; I^r, \ell^r)$.

Step 4. If $\mathsf{len}(L^r; \ell) \ge 0$, then ouput I^r and halt.

Step 5. Define $I^{r+1} := (I^r \cup \mathbf{p}(L^r)) \setminus \mathbf{n}(L^r)$.

Step 6. Define a function $\pi^{r+1} \colon \mathsf{V} \to \mathbb{R}$ by $\pi^{r+1}(\mathsf{v}) := \pi^r(\mathsf{v}) + \mathsf{mp}_2(\mathsf{v}; I^r, \ell^r)$.

Step 7. Update $r := r + 1$, and go back to **Step 2**.

End of Algorithm

Since π^1 is a compatible potential function for $\mathsf{D}_2(I^1)$, $\ell^1(a) \ge 0$ holds for every arc a in $\mathsf{A}_2(I^1)$. Thus, we have $\mathsf{mc}_2(I^1, \ell^1) \ge 0$. This implies that the first iteration of **Algorithm 2** is well-defined. Furthermore, since $I_1 \in \mathcal{O}_1^*$, Lemma 18 implies that $\pi_1(\mathsf{q}[u]) \ge 0$ holds for every element u in I_1. Assume that the kth iteration of **Algorithm 2** is well-defined for some positive integer k. That is, we assume that $I^k \in \mathcal{O}_2$ and π^k is a compatible potential function for $\mathsf{D}_2(I^k)$. Furthermore, we assume that $\pi^k(\mathsf{q}[u]) \ge 0$ holds for every element u in I^k. Then, we prove that if **Algorithm 2** does not halt in the kth iteration, then the $(k+1)$st iteration of **Algorithm 2** is well-defined. Notice that since $\mathsf{mp}_2(\mathsf{s}; I^k, \ell^k) \neq \infty$ holds, there exists a directed path in $\mathsf{D}_2(I^k)$ from v to t for every vertex v in V. The following lemma implies that the $(k+1)$st iteration of **Algorithm 2** is well-defined. This lemma is well known (see, e.g., [14, Theorem 2]). For completeness, we give a proof in Section 4.1.

Lemma 20. *$I^{k+1} \in \mathcal{O}_2$ and π^{k+1} is a compatible potential function for $\mathsf{D}_2(I^{k+1})$. Furthermore, $\pi^{k+1}(\mathsf{q}[u]) \ge 0$ for every element u in I^{k+1}.*

Next we prove that the number of iteration of **Algorithm 2** is finite.

Lemma 21. *The number of iterations of* **Algorithm 2** *is finite.*

Proof. Assume that for some positive integer k, **Algorithm 2** does not halt in the kth iteration. Then, we have $\mathsf{len}(L^k; \ell) < 0$. Thus, since $\mathsf{c}(I^k) - \mathsf{c}(I^{k+1}) = -\mathsf{len}(L^k; \ell)$ and $\xi(u)$ is an integer for every element u in S, $\mathsf{c}(I^k) - \mathsf{c}(I^{k+1}) \ge 1$. Define $\Delta := \min\{\xi(u) \mid u \in S\}$. For every member I in $\mathcal{O}_2$, $\mathsf{c}(I) \ge |S|\Delta$. Thus, this lemma follows from the fact that $\mathsf{c}(I^k) - \mathsf{c}(I^{k+1}) \ge 1$. □

Assume that **Algorithm 2** halts when $r = o$.

Lemma 22. *$I^o \in \mathcal{O}_2^*$.*

Proof. For every directed cycle C in $\mathsf{D}_2(I^o)$, we have $\mathsf{len}(C; \ell) = \mathsf{len}(C; \ell^o)$. Since π^o is a compatible potential function for $\mathsf{D}_2(I^o)$, we have $\ell^o(a) \ge 0$ for every arc a in $\mathsf{A}_2(I^o)$. Thus, $\mathsf{mc}_2(I^o, \ell) \ge 0$. If $\mathsf{mp}_2(\mathsf{s}; I^o, \ell^o) = \infty$, then $\mathsf{mp}_2(\mathsf{s}; I^o, \ell) = \infty$ clearly holds. Otherwise, for every directed path L from s to t in $\mathsf{D}_2(I^o)$, we have $\mathsf{len}(L; \ell) = \mathsf{len}(L; \ell^o) + \pi^o(\mathsf{s}) - \pi^o(\mathsf{t})$. This implies that $\mathcal{L}_2(\mathsf{s}; I^o, \ell) = \mathcal{L}_2(\mathsf{s}; I^o, \ell^o)$. Furthermore, since **Algorithm 2** halts when $r = o$, $\mathsf{len}(L^o; \ell) \ge 0$. This implies that $\mathsf{mp}_2(\mathsf{s}; I^o, \ell) \ge 0$. This lemma follows from this observation and Lemma 18. □

Lemma 23. *For every integer i in $\{1, 2, \ldots, d\}$ such that $|I_1 \cap S_i| < q_1(i)$, $I^o \cap S_i \subseteq I_1 \cap S_i$*

Proof. For proving this lemma, it suffices to prove that for every integer r in $\{1, 2, \ldots, o-1\}$ and every integer i in $\{1, 2, \ldots, d\}$ such that $|I_1 \cap S_i| < q_1(i)$, we have $(S_i \setminus I_1) \cap \mathbf{p}(L^r) = \emptyset$.

For every integer r in $\{1, 2, \ldots, o\}$, since π^r is a compatible potential function for $\mathsf{D}_2(I^r)$, we have $\ell^r(a) \ge 0$ for every arc a in $\mathsf{A}_2(I^r)$. This implies that $\mathsf{mp}_2(\mathsf{v}; I^r, \ell^r) \ge 0$ for every vertex v in V. Thus, the definition of **Algorithm 2** implies that $\pi^{r+1}(\mathsf{v}) \ge \pi^r(\mathsf{v})$ holds for every integer r in

$\{1, 2, \ldots, o-1\}$ and every vertex v in V. This and Lemma 19 imply that $\pi^r(\mathsf{p}[i]) \geq 0$ for every integer r in $\{1, 2, \ldots, o\}$ and every integer i in $\{1, 2, \ldots, d\}$ such that $|I_1 \cap S_i| < q_1(i)$.

Assume that there exist integers i in $\{1, 2, \ldots, d\}$ and r in $\{1, 2, \ldots, o-1\}$ such that $(S_i \setminus I_1) \cap \mathbf{p}(L^r) \neq \emptyset$. Assume that $L^r = (a_1, a_2, \ldots, a_h)$, and $a_p = (\mathsf{v}_p, \mathsf{w}_p)$ for each integer p in $\{1, 2, \ldots, h\}$. Furthermore, we assume that $v \in (S_i \setminus I_1) \cap \mathbf{p}(L^r)$ and $a_z = \mathsf{a}[v]$. Since $r < o$ holds, $\mathsf{len}(L^r; \ell) < 0$. Define a directed path $L^\bullet$ in $\mathsf{D}_2(I^r)$ by $L^\bullet := (a_z, a_{z+1}, \ldots, a_h)$. Since π^r is a compatible potential function for $\mathsf{D}_2(I^r)$, $\ell(a_p) \geq \pi^r(\mathsf{v}_p) - \pi^r(\mathsf{w}_p)$ for every integer p in $\{z, z+1, \ldots, h\}$. Since $\pi^r(\mathsf{t}) = 0$ clearly holds and $\mathsf{w}_h = \mathsf{t}$,

$$\mathsf{len}(L^\bullet; \ell) := \ell(a_z) + \ell(a_{z+1}) + \cdots + \ell(a_h) \geq \pi^r(\mathsf{v}_z) = \pi^r(\mathsf{p}[i]) \geq 0.$$

This implies that $z > 1$. Since $\mathsf{w}_{z-1} = \mathsf{v}_z = \mathsf{p}[i]$, $|S_i \cap I^r| > 0$. This implies that there exists an arc $(\mathsf{p}[i], \mathsf{t})$ in $\mathsf{A}_2(I^r)$. Define a directed path L° in $\mathsf{D}_2(I^r)$ by $L^\circ := (a_1, a_2, \ldots, a_{z-1}, (\mathsf{p}[i], \mathsf{t}))$. Then, $\mathsf{nb}(L^\circ) < \mathsf{nb}(L^r)$ and $\mathsf{len}(L^\circ; \ell) \leq \mathsf{len}(L^r; \ell)$. This contradicts the definition of L^r. □

We are now ready to prove the main result of this section.

Proof of Lemma 3. This lemma immediately follows from Lemmas 22 and 23. □

4.1 Omitted proofs

In this subsection, we give omitted proofs. Again, we emphasize that the following proofs are not new results, and we give proofs because our setting is slightly different from the original settings and we have to modify original proofs so that they fit our setting.

Theorem 24 (Krogdahl [24, 25, 26] (see also [31, Theorem 39.13]))**.** *Assume that we are given a matroid $\mathbf{M} = (U, \mathcal{I})$, an independent set I of $\mathbf{M}$, and a subset J of U such that $|I| = |J|$. Then, if there exists a unique bijective mapping $\mathsf{f} \colon I \setminus J \to J \setminus I$ such that $(I \cup \{\mathsf{f}(u)\}) \setminus \{u\} \in \mathcal{I}$ for every element u in $I \setminus J$, then J is an independent set of $\mathbf{M}$.*

Theorem 25 (Brualdi [2] (see also [31, Corollary 39.12a]))**.** *Assume that we are given a matroid $\mathbf{M} = (U, \mathcal{I})$ and independent sets I, J of $\mathbf{M}$ such that $|I| = |J|$. Then, there exists a bijective mapping $\mathsf{f} \colon I \setminus J \to J \setminus I$ such that $(I \cup \{\mathsf{f}(u)\}) \setminus \{u\} \in \mathcal{I}$ for every element u in $I \setminus J$.*

Theorem 26 (See, e.g., [14, Lemma 2])**.** *Assume that we are given a matroid $\mathbf{M} = (U, \mathcal{I})$ and an independent set I of $\mathbf{M}$. Furthermore, we assume that we are given distinct elements $u_1, u_2, \ldots, u_h$ in I and $v_1, v_2, \ldots, v_h$ in $U \setminus I$ satisfying the following conditions.*

- *$v_i \in \mathbf{sp}_{\mathbf{M}}(I)$ for every integer i in $\{1, 2, \ldots, h\}$.*
- *$u_i \in \mathsf{C}_{\mathbf{M}}(v_i, I)$ for every integer i in $\{1, 2, \ldots, h\}$.*
- *$u_i \notin \mathsf{C}_{\mathbf{M}}(v_j, I)$ for any pair of integers i, j in $\{1, 2, \ldots, h\}$ such that $i < j$.*

Define $J := (I \cup \{v_1, v_2, \ldots, v_h\}) \setminus \{u_1, u_2, \ldots, u_h\}$. Then, $J \in \mathcal{I}$ and $\mathbf{cl}_{\mathbf{M}}(I) = \mathbf{cl}_{\mathbf{M}}(J)$.

For every matroid $\mathbf{M} = (U, \mathcal{I})$, every independent set I of $\mathbf{M}$, and every pair of elements u in I and v in $\mathbf{sp}_{\mathbf{M}}(I)$ such that $u \in \mathsf{C}_{\mathbf{M}}(v, I)$, Theorem 26 implies that $\mathbf{sp}_{\mathbf{M}}(I) \setminus \{v\} = \mathbf{sp}_{\mathbf{M}}(J) \setminus \{u\}$.

Theorem 27 (See, e.g., [14, Lemma 3])**.** *Assume that we are given a matroid $\mathbf{M} = (U, \mathcal{I})$, an independent set I of $\mathbf{M}$, and elements u in I and v in $\mathbf{sp}_{\mathbf{M}}(I)$ such that $u \in \mathsf{C}_{\mathbf{M}}(v, I)$. Furthermore, we are given elements u' in $I \setminus \{u\}$ and v' in $\mathbf{sp}_{\mathbf{M}}(I) \setminus \{v\}$. Define $J := (I \cup \{v\}) \setminus \{u\}$. Then, the following statements hold.*

- *$u \in \mathbf{sp}_{\mathbf{M}}(J)$ and $v \in \mathsf{C}_{\mathbf{M}}(u, J)$.*
- *If $v \in \mathsf{C}_{\mathbf{M}}(v', J)$, then $u \in \mathsf{C}_{\mathbf{M}}(v', I)$.*
- *If $u' \in \mathsf{C}_{\mathbf{M}}(u, J)$, then $u' \in \mathsf{C}_{\mathbf{M}}(v, I)$.*
- *If $u' \in \mathsf{C}_{\mathbf{M}}(v', J)$ and $u' \notin \mathsf{C}_{\mathbf{M}}(v', I)$, then $u' \in \mathsf{C}_{\mathbf{M}}(v, I)$ and $u \in \mathsf{C}_{\mathbf{M}}(v', I)$.*

4.1.1 Proof of Lemma 18

The following proof is based on the proof of [31, Theorem 41.5].

Lemma 28. *Assume that we are given an integer t in $\{1,2\}$, a member I in $\mathcal{O}_t$, and $\mathsf{mc}_t(I,\ell) < 0$. Let C be a directed cycle in $\mathsf{D}_t(I)$ satisfying the condition that $\mathsf{len}(C;\ell) < 0$ and $\mathsf{nb}(C) \leq \mathsf{nb}(C')$ for every directed cycle C' in $\mathsf{D}_t(I)$ such that $\mathsf{len}(C';\ell) < 0$. Then, $(I \cup \mathbf{p}(C)) \setminus \mathbf{n}(C) \in \mathcal{O}_t$.*

Proof. Define $J := (I \cup \mathbf{p}(C)) \setminus \mathbf{n}(C)$. Then, $|I| = |J|$. For proving this lemma by contradiction, we assume that $J \notin \mathcal{O}_t$. Since $I \in \mathcal{O}_t$, $|J \cap S_i| \leq q_t(i)$ for every integer i in $\{1,2,\ldots,d\}$. Thus, $J \notin \mathcal{J}$ holds. Assume that $C = (a_1, a_2, \ldots, a_{3h})$, $a_p \in \mathsf{A}^+(I)$ for each integer p in $\{1,4,\ldots,3h-2\}$, and $a_p \in \mathsf{A}^-(I)$ for each integer p in $\{3,6,\ldots,3h\}$. Assume that $a_{3p-2} = \mathsf{a}[v_p]$ and $a_{3p} = \mathsf{b}[u_p]$ for each integer p in $\{1,2,\ldots,h\}$. Then, $I \setminus J = \{u_1, u_2, \ldots, u_h\}$ and $J \setminus I = \{v_1, v_2, \ldots, v_h\}$. For every integer p in $\{1,2,\ldots,h\}$, since $(\mathsf{q}[v_p], \mathsf{q}[u_p]) \in \mathsf{A}_t(I)$, $v_p \in \mathbf{sp}_{\mathbf{N}}(I)$. Define $\mathsf{f}\colon I \setminus J \to J \setminus I$ by $\mathsf{f}(u_p) := v_p$ for each integer p in $\{1,2,\ldots,h\}$. Then, for every integer p in $\{1,2,\ldots,h\}$, since $(\mathsf{q}[v_p], \mathsf{q}[u_p]) \in \mathsf{A}_t(I)$, we have $(I \cup \{\mathsf{f}(u_p)\}) \setminus \{u_p\} \in \mathcal{J}$. Thus, since $J \notin \mathcal{J}$, Theorem 24 implies that there exists a bijective mapping $\mathsf{g}\colon I \setminus J \to J \setminus I$ such that $(I \cup \{\mathsf{g}(u)\}) \setminus \{u\} \in \mathcal{J}$ for every element u in $I \setminus J$ and $\mathsf{g} \neq \mathsf{f}$. Notice that $u \in \mathsf{C}_{\mathbf{N}}(\mathsf{g}(u), I)$ for every element u in $I \setminus J$. This implies that $(\mathsf{q}[\mathsf{g}(u)], \mathsf{q}[u]) \in \mathsf{A}_t(I)$ for every element u in $I \setminus J$. Define

$$\mathsf{A}^{\circ} := \{a_p \mid p \in \{1,3,4,6,\ldots,3h-2,3h\}\}$$
$$\mathsf{A}^{\bullet} := \{a_p \mid p \in \{2,5,\ldots,3h-1\}\} \cup \{(\mathsf{q}[\mathsf{g}(u)], \mathsf{q}[u]) \mid u \in I \setminus J\}.$$

Since $\mathsf{g} \neq \mathsf{f}$, there exist distinct integers p, p' in $\{1,2,\ldots,h\}$ such that $\mathsf{g}(u_p) = v_{p'}$. Define a directed cycle C_1 in $\mathsf{D}_t(I)$ by

$$C_1 := \begin{cases} (a_{3p}, a_{3p+1}, \ldots, a_{3p'-2}, (\mathsf{q}[v_{p'}], \mathsf{q}[u_p])) & \text{if } p < p' \\ (a_{3p}, a_{3p+1}, \ldots, a_{3h}, a_1, a_2, \ldots, a_{3p'-2}, (\mathsf{q}[v_{p'}], \mathsf{q}[u_p])) & \text{if } p > p'. \end{cases}$$

Notice that $\mathsf{nb}(C_1) < \mathsf{nb}(C)$. It is not difficult to see that there exist directed cycles $C_1, C_2, \ldots, C_n$ in $\mathsf{D}_t(I)$ satisfying the following conditions.

- For every arc a in $\mathsf{A}_t(I)$, if $a \in C_x$ for some integer x in $\{1,2,\ldots,n\}$, then $a \in \mathsf{A}^{\circ} \cup \mathsf{A}^{\bullet}$. Thus, for every integer x in $\{1,2,\ldots,n\}$, $\mathsf{nb}(C_x) \leq \mathsf{nb}(C)$.
- For every arc a in A°, there exist exactly two distinct integers x, y in $\{1,2,\ldots,n\}$ such that $a \in C_x$ and $a \in C_y$.
- For every arc a in $\mathsf{A}^{\bullet}$, there exists exactly one integer x in $\{1,2,\ldots,n\}$ such that $a \in C_x$.
- There exists at most one integer x in $\{1,2,\ldots,n\}$ such that $\mathsf{nb}(C_x) = \mathsf{nb}(C)$. Furthermore, for such an integer x, we have $\mathsf{len}(C_x;\ell) = \mathsf{len}(C;\ell)$.

Since $\mathsf{len}(C;\ell) < 0$, $\sum\{\mathsf{len}(C_x;\ell) \mid x \in \{1,2,\ldots,n\}, C_x \neq C\} < 0$. Thus, there exists an integer x in $\{1,2,\ldots n\}$ such that $\mathsf{len}(C_x;\ell) < 0$ and $\mathsf{nb}(C_x) < \mathsf{nb}(C)$. This contradicts the definition of C. This completes the proof. □

Lemma 29. *Assume that we are given an integer t in $\{1,2\}$, a member I in $\mathcal{O}_t$, and a function $\nu\colon \mathsf{A} \to \mathbb{R}$ such that $\mathsf{mc}_t(I,\nu) \geq 0$. Let L be a directed path in $\mathcal{L}_t(\mathsf{s}; I, \nu)$ such that $\mathsf{nb}(L) \leq \mathsf{nb}(L')$ for every directed path L' in $\mathcal{L}_t(\mathsf{s}; I, \nu)$. Then, $(I \cup \mathbf{p}(L)) \setminus \mathbf{n}(L) \in \mathcal{O}_t$.*

Proof. Define $J := (I \cup \mathbf{p}(L)) \setminus \mathbf{n}(L)$. Assume that $J \notin \mathcal{O}_t$. Since $I \in \mathcal{O}_t$, $|J \cap S_i| \leq q_t(i)$ for every integer i in $\{1,2,\ldots,d\}$. Thus, $J \notin \mathcal{J}$. We divide the proof into the following two cases.

Case 1. $L = (a_1, a_2, \ldots, a_h)$ and $a_1 \in \mathsf{A}^{\mathsf{p}}$.

Case 2. $L = (a_0, a_1, \ldots, a_h)$ and $a_0 \in \mathsf{A}^{\mathsf{q}}$.

Let z be a positive integer such that

$$h = \begin{cases} 3z+2 & \text{if } a_h \in \mathsf{A}^{\mathsf{p}} \\ 3z+3 & \text{if } a_h \in \mathsf{A}^{\mathsf{q}}. \end{cases}$$

Assume that $a_{3p-1} = \mathsf{a}[v_p]$ and $a_{3p+1} = \mathsf{b}[u_p]$ for each positive integer p in $\{1, 2, \ldots, z\}$. Then, we have $I \setminus J = \{u_1, u_2, \ldots, u_z\}$ and $J \setminus I = \{v_1, v_2, \ldots, v_z\}$.

Case 1. We first consider the case where $a_h \in \mathsf{A}^{\mathsf{p}}$. In this case, $|I| = |J|$. For every integer p in $\{1, 2, \ldots, z\}$, since $(\mathsf{q}[v_p], \mathsf{q}[u_p]) \in \mathsf{A}_t(I)$, $v_p \in \mathbf{sp}_{\mathbf{N}}(I)$. Define $\sigma \colon I \setminus J \to J \setminus I$ by $\mathsf{f}(u_p) := v_p$ for each integer p in $\{1, 2, \ldots, z\}$. For every integer p in $\{1, 2, \ldots, z\}$, since $(\mathsf{q}[v_p], \mathsf{q}[u_p]) \in \mathsf{A}_t(I)$ holds, $(I \cup \{\mathsf{f}(u_p)\}) \setminus \{u_p\} \in \mathcal{J}$. Thus, since $J \notin \mathcal{J}$ holds, Theorem 24 implies that there exists a bijective mapping $\mathsf{g} \colon I \setminus J \to J \setminus I$ such that $(I \cup \{\mathsf{g}(u)\}) \setminus \{u\} \in \mathcal{J}$ for every element u in $I \setminus J$ and $\mathsf{g} \neq \mathsf{f}$. Notice that $u \in \mathsf{C}_{\mathbf{N}}(\mathsf{g}(u), I)$ for every element u in $I \setminus J$. This implies that $(\mathsf{q}[\mathsf{g}(u)], \mathsf{q}[u]) \in \mathsf{A}_t(I)$ for every element u in $I \setminus J$. Define

$$\mathsf{A}^{\circ} := \{a_p \mid p \in \{1, 3z+2\} \cup \{2, 4, \ldots, 3z-1, 3z+1\}\}$$
$$\mathsf{A}^{\bullet} := \{a_p \mid p \in \{3, 6, \ldots, 3z\}\} \cup \{(\mathsf{q}[\mathsf{g}(u)], \mathsf{q}[u]) \mid u \in I \setminus J\}.$$

Since $\mathsf{g} \neq \mathsf{f}$, there exist distinct integers p, p' in $\{1, 2, \ldots, z\}$ such that $\mathsf{g}(u_p) = v_{p'}$. Thus, in the similar way as in the proof of Lemma 28, (by regrading s and t as the same vertex) we can see that there exist simple directed paths L_1, L_2 in $\mathsf{D}_t(I)$ from s to t and directed cycles $L_3, L_4, \ldots, L_n$ in $\mathsf{D}_t(I)$ satisfying the following conditions.

- For every arc a in $\mathsf{A}_t(I)$, if $a \in L_x$ for some integer x in $\{1, 2, \ldots, n\}$, then $a \in \mathsf{A}^{\circ} \cup \mathsf{A}^{\bullet}$. Thus, $\mathsf{nb}(L_1) \leq \mathsf{nb}(L)$ and $\mathsf{nb}(L_2) \leq \mathsf{nb}(L)$.
- For every arc a in A°, there exist exactly two distinct integers x, y in $\{1, 2, \ldots, n\}$ such that $a \in L_x$ and $a \in L_y$.
- For every arc a in $\mathsf{A}^{\bullet}$, there exists exactly one integer x in $\{1, 2, \ldots, n\}$ such that $a \in L_x$.
- There exists at most one integer x in $\{1, 2\}$ such that $\mathsf{nb}(L_x) = \mathsf{nb}(L)$.

Since $\mathsf{mc}_t(I, \nu) \geq 0$, $\mathsf{len}(L_1; \ell) + \mathsf{len}(L_2; \ell) \leq 2 \cdot \mathsf{len}(L; \ell)$. This implies that there exists an integer x in $\{1, 2\}$ such that $L_x \in \mathcal{L}_t(\mathsf{s}; I, \nu)$ and $\mathsf{nb}(L_x) < \mathsf{nb}(L)$. This contradicts the definition of L.

Next we consider the case where $a_h \in \mathsf{A}^{\mathsf{q}}$. Define $I' := I \cup \{v_{z+1}\}$. Then, $a_h \in \mathsf{A}^{\mathsf{q}}$ implies that $I' \in \mathcal{J}$, and Theorem 1 implies that $v_p \in \mathbf{sp}_{\mathbf{N}}(I')$ and $\mathsf{C}_{\mathbf{N}}(v_p, I) = \mathsf{C}_{\mathbf{N}}(v_p, I')$ for every integer p in $\{1, 2, \ldots, z\}$. Define $\mathsf{f} \colon I' \setminus J \to J \setminus I'$ by $\sigma(u_p) := v_p$ for each integer p in $\{1, 2, \ldots, z\}$. Then, for every integer p in $\{1, 2, \ldots, z\}$, since $(\mathsf{q}[v_p], \mathsf{q}[u_p]) \in \mathsf{A}_t(I)$ and $\mathsf{C}_{\mathbf{N}}(v_p, I) = \mathsf{C}_{\mathbf{N}}(v_p, I')$, we have $(I' \cup \{\mathsf{f}(u_p)\}) \setminus \{u_p\} \in \mathcal{J}$. Thus, since $J \notin \mathcal{J}$ holds, Theorem 24 implies that there exists a bijective mapping $\mathsf{g} \colon I' \setminus J \to J \setminus I'$ such that $(I' \cup \{\mathsf{g}(u)\}) \setminus \{u\} \in \mathcal{J}$ holds for every element u in $I' \setminus J$ and $\mathsf{g} \neq \mathsf{f}$. Notice that $u \in \mathsf{C}_{\mathbf{N}}(\mathsf{g}(u), I') = \mathsf{C}_{\mathbf{N}}(\mathsf{g}(u), I)$ for every element u in $I' \setminus J$. This implies that $(\mathsf{q}[\mathsf{g}(u)], \mathsf{q}[u]) \in \mathsf{A}_t(I)$ for every element u in $I' \setminus J$. Define

$$\mathsf{A}^{\circ} := \{a_p \mid p \in \{1, 3z+2, 3z+3\} \cup \{2, 4, \ldots, 3z-1, 3z+1\}\}$$
$$\mathsf{A}^{\bullet} := \{a_p \mid p \in \{3, 6, \ldots, 3z\}\} \cup \{(\mathsf{q}[\mathsf{g}(u)], \mathsf{q}[u]) \mid u \in I' \setminus J\}.$$

The rest of the proof is the same as the previous case.

Case 2. Assume that $a_1 = \mathsf{b}[u_0]$. Define $J' := J \cup \{u_0\}$. Since $J \notin \mathcal{J}$, $J' \notin \mathcal{J}$.

We first assume that $a_h \in \mathsf{A}^{\mathsf{p}}$. In this case, $|I| = |J'|$. For every integer p in $\{1, 2, \ldots, z\}$, since $(\mathsf{q}[v_p], \mathsf{q}[u_p]) \in \mathsf{A}_t(I)$, $v_p \in \mathbf{sp}_{\mathbf{N}}(I)$. Define $\mathsf{f} \colon I \setminus J' \to J' \setminus I$ by $\mathsf{f}(u_p) := v_p$ for each integer p in $\{1, 2, \ldots, z\}$. Then, for every integer p in $\{1, 2, \ldots, z\}$, since $(\mathsf{q}[v_p], \mathsf{q}[u_p]) \in \mathsf{A}_t(I)$ holds, we have $(I \cup \{\mathsf{f}(u_p)\}) \setminus \{u_p\} \in \mathcal{J}$. Since $J' \notin \mathcal{J}$, Theorem 24 implies that there exists a bijective mapping $\mathsf{g} \colon I \setminus J' \to J' \setminus I$ such that $(I \cup \{\mathsf{g}(u)\}) \setminus \{u\} \in \mathcal{J}$ for every element u in $I \setminus J'$ and $\mathsf{g} \neq \mathsf{f}$. Notice that $u \in \mathsf{C}_{\mathbf{N}}(\mathsf{g}(u), I)$ for every element u in $I \setminus J'$. This implies that $(\mathsf{q}[\mathsf{g}(u)], \mathsf{q}[u]) \in \mathsf{A}_t(I)$ for every element u in $I \setminus J'$. Define

$$\mathsf{A}^{\circ} := \{a_p \mid p \in \{0, 1, 3z+2\} \cup \{2, 4, \ldots, 3z-1, 3z+1\}\}$$
$$\mathsf{A}^{\bullet} := \{a_p \mid p \in \{3, 6, \ldots, 3z\}\} \cup \{(\mathsf{q}[\mathsf{g}(u)], \mathsf{q}[u]) \mid u \in I \setminus J'\}.$$

The rest of the proof is the same as **Case 1**.

Next we assume that $a_h \in \mathsf{A}^{\mathsf{q}}$. Assume that $a_{3z+2} = \mathsf{a}[v_{z+1}]$. Define $I' := I \cup \{v_{z+1}\}$. Then, $a_h \in \mathsf{A}^{\mathsf{q}}$ implies that $I' \in \mathcal{J}$, and Theorem 1 implies that $v_p \in \mathbf{sp}_{\mathbf{N}}(I')$ and $\mathsf{C}_{\mathbf{N}}(v_p, I) = \mathsf{C}_{\mathbf{N}}(v_p, I')$ for every integer p in $\{1, 2, \ldots, z\}$. Define $\mathsf{f} \colon I' \setminus J' \to J' \setminus I'$ by $\mathsf{f}(u_p) := v_p$ for each integer p in $\{1, 2, \ldots, z\}$. Then, for every integer p in $\{1, 2, \ldots, z\}$, since $(\mathsf{q}[v_p], \mathsf{q}[u_p]) \in \mathsf{A}_t(I)$ and $\mathsf{C}_{\mathbf{N}}(v_p, I') = \mathsf{C}_{\mathbf{N}}(v_p, I)$, we have $(I' \cup \{\mathsf{f}(u_p)\}) \setminus \{u_p\} \in \mathcal{J}$. Since $J' \notin \mathcal{J}$ holds, Theorem 24 implies that there exists a bijective mapping $\mathsf{g} \colon I' \setminus J' \to J' \setminus I'$ such that $(I' \cup \{\mathsf{g}(u)\}) \setminus \{u\} \in \mathcal{J}$ for every element u in $I' \setminus J'$ and $\mathsf{g} \neq \mathsf{f}$. Notice that $u \in \mathsf{C}_{\mathbf{N}}(\mathsf{g}(u), I) = \mathsf{C}_{\mathbf{N}}(\mathsf{g}(u), I')$ holds for every element u in $I' \setminus J'$. This implies that $(\mathsf{q}[\mathsf{g}(u)], \mathsf{q}[u]) \in \mathsf{A}_t(I)$ for every element u in $I' \setminus J'$. Define

$$\mathsf{A}^{\circ} := \{a_p \mid p \in \{0, 1, 3z+2, 3z+3\} \cup \{2, 4, \ldots, 3z-1, 3z+1\}\}$$
$$\mathsf{A}^{\bullet} := \{a_p \mid p \in \{3, 6, \ldots, 3z\}\} \cup \{(\mathsf{q}[\mathsf{g}(u)], \mathsf{q}[u]) \mid u \in I' \setminus J'\}.$$

The rest of the proof is the same as **Case 1**. □

Proof of Lemma 18. We first prove the "only if" part. Assume that $I \in \mathcal{O}_t^*$ and $\mathsf{mc}_t(I, \ell) < 0$. Let C be a directed cycle in $\mathsf{D}_t(I)$ satisfying the condition that $\mathsf{len}(C; \ell) < 0$ and $\mathsf{nb}(C) \leq \mathsf{nb}(C')$ for every directed cycle C' in $\mathsf{D}_t(I)$ such that $\mathsf{len}(C'; \ell) < 0$. Define $J := (I \cup \mathbf{p}(C)) \setminus \mathbf{n}(C)$. Lemma 28 implies that $J \in \mathcal{O}_t$ and $\mathsf{c}(J) = \mathsf{c}(I) + \mathsf{len}(C; \ell) < \mathsf{c}(I)$, which contradicts the fact that $I \in \mathcal{O}_t^*$. Next we assume that $\mathsf{mc}_t(I, \ell) \geq 0$ and $\mathsf{mp}_t(\mathsf{s}; I, \ell) < 0$. Let L be a member in $\mathcal{L}_t(\mathsf{s}; I, \ell)$ such that $\mathsf{nb}(L) \leq \mathsf{nb}(L')$ for every directed path L' in $\mathcal{L}_t(\mathsf{s}; I, \ell)$. Define $J := (I \cup \mathbf{p}(L)) \setminus \mathbf{n}(L)$. Lemma 29 implies that $J \in \mathcal{O}_t$ and $\mathsf{c}(J) = \mathsf{c}(I) + \mathsf{len}(L; \ell) < \mathsf{c}(I)$. This contradicts the fact that $I \in \mathcal{O}_t^*$.

In what follows, we prove the "if" part. Assume that $\mathsf{mc}_t(I, \ell) \geq 0$ and $\mathsf{mp}_t(\mathsf{s}; I, \ell) \geq 0$. Let J be a member in $\mathcal{O}_t$. Furthermore, we define I' (resp., J') as a maximal subset of $I \cup J$ satisfying the condition that $I' \in \mathcal{J}$ and $I \subseteq I'$ (resp, $J' \in \mathcal{J}$ and $J \subseteq J'$). Then, (I2) implies that $|I'| = |J'|$. Assume that $I' \setminus J' = \{u_1, u_2, \ldots, u_h\}$ and $J' \setminus I' = \{v_1, v_2, \ldots, v_h\}$. Theorem 25 implies that there exists a bijective mapping $\mathsf{f} \colon I' \setminus J' \to J' \setminus I'$ such that $(I' \cup \{\mathsf{f}(u)\}) \setminus \{u\} \in \mathcal{I}$ for every element u in $I' \setminus J'$. For every element u in $I' \setminus J'$ such that $\mathsf{f}(u) \in \mathbf{sp}_{\mathbf{N}}(I)$, since Theorem 1 implies that $\mathsf{C}_{\mathbf{N}}(\mathsf{f}(u), I) = \mathsf{C}_{\mathbf{N}}(\mathsf{f}(u), I')$, we have $(\mathsf{q}[\mathsf{f}(u)], \mathsf{q}[u]) \in \mathsf{A}_t(I)$. Define

$$\begin{aligned}
\mathsf{A}' &:= \{\mathsf{a}[v_p], \mathsf{b}[u_p] \mid p \in \{1, 2, \ldots, h\}\} \cup \{(\mathsf{q}[u], \mathsf{t}) \mid u \in I' \setminus I\} \cup \{(\mathsf{s}, \mathsf{q}[u]) \mid u \in J' \setminus J\} \\
&\quad \cup \{(\mathsf{q}[\mathsf{f}(u)], \mathsf{q}[u]) \mid u \in I' \setminus J', \mathsf{f}(u) \in \mathbf{sp}_{\mathbf{N}}(I)\} \\
&\quad \cup \{(\mathsf{q}[\mathsf{f}(u)], \mathsf{t}), (\mathsf{s}, \mathsf{q}[u]) \mid u \in I' \setminus J', \mathsf{f}(u) \notin \mathbf{sp}_{\mathbf{N}}(I)\} \\
\mathsf{A}^{\blacksquare} &:= \{(\mathsf{s}, \mathsf{p}[i]) \mid i \in \{1, 2, \ldots, d\}, |I \cap S_i| < |J \cap S_i|\} \\
\mathsf{A}^{\square} &:= \{(\mathsf{p}[i], \mathsf{t}) \mid i \in \{1, 2, \ldots, d\}, |I \cap S_i| > |J \cap S_i|\}.
\end{aligned}$$

Then, it is not difficult to see that there exist simple directed paths $L_1, L_2, \ldots, L_n$ in $\mathsf{D}_t(I)$ from s to t and directed cycles $L_{n+1}, L_{n+2}, \ldots, L_m$ in $\mathsf{D}_t(I)$ satisfying the following conditions.

- For every arc a in $\mathsf{A}_t(I)$, if $a \in L_x$ for some integer x in $\{1, 2, \ldots, m\}$, then $a \in \mathsf{A}' \cup \mathsf{A}^{\blacksquare} \cup \mathsf{A}^{\square}$.
- For every arc a in A', there exists exactly one integer x in $\{1, 2, \ldots, m\}$ such that $a \in L_x$.
- For every arc $(\mathsf{s}, \mathsf{p}[i])$ in $\mathsf{A}^{\blacksquare}$, $|\{x \in \{1, 2, \ldots, n\} \mid (\mathsf{s}, \mathsf{p}[i]) \in L_x\}| = |J \cap S_i| - |I \cap S_i|$.
- For every arc $(\mathsf{p}[i], \mathsf{t})$ in $\mathsf{A}^{\square}$, $|\{x \in \{1, 2, \ldots, n\} \mid (\mathsf{p}[i], \mathsf{t}) \in L_x\}| = |I \cap S_i| - |J \cap S_i|$.

Thus, we have $\mathsf{c}(J) - \mathsf{c}(I) = \sum_{i=1}^{m} \mathsf{len}(L_i; \ell) \geq 0$. This completes the proof. □

4.1.2 Proof of Lemma 20

The following proof is based on the original proof of [14, Theorem 2]. Since π^k is a compatible potential function for $\mathsf{D}_2(I^k)$, $\ell^k(a) \geq 0$ holds for every arc a in $\mathsf{A}_2(I^k)$. Thus, the first statement follows from Lemma 29. In what follows, we assume that $L^k = (a_1, a_2, \ldots, a_h)$ and $a_p = (\mathsf{v}_p, \mathsf{w}_p)$ for each integer p in $\{1, 2, \ldots, h\}$. It is not difficult to see that $\mathsf{mp}_2(\mathsf{v}; I^k, \ell) \geq \pi^k(\mathsf{v})$ holds for every vertex v in V. Thus, since $\pi^k(\mathsf{q}[u]) \geq 0$ holds for every element u in I^k and $\mathsf{len}(L^k; \ell) < 0$, we have $a_1 \in \mathsf{A}^{\mathsf{p}}$. Furthermore, it is not difficult to see that for every vertex v in V,

$$\pi^{k+1}(\mathsf{v}) = \pi^k(\mathsf{v}) + \mathsf{mp}_2(\mathsf{v}; I^k, \ell^k) = \pi^k(\mathsf{v}) + \mathsf{mp}_2(\mathsf{v}; I^k, \ell) - \pi^k(\mathsf{v}) = \mathsf{mp}_2(\mathsf{v}; I^k, \ell). \tag{3}$$

Lemma 30. *π^{k+1} is a compatible potential function for $\mathsf{D}_2(I^k)$.*

Proof. For every arc $a = (\mathsf{v}, \mathsf{w})$ in $\mathsf{A}_2(I^k)$, $\mathsf{mp}_2(\mathsf{v}; I^k, \ell) \leq \mathsf{mp}_2(\mathsf{w}; I^k, \ell) + \ell(a)$. Thus, for every arc $a = (\mathsf{v}, \mathsf{w})$ in $\mathsf{A}_2(I^k)$, (3) implies that

$$\pi^{k+1}(\mathsf{v}) - \pi^{k+1}(\mathsf{w}) = \mathsf{mp}_2(\mathsf{v}; I^k, \ell) - \mathsf{mp}_2(\mathsf{w}; I^k, \ell) \leq \ell(a).$$

This completes the proof. □

Lemma 31. *For every integer p in $\{1, 2, \ldots, h\}$, we have $\ell(a_p) = \pi^{k+1}(\mathsf{v}_p) - \pi^{k+1}(\mathsf{w}_p)$.*

Proof. Lemma 30 implies that $\ell(a_p) \geq \pi^{k+1}(\mathsf{v}_p) - \pi^{k+1}(\mathsf{w}_p)$ for every integer p in $\{1, 2, \ldots, h\}$. In addition, we have $\mathcal{L}_2(\mathsf{s}; I^k, \ell^k) = \mathcal{L}_2(\mathsf{s}; I^k, \ell)$ and $\pi^{k+1}(\mathsf{t}) = 0$. Thus, (3) implies that

$$\pi^{k+1}(\mathsf{s}) = \mathsf{mp}_2(\mathsf{v}; I^k, \ell) = \mathsf{len}(L^k; \ell) = \sum_{p=1}^{h} \ell(a_p) \geq \sum_{p=1}^{h} (\pi^{k+1}(\mathsf{v}_p) - \pi^{k+1}(\mathsf{w}_p)) = \pi^{k+1}(\mathsf{s}).$$

This implies that $\ell(a_p) = \pi^{k+1}(\mathsf{v}_p) - \pi^{k+1}(\mathsf{w}_p)$ for every integer p in $\{1, 2, \ldots, h\}$. □

The third statement of Lemma 20 follows from the following lemma.

Lemma 32. *For every element u in I^{k+1}, we have $\pi^{k+1}(\mathsf{q}[u]) \geq 0$.*

Proof. For every element u in $I^k \cap I^{k+1}$, since $\mathsf{mp}_2(\mathsf{q}[u]; I^k, \ell^k) \geq 0$ holds, $\pi^{k+1}(\mathsf{q}[u]) \geq 0$ holds. Let u be an element in $I^{k+1} \setminus I^k$. It is not difficult to see that there exists an integer p in $\{1, 2, \ldots, h\}$ such that $a_p = \mathsf{a}[u]$. If $p = h - 1$, then Lemma 31 implies that $\pi^{k+1}(\mathsf{q}[u]) = \pi^{k+1}(\mathsf{t}) = 0$. Assume that $p < h - 1$ and $a_{p+1} = (\mathsf{q}[u], \mathsf{q}[v])$. Then, the induction hypothesis implies that $\pi^{k+1}(\mathsf{q}[v]) = \pi^k(\mathsf{q}[v]) + \mathsf{mp}_2(\mathsf{q}[v]; I^k, \ell^k) \geq 0$. Thus, Lemma 31 implies that $\pi^{k+1}(\mathsf{q}[u]) = \pi^{k+1}(\mathsf{q}[v]) \geq 0$. □

In what follows, we define z as a positive integer such that

$$h = \begin{cases} 3z + 2 & \text{if } a_h \in \mathsf{A}^{\mathsf{p}} \\ 3z + 3 & \text{if } a_h \in \mathsf{A}^{\mathsf{q}}. \end{cases}$$

Assume that $a_{3p-1} = \mathsf{a}[v_p]$ and $a_{3p+1} = \mathsf{b}[u_p]$ for each positive integer p in $\{1, 2, \ldots, z\}$. Furthermore, we define elements $\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_z$ in I^k and $\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_z$ in $S \setminus I^k$ as follows.

- $\{u_1, u_2, \ldots, u_z\} = \{\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_z\}$ and $\{v_1, v_2, \ldots, v_z\} = \{\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_z\}$.
- For every pair of integers m, n in $\{1, 2, \ldots, z\}$, $u_m = \overline{u}_n$ if and only if $v_m = \overline{v}_n$
- $\pi^{k+1}(\mathsf{q}[\overline{u}_m]) \leq \pi^{k+1}(\mathsf{q}[\overline{u}_n])$ for every pair of integers m, n in $\{1, 2, \ldots, z\}$ such that $m < n$.
- For every pair of integers m, n in $\{1, 2, \ldots, z\}$ such that $m < n$, if $\pi^{k+1}(\mathsf{q}[\overline{u}_m]) = \pi^{k+1}(\mathsf{q}[\overline{u}_n])$ holds, then there exist integers m', n' in $\{1, 2, \ldots, z\}$ such that $m' > n'$ such that $\overline{u}_m = u_{m'}$ and $\overline{u}_n = u_{n'}$.

Notice that Lemma 31 implies that $\pi^{k+1}(\mathsf{q}[\overline{u}_p]) = \pi^{k+1}(\mathsf{q}[\overline{v}_p])$ for every integer p in $\{1, 2, \ldots, z\}$.

Lemma 33. $\overline{u}_m \notin \mathsf{C}_{\mathbf{N}}(\overline{v}_n, I^k)$ *for any pair of integers* m, n *in* $\{1, 2, \ldots, z\}$ *such that* $m < n$.

Proof. Assume that there exist integers m, n in $\{1, 2, \ldots, z\}$ such that $m < n$ and $\overline{u}_m \in \mathsf{C}_{\mathbf{N}}(\overline{v}_n, I^k)$. Since $(\mathsf{q}[\overline{v}_n], \mathsf{q}[\overline{u}_m]) \in \mathsf{A}_2(I^k)$, Lemma 30 implies that $\pi^{k+1}(\mathsf{q}[\overline{u}_m]) \geq \pi^{k+1}(\mathsf{q}[\overline{v}_n]) = \pi^{k+1}(\mathsf{q}[\overline{u}_n])$. Thus, $\pi^{k+1}(\mathsf{q}[\overline{u}_m]) = \pi^{k+1}(\mathsf{q}[\overline{u}_n])$. This implies that there exist integers m', n' in $\{1, 2, \ldots, z\}$ such that $m' > n'$ such that $\overline{u}_m = u_{m'}$ and $\overline{u}_n = u_{n'}$. Define $L := (a_{3n'}, a_{3n'+1}, \ldots, a_{3m'})$. Lemma 30 implies that $\mathsf{len}(L; \ell) \geq \pi^{k+1}(\mathsf{q}[\overline{v}_n]) - \pi^{k+1}(\mathsf{q}[\overline{u}_m]) = 0$. This contradicts the definition of L^k (i.e., there exists a shortcut from $\mathsf{q}[v_{n'}]$ to $\mathsf{q}[u_{m'}]$). This contradicts the definition of L^k. □

Lemma 34. $\mathbf{cl_N}(I^k) \subseteq \mathbf{cl_N}(I^{k+1})$. *Furthermore, if* $a_h \in \mathsf{A}^{\mathsf{p}}$, *then* $\mathbf{cl_N}(I^k) = \mathbf{cl_N}(I^{k+1})$.

Proof. If $a_h \in \mathsf{A}^{\mathsf{p}}$, then the second statement of this lemma follows from Theorem 26 and Lemma 33. Next we assume that $a_h \in \mathsf{A}^{\mathsf{q}}$. Define $L := (a_1, a_2, \ldots, a_{h-2})$ and $J := (I^k \cup \mathbf{p}(L)) \setminus \mathbf{n}(L)$. Then, Theorem 26 and Lemma 33 imply that $J \in \mathcal{J}$ and $\mathbf{cl_N}(I^k) = \mathbf{cl_N}(J)$. Thus, since $J \subseteq I^{k+1}$, we have $\mathbf{cl_N}(J) \subseteq \mathbf{cl_N}(I^{k+1})$. This completes the proof. □

Proof of Lemma 20. Since we have already proved the first and third statements, what remains is to prove that π^{k+1} is a compatible potential function for $\mathsf{D}_2(I^{k+1})$. That is, we prove that for every arc $a = (\mathsf{v}, \mathsf{w})$ in $\mathsf{A}_2(I^{k+1})$, $\ell(a) \geq \pi^{k+1}(\mathsf{v}) - \pi^{k+1}(\mathsf{w})$ holds. Notice that Lemma 30 implies that for every arc $a = (\mathsf{v}, \mathsf{w})$ in $\mathsf{A}_2(I^k)$, $\ell(a) \geq \pi^{k+1}(\mathsf{v}) - \pi^{k+1}(\mathsf{w})$ holds.

We consider arcs in A^+. Let $\mathsf{a}[u]$ be an arc in $\mathsf{A}^+(I^{k+1}) \setminus \mathsf{A}^+(I^k)$. Then, it is not difficult to see that there exists an integer p in $\{1, 2, \ldots, h\}$ such that $\mathsf{b}[u] = a_p$. Thus, Lemma 31 implies that $\pi^{k+1}(\mathsf{q}[u]) - \pi^{k+1}(\mathsf{p}[\mathsf{i}(u)]) = -\xi(u)$, which implies that $\pi^{k+1}(\mathsf{p}[\mathsf{i}(u)]) - \pi^{k+1}(\mathsf{q}[u]) = \xi(u)$.

We consider arcs in A^-. Let $\mathsf{b}[u]$ be an arc in $\mathsf{A}^-(I^{k+1}) \setminus \mathsf{A}^-(I^k)$. Then, it is not difficult to see that there exists an integer p in $\{1, 2, \ldots, h\}$ such that $\mathsf{a}[u] = a_p$. Thus, Lemma 31 implies that $\pi^{k+1}(\mathsf{p}[\mathsf{i}(u)]) - \pi^{k+1}(\mathsf{q}[u]) = \xi(u)$, which implies that $\pi^{k+1}(\mathsf{q}[u]) - \pi^{k+1}(\mathsf{p}[\mathsf{i}(u)]) = -\xi(u)$.

We consider arcs in A^{p}. Let $(\mathsf{s}, \mathsf{p}[i])$ be an arc in $\mathsf{A}_2^{\mathsf{p}}(I^{k+1}) \setminus \mathsf{A}_2^{\mathsf{p}}(I^k)$. In this case, $a_h \in \mathsf{A}^{\mathsf{p}}$ and $a_h = (\mathsf{p}[i], \mathsf{t})$. Since $L^k \in \mathcal{L}_2(\mathsf{s}; I^k, \ell^k)$ holds, $(a_h) \in \mathcal{L}_2(\mathsf{p}[i]; I^k, \ell^k)$. Thus, $\mathsf{len}(L; \ell^k) \geq \mathsf{len}((a_h); \ell^k) = -\pi^k(\mathsf{p}[i])$ for every directed path L in $\mathsf{D}_2(I^k)$ from $\mathsf{p}[i]$ to t. This implies that $\mathsf{len}(L; \ell) \geq 0$ holds for every directed path L in $\mathsf{D}_2(I^k)$ from $\mathsf{p}[i]$ to t, and $\mathsf{mp}_2(\mathsf{p}[i]; I^k, \ell) = 0$. Thus, (3) implies that $\pi^{k+1}(\mathsf{p}[i]) = \mathsf{mp}_2(\mathsf{p}[i]; I^k, \ell) = 0$. Since we have $\mathsf{len}(L^k; \ell) < 0$, (3) implies that $\pi^{k+1}(\mathsf{s}) < 0$ holds. Thus, $\pi^{k+1}(\mathsf{s}) - \pi^{k+1}(\mathsf{p}[i]) \leq 0 = \ell((\mathsf{s}, \mathsf{p}[i]))$. Let $(\mathsf{p}[i], \mathsf{t})$ be an arc in $\mathsf{A}_2^{\mathsf{p}}(I^{k+1}) \setminus \mathsf{A}_2^{\mathsf{p}}(I^k)$. In this case, $a_1 = (\mathsf{s}, \mathsf{p}[i])$ holds. Thus, since (3) implies that $\pi^{k+1}(\mathsf{p}[i]) = \mathsf{mp}_2(\mathsf{p}[i]; I^k, \ell) = \mathsf{len}(L^k; \ell) < 0$, we have $\pi^{k+1}(\mathsf{p}[i]) - \pi^{k+1}(\mathsf{t}) \leq 0 = \ell((\mathsf{p}[i], \mathsf{t}))$.

We consider arcs in A^{q}. For every arc $(\mathsf{s}, \mathsf{q}[u])$ in $\mathsf{A}^{\mathsf{q}}(I^{k+1})$, Lemma 32 implies that $\pi^{k+1}(\mathsf{s}) - \pi^{k+1}(\mathsf{q}[u]) \leq 0 = \ell((\mathsf{s}, \mathsf{q}[u]))$. Let $(\mathsf{q}[u], \mathsf{t})$ be an arc in $\mathsf{A}^{\mathsf{q}}(I^{k+1})$. Since $u \notin \mathbf{cl_N}(I^{k+1})$, Lemma 34 implies that $u \notin \mathbf{cl_N}(I^k)$. This implies that $(\mathsf{q}[u], \mathsf{t}) \in \mathsf{A}^{\mathsf{q}}(I^k)$ holds. What remains is to prove that $\pi^{k+1}(\mathsf{q}[u]) \leq \pi^{k+1}(\mathsf{q}[v])$ for every arc $(\mathsf{q}[v], \mathsf{q}[u])$ in $\mathsf{A}^{\mathsf{q}}(I^{k+1})$. We first assume that $a_h \in \mathsf{A}^{\mathsf{p}}$. For each integer p in $\{1, 2, \ldots, z\}$, we define

$$J_p := (I^k \cup \{\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_p\}) \setminus \{\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_p\}.$$

Define $J_0 := I^k$. Then, Theorem 26 and Lemma 33 imply that $J_p \in \mathcal{J}$ and $\mathbf{cl_N}(I^k) = \mathbf{cl_N}(J_p)$ for every integer p in $\{1, 2, \ldots, z\}$. For every pair of integers m, n in $\{1, 2, \ldots, z\}$ such that $m < n$, Lemma 33 implies that $\mathsf{C}_{\mathbf{N}}(\overline{v}_n, J_m) = \mathsf{C}_{\mathbf{N}}(\overline{v}_n, I^k)$. Thus, every pair of integers m, n in $\{1, 2, \ldots, z\}$ such that $m < n$, we have $\overline{u}_n \in \mathsf{C}_{\mathbf{N}}(\overline{v}_n, J_m)$. Let p be an integer in $\{1, 2, \ldots, z\}$. Then, we assume that $\pi^{k+1}(\mathsf{q}[v]) \leq \pi^{k+1}(\mathsf{q}[u])$ for every arc $(\mathsf{q}[v], \mathsf{q}[u])$ in $\mathsf{A}^{\mathsf{q}}(J_{p-1})$, and we prove that $\pi^{k+1}(\mathsf{q}[v]) \leq \pi^{k+1}(\mathsf{q}[u])$ for every arc $(\mathsf{q}[v], \mathsf{q}[u])$ in $\mathsf{A}^{\mathsf{q}}(J_p)$. Let $(\mathsf{q}[v], \mathsf{q}[u])$ be an arc in $\mathsf{A}^{\mathsf{q}}(J_p) \setminus \mathsf{A}^{\mathsf{q}}(J_{p-1})$. Notice that $\overline{u}_p \in \mathsf{C}_{\mathbf{N}}(\overline{v}_p, J_{p-1})$. Since $v \in \mathbf{cl_N}(J_p) = \mathbf{cl_N}(J_{p-1}) = \mathbf{cl_N}(I^k)$, $v = \overline{u}_p$ or $v \in \mathbf{sp_N}(J_{p-1}) \setminus \{\overline{v}_p\}$. In what follows, we will use Theorem 27 by setting $I := J_{p-1}$, $J := J_p$, $u = \overline{u}_p$, $v = \overline{v}_p$, $u' := u$ (if $u \neq \overline{v}_p$), and $v' := v$ (if $v \neq \overline{u}_p$).

- If $v = \overline{u}_p$ and $u = \overline{v}_p$, then Lemma 31 implies that $\pi^{k+1}(\mathsf{q}[\overline{u}_p]) = \pi^{k+1}(\mathsf{q}[\overline{v}_p])$.
- Assume that $v \neq \overline{u}_p$ and $u = \overline{v}_p$. Theorem 27 implies that $(\mathsf{q}[v], \mathsf{q}[\overline{u}_p]) \in \mathsf{A}^{\mathsf{q}}(J_{p-1})$. Thus, we have $\pi^{k+1}(\mathsf{q}[v]) \leq \pi^{k+1}(\mathsf{q}[\overline{u}_p]) = \pi^{k+1}(\mathsf{q}[\overline{v}_p])$.
- Assume that $v = \overline{u}_p$ and $u \neq \overline{v}_p$. Theorem 27 implies that $(\mathsf{q}[\overline{v}_p], \mathsf{q}[u]) \in \mathsf{A}^{\mathsf{q}}(J_{p-1})$. Thus, we have $\pi^{k+1}(\mathsf{q}[\overline{u}_p]) = \pi^{k+1}(\mathsf{q}[\overline{v}_p]) \leq \pi^{k+1}(\mathsf{q}[u])$.
- Assume that $v \neq \overline{u}_p$ and $u \neq \overline{v}_p$. Then, Theorem 27 implies that $(\mathsf{q}[v], \mathsf{q}[\overline{u}_p]) \in \mathsf{A}^{\mathsf{q}}(J_{p-1})$ and $(\mathsf{q}[\overline{v}_p], \mathsf{q}[u]) \in \mathsf{A}^{\mathsf{q}}(J_{p-1})$. Thus, $\pi^{k+1}(\mathsf{q}[v]) \leq \pi^{k+1}(\mathsf{q}[\overline{u}_p]) = \pi^{k+1}(\mathsf{q}[\overline{v}_p]) \leq \pi^{k+1}(\mathsf{q}[u])$.

Next we assume that $a_h \in \mathsf{A}^{\mathsf{q}}$. Assume that $a_{3z+2} = \mathsf{a}[v_{z+1}]$. Define

$$J := (I^k \cup \{v_1, v_2, \ldots, v_z\}) \setminus \{u_1, u_2, \ldots, u_z\}.$$

Then, $I^{k+1} = J \cup \{v_{z+1}\}$. In the same way as above, we can prove that $J \in \mathcal{J}$ and $\pi^{k+1}(\mathsf{q}[v]) \leq \pi^{k+1}(\mathsf{q}[u])$ for every arc $(\mathsf{q}[v], \mathsf{q}[u])$ in $\mathsf{A}^{\mathsf{q}}(J)$. Let $(\mathsf{q}[v], \mathsf{q}[u])$ be an arc in $\mathsf{A}^{\mathsf{q}}(I^{k+1}) \setminus \mathsf{A}^{\mathsf{q}}(J)$. Then, Theorem 1 implies that if $v \in \mathbf{sp}_{\mathbf{N}}(J)$, then $\mathsf{C}_{\mathbf{N}}(v, J) = \mathsf{C}_{\mathbf{N}}(v, I^{k+1})$, i.e., $(\mathsf{q}[v], \mathsf{q}[u]) \in \mathsf{A}^{\mathsf{q}}(I^{k+1}) \cap \mathsf{A}^{\mathsf{q}}(J)$. Thus, $v \notin \mathbf{cl}_{\mathbf{N}}(J)$ holds. Since Theorem 26 and Lemma 33 imply that $\mathbf{cl}_{\mathbf{N}}(I^k) = \mathbf{cl}_{\mathbf{N}}(J)$, we have $v \notin \mathbf{cl}_{\mathbf{N}}(I^k)$. This implies that $(\mathsf{q}[v], \mathsf{t}) \in \mathsf{A}_2(I^k)$, and Lemma 30 implies that $\pi^{k+1}(\mathsf{q}[v]) \leq \pi^{k+1}(\mathsf{t}) = 0$. Thus, Lemma 32 implies that $\pi^{k+1}(\mathsf{q}[v]) \leq 0 \leq \pi^{k+1}(\mathsf{q}[u])$. □

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